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Spatial Numerical Range in Non-unital, Normed Algebras II

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Abstract: Let $(A, \|\cdot\|)$ be a non-unital, normed algebra. In this paper, we study the geometry of the spatial numerical range $V_A(a)$ of an element a of A. This is in the continuation of our paper [3]. In [2, Section 10], all results on the (spatial) numerical range are proved for unital normed algebras. In this paper, we study these results for non-unital algebras.

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Keywords: Banach Algebra, Numerical Range, Spatial Numerical Range.

1. Introduction

Let $(A, \|\cdot\|)$ be a non-unital, normed algebra. Let $S(A) = \{x \in A : \|x\| = 1\}$ be the unit sphere in A and the dual set $D_A(x) = \{\varphi \in A^* : \|\varphi\| = 1 = \varphi(x)\}$ for each $x \in S(A)$. Let $V_A(a; x) = \{\varphi(ax) : \varphi \in D_A(x)\}$ for $a \in A$. Then the set $V_A(a) = \cup\{V_A(a; x) : x \in S(A)\}$ is the *spatial numerical range (SNR)* of a in $(A, \|\cdot\|)$. If A is a unital normed algebra, then the spatial numerical range is exactly the numerical range defined in [2]. The idea of the spatial numerical range in non-unital, normed algebras came from operator theory. It is very well studied and used in operator theory. We have applied this concept in proving some results on the spectral extension property (SEP) in non-unital Banach algebras [4]. Consider $A_e = A + \mathbb{C}1$ be the unitization of A with the identity 1. Then $\sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda 1$ is not invertible in $A_e\}$ is the *spectrum* of a in A. In the case of unital Banach algebras, the spectrum is contained in the numerical range. The natural question arises that, is it true in the non-complete, normed algebras? Here we answer this question in negation; we give counter examples for the non-unital as well as the unital case. We shall refer [3] for notation and basic results on spatial numerical range. To the best of our knowledge, the results proved in this article are not proved by others in non-unital normed algebras.

2. Main Results

The following lemma expresses $V(a; x)$ as an intersection of closed discs in the complex plane. We use the notation $B(z, r) = \{\lambda \in \mathbb{C} : |z - \lambda| \leq r\}$ for the closed disc with center z and radius r. It is a non-unital analogue of [2, Lemma 5, P.52]. In the case of unital algebra, taking identity in place of x, we get the result about the numerical range of an element $a \in A$.

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Proposition 2.1. *Let A be a non-unital normed algebra. Let $a \in A$ and $x \in S(A)$. Then*

(i). $V_A(a;x) = \cap_{z\in\mathbb{C}} B(z, \|zx - ax\|)$;

(ii). $V_A(a) = \cup_{x\in S(A)}\{\cap_{z\in\mathbb{C}} B(z, \|zx - ax\|)\}$;

(iii). $\inf\{|\lambda| : \lambda \in V_A(a)\} \le \inf\{\|ax\| : x \in S(A)\}$.

Proof. (i) Let $\lambda \in V_A(a;x)$. Then $\lambda = \varphi(ax)$ and $\varphi \in D_A(x)$. Now, for any $z \in \mathbb{C}$,

$$|\lambda - z| = |\varphi(ax) - z| = |\varphi(ax - zx)| \le \|ax - zx\|.$$

Hence $\lambda \in \cap_{z\in\mathbb{C}} B(z, \|zx - ax\|)$. Conversely, let $\lambda \in \cap_{z\in\mathbb{C}} B(z, \|zx - ax\|)$. Then $|z - \lambda\| \le \|zx - ax\|$, for every $z \in \mathbb{C}$. Suppose that x and ax are linearly dependents, i.e, $ax = \alpha x$ for some $\alpha \in \mathbb{C}$. Therefore $\alpha = \lambda$. Let $Y = span\{ax\}$. Define $\varphi(zax) = z\lambda\,(zax \in Y)$. Then

$$\varphi(x) = \varphi(\frac{1}{\alpha}ax) = \frac{1}{\alpha}\lambda = \frac{1}{\alpha}\alpha = 1.$$

Also we have $|\varphi(zax)| = |z\lambda| = |z\lambda|\|x\| = \|z\alpha x\| = \|zax\|$. Therefore $\|\varphi\| = 1$. And $\varphi(ax) = \lambda \in V_A(a;x)$. Thus $\cap_{z\in\mathbb{C}} B(z, \|zx - ax\|) \subset V_A(a)$. Now suppose that a and ax are linearly independent. Define $\varphi(\alpha x + \beta ax) = \alpha + \beta\lambda$. Suppose that $\beta \ne 0$. Since $\lambda \in \cap_{z\in\mathbb{C}} B(z, \|zx - ax\|)$, $|z - \lambda| \le \|zx - ax\|(z \in \mathbb{C})$. Take $z = -\frac{\alpha}{\beta}$. Then we get $|-\frac{\alpha}{\beta} - \lambda| \le \|-\frac{\alpha}{\beta}x - ax\|$, i.e., $|\alpha + \beta\lambda| \le \|\alpha x + \beta ax\|$. Now if $\beta = 0$, then $|\varphi(\alpha x)| = \|\alpha\|$. Therefore $\|\varphi\| \le 1$. Also we have $\varphi(x) = 1$. So we get $\|\varphi\| = 1$. Hence $\lambda = \varphi(ax) \in V_A(a;x)$.

(ii) This immediately follows from (i) above.

(iii) By the definition of the spatial numerical range,

$$\begin{aligned}
\inf\{|\lambda| : \lambda \in V_A(a)\} &= \inf\{\cup_{x\in S(A)}\{|\lambda| : \lambda \in V_A(a;x)\}\} \\
&= \inf\{\cup_{x\in S(A)}\{|f(ax)| : f \in D_A(x)\}\} \\
&\le \inf\{\|ax\| : x \in S(A)\}.
\end{aligned}$$

□

Next result is a non-unital analogue of [2]. The proof is a slight modification of the proof given in [2]. For $z \in \mathbb{C}$, Rez is the real part of z.

Theorem 2.2. *Let A be a non-unital normed algebra and let $a \in A$. Then*

$$\begin{aligned}
\max\{Re\lambda : \lambda \in V_A(a)\} &= \inf_{\alpha>0} \frac{1}{\alpha} \sup_{x\in S(A)} \{\|x + \alpha ax\| - 1\} \\
&= \lim_{\alpha\longrightarrow 0^+} \frac{1}{\alpha} \sup_{x\in S(A)} \{\|x + \alpha ax\| - 1\}
\end{aligned}$$

Proof. If $a = 0$, then the identity is clearly true. So we assume that $a \ne 0$. Set $\mu = \max\{Re\lambda : \lambda \in V_A(a)\}$. Let $x \in S(A)$ and $\varphi \in D_A(x)$. Then, for any $\alpha > 0$, we have

$$\begin{aligned}
\varphi(ax) &= \frac{1}{\alpha}\{\varphi(x + \alpha ax) - 1\} \\
\Longrightarrow Re\varphi(ax) &= \frac{1}{\alpha} Re\{\varphi(x + \alpha ax) - 1\} \\
\Longrightarrow Re\varphi(ax) &\le \frac{1}{\alpha} \sup_{x\in S(A)} \{\|x + \alpha ax\| - 1\} \\
\Longrightarrow \qquad \mu &\le \frac{1}{\alpha} \sup_{x\in S(A)} \{\|x + \alpha ax\| - 1\}.
\end{aligned}$$

Since $\alpha > 0$ is arbitrary, we get

$$\mu \leq \inf_{\alpha>0} \frac{1}{\alpha} \sup_{x\in S(A)} \{\|x + \alpha ax\| - 1\} \leq \limsup_{\alpha\longrightarrow 0^+} \frac{1}{\alpha} \sup_{x\in S(A)} \{\|x + \alpha ax\| - 1\}. \tag{1}$$

For the reverse inequality, let $0 < \alpha < \|a\|^{-1}$. Given $x \in S(A)$ and $\varphi \in D_A(x)$, we have

$$\begin{aligned}
& Re\varphi(ax) \leq \mu \leq \|a\| \\
\Longrightarrow\ & Re\varphi(\alpha ax) \leq \alpha\mu \leq \alpha\|a\| \\
\Longrightarrow\ & 0 < 1 - \alpha\|a\| \leq 1 - \alpha\mu \leq 1 - Re\varphi(\alpha ax) \\
\Longrightarrow\ & 0 < 1 - \alpha\mu \leq Re\varphi(x - \alpha ax) \leq \|x - \alpha ax\|.
\end{aligned} \tag{2}$$

Now for any $y \in A$, taking $x = \frac{y}{\|y\|}$ in the inequality (2) above, we get

$$(1 - \alpha\mu)\|y\| \leq \|y - \alpha ay\| \quad (y \in A). \tag{3}$$

Finally, let $x \in S(A)$ be arbitrary. Now taking $y = x + \alpha ax$, in inequality (3) above, we obtain

$$\begin{aligned}
& (1 - \alpha\mu)\|x + \alpha ax\| \leq \|x + \alpha ax - \alpha a(x + \alpha ax)\| = \|x - \alpha^2 a^2 x\| \\
\Longrightarrow\ & \|x + \alpha ax\| \leq \frac{\|x - \alpha^2 a^2 x\|}{1 - \alpha\mu} \leq \frac{1 + \alpha^2\|a^2\|}{1 - \alpha\mu} \\
\Longrightarrow\ & \|x + \alpha ax\| - 1 \leq \frac{1 + \alpha^2\|a^2\|}{1 - \alpha\mu} - 1 = \frac{\alpha^2\|a^2\| + \alpha\mu}{1 - \alpha\mu} \\
\Longrightarrow\ & \sup_{x\in S(A)} \frac{1}{\alpha}(\|x + \alpha ax\| - 1) \leq \frac{\alpha\|a^2\| + \mu}{1 - \alpha\mu} \\
\Longrightarrow\ & \liminf_{\alpha\longrightarrow 0^+} \frac{1}{\alpha} \sup_{x\in S(A)} (\|x + \alpha ax\| - 1) \leq \liminf_{\alpha\longrightarrow 0^+} \frac{\mu + \alpha\|a^2\|}{1 - \alpha\mu} = \mu.
\end{aligned} \tag{4}$$

Hence, by inequalities (1) and (4), the result follows. □

The next result should be compared with [2, Proposition 7, P.53]. We do not know whether equality holds or not.

Proposition 2.3. *Let J be a closed bi-ideal in A, and $a \in A$. Then*

$$V_{A/J}(a + J) \subset \cap\{V_A(a + j) : j \in J\}.$$

Proof. Let $\lambda \in V_{A/J}(a + J)$. By Proposition 2.1($ii$), there exists $x + J \in S(A/J)$ such that

$$\begin{aligned}
& \lambda \in \cap_{z\in\mathbb{C}} B(z, |||z(x + J) - (a + J)(x + J)|||); \\
\Longrightarrow\ & \lambda \in \cap_{z\in\mathbb{C}} B(z, |||(zx - ax) + J|||); \\
\Longrightarrow\ & |z - \lambda| \leq |||(zx - ax) + J||| \quad (z \in \mathbb{C}); \\
\Longrightarrow\ & |z - \lambda| \leq ||zx - ax - jx|| \quad (z \in \mathbb{C}, j \in J); \\
\Longrightarrow\ & |z - \lambda| \leq ||zx - (a + j)x|| \quad (z \in \mathbb{C}, j \in J); \\
\Longrightarrow\ & \lambda \in B(z, \|zx - (a + j)x\|) \quad (z \in \mathbb{C}, j \in J); \\
\Longrightarrow\ & \lambda \in \cap_{j\in J} \cap_{z\in\mathbb{C}} B(z, \|zx - (a + j)x\|); \\
\Longrightarrow\ & \lambda \in \cap_{j\in J} V_A(a + j; x) \quad (\text{ by Proposition } 2.1(i)); \\
\Longrightarrow\ & \lambda \in \cap_{j\in J} V_A(a + j).
\end{aligned}$$

This completes the proof. □

Example 2.4. *In the case of unital Banach algebras, the spectrum is contained in the numerical range [1, Section 2, P.19]. However, this is not true in normed algebras. Following are counter examples.*
(*i*) *Consider* $A = (P[0,1], \|\cdot\|_\infty)$, *where* $P[0,1]$ *is a set of all polynomials on the closed interval* $[0,1]$. *Then* A *is a unital normed algebra. Then the spectrum* $\sigma_A(p) = \mathbb{C}$, *where* p *is any non-constant polynomial. But the spatial numerical range* $V_A(p)$ *is always bounded. Hence the spectrum* $\sigma_A(a)$ *not contained in the spatial numerical range* $V_A(p)$.
(*ii*) *Consider* $A = (P_{\frac{1}{2}}[0,1], \|\cdot\|_\infty)$, *where* $P_{\frac{1}{2}}[0,1] = \{p \in P[0,1] : p(\frac{1}{2}) = 0\}$. *Then* A *is a non-unital normed algebra. As above,* $\sigma_A(p) = \mathbb{C}$ *and* $V_A(a)$ *is bounded, and hence* $\sigma_A(p)$ *is not contained in* $V_A(p)$.

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