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# On Study Generalized $\mathcal{BR}$-Recurrent Finsler Space

**Research Article**

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**Abstract:** In this paper, we defined the generalized $\mathcal{B}R-$recurrent space which characterized by the following condition

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right), \qquad R^i_{jkh} \neq 0,$$

where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to $x^m$, $\lambda_m$ and $\mu_m$ are known as recurrence vectors. The purpose of the present paper to obtain the necessary and sufficient condition for (i) Berwald curvature tensor $H^i_{jkh}$, its associative $H_{jpkh}$ and Cartan's fourth curvature tensor to be generalized recurrent, (ii) the tensor $(H_{hk} - H_{kh})$ and $H-$Ricci tensor $H_{kh}$ are to be non-vanishing and (iii) the torsion tensor $K^i_{jk}$, the deviation tensor $K^i_h$, $K-$Ricci tensor $K_{jk}$, the curvature vectors $K_k$, $R_j$ and the curvature scalar $H$ to behave as recurrent. Also to study the covariant vectors $\lambda_{\mathrm{m}}$ and $\mu_{\mathrm{m}}$.

**Keywords:** Finsler Space, Generalized $\mathcal{B}R$-Recurrent Space, Landsberg Space.



## 1. Introduction

For the first time, the idea of recurrent of curvature tensor was study in Finsler space by A.Moor [1]. Due to different connections of Finsler space, the recurrent of Cartan's third curvature tensor $R^i_{jkh}$ have been discussed by R.Verma [9]. P.N.Pandey, S.Saxena and A.Goswami [8] introduced a generalized H-recurrent Finsler space, F.Y.A.Qasem and A.M.A.Al-qashbari ([2, 4]) introduced a generalized $H^h$-recurrent space and studied some types of this space, also they defined a generalized $R^h$-recurrent space and obtained some identities satisfied in such space [3]. F.Y.A. Qasem and A.A.A. Abdallah [5] studied certain types of generalized $\mathcal{B}R$-recurrent space.

Let $F_n$ be an n-dimensional Finsler space equipped with the metric function F(x, y) satisfying the request conditions [6]. The vector $y_i$ is defined by

$$a)\ \ y_i = g_{ij}(x,y)\, y^j \quad \text{and} \quad b)\ \ y_i y^i = F^2. \tag{1}$$

The two sets of quantities $g_{ij}$ and its associative $g^{ij}$, which are components of a metric tensor connected by

$$g_{ij} g^{ik} = \delta^k_j = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases} \tag{2}$$

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In view of (1) and (2), we have

$$a)\ \delta^i_k y_i = y_k,\ b)\ \delta^i_k y^k = y^i,\ c)\ \dot{\partial}_j y^i = \delta^i_j,\ d)\ \delta^i_j g_{ir} =\ g_{jr},\ e)\ \dot{\partial}_i y_j = g_{ij},\ f)\ \delta^i_j g^{jk} = g^{ik},\ g)\ \delta^i_i = n \text{ and } h)\ \delta^i_j \delta^j_k = \delta^i_k. \tag{3}$$

The tensor $C_{ijk}$ is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij} \tag{4}$$

which is positively homogeneous of degree -1 in $y^i$ and symmetric in all its indices and called (h)hv-torsion tensor [7] and its associative $C^i_{jk}$ is positively homogeneous of degree -1 in $y^i$ and symmetric in its lower indices and called (v)hv-torsion tensor. According to Euler's theorem on homogeneous functions, these tensors satisfy the following:

$$\begin{aligned} &a)\ C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0,\ b)\ C^i_{jk}y^k = 0 = C^i_{kj}y^k,\ \ c)\ C^i_{jr}g^{jr} = C^i, \\ &d)\ C^i_{jr}\delta^r_k = C^i_{jk},\ \ e)\ C^i_{ir} = C_r \ \text{ and }\ f)\ C^i y_i = 0. \end{aligned} \tag{5}$$

Berwald connected parameter $G^i_{jk}$, the functions $G^i_k$ and $G^i$ are homogeneous of degree zero, one and two, respectively [6] and satisfy the following :

$$a)\ G^i_{jkh} = \partial_j \partial_k G^i,\quad b)\ G^i_{jk}y^j = G^i_k \ \text{ and }\ c)\ \dot{\partial}_j G^i_k = G^i_{jk}. \tag{6}$$

The tensor $G^i_{jkh}$ is positively homogeneous of degree -1 in $y^i$ and symmetric in all its lower indices. In view of (4) and (6a) and due to Euler's theorem, we have

$$G^i_{jkh}y^j = G^i_{kjh}y^j = G^i_{khj}y^j = 0. \tag{7}$$

Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field $T^i_j$ with respect to $x^k$ is given by

$$\mathcal{B}_k T^i_j := \partial_k T^i_j - \left(\dot{\partial}_r T^i_j\right) G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

The processes of Berwald covariant differentiation with respect to$x^h$ and the partial differentiation with respect to $y^k$ commute according to

$$\left(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_k \dot{\partial}_h\right) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj} \tag{8}$$

for an arbitrary tensor field $T^i_j$. In most of existing literature, Berwald covariant derivative $\mathcal{B}_k T^i_j$ appears as $T^i_{j(k)}$. Berwald covariant derivative of the vector $y^i$ vanish identically, i.e.

$$\mathcal{B}_k y^i = 0. \tag{9}$$

But, in general, Berwald covariant derivative of the metric tensor $g_{ij}$ does not vanish and given by

$$\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h \mathcal{B}_h C_{ijk}. \tag{10}$$

The h-curvature tensor (Cartan's third curvature tensor) is defined by

$$R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + \left(\partial_l \Gamma^{*i}_{jk}\right) G^l_h + C^i_{jm}\left(\partial_k G^m_h - G^m_{kl}G^l_h\right) + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - k/h\ ^*.$$

The curvature tensor $R^i_{jkh}$ and the h(v)-torsion tensor $H^i_{kh}$ are related by

$$R^i_{jkh}y^j = H^i_{kh} = K^i_{jkh}y^j. \tag{11}$$

The h(v)-torsion tensor satisfies the relation

$$H^i_{kh} y^k = H^i_h = -H^i_{hk} y^k. \tag{12}$$

Berwald curvature tensor $H^i_{jkh}$ and the h(v)-torsion tensor $H^i_{kh}$ are skew-symmetric in the lower indices k and h and they are positively homogenous of degree zero and one in $y^i$, respectively. They are satisfy the following:

$$a)\;\; \dot{\partial}_r H^i_{kh} = H^i_{rkh},\;\; b)\;\; H^r_{jkr} = H_{jk},\;\; c)\;\; H_{ijkh} := g_{jr} H^r_{ikh},\;\; d)\;\; H^i_{ki} = H_k \text{ and } e)\;\; \dot{\partial}_h H_k = H_{kh}. \tag{13}$$

The deviation tensor $H^i_h$ is positively homogeneous of degree two in $y_i$. In view of Euler's theorem on homogeneous functions we have the following relations

$$a)\;\; H = \frac{1}{n-1} H^r_r,\;\; b)\;\; H_k y^k = (n-1)\,H \;\text{ and }\; c)\;\; H^i_{ikh} = H_{kh} - H_{hk}. \tag{14}$$

where $H$ is the curvature scalar. The tensor $K^i_{jkh}$ is called Cartan's is fourth curvature tensor defined as follows:

$$K^i_{jkh} := \partial_h \Gamma^{*i}_{kj} + \left(\dot{\partial}_l \Gamma^{*i}_{jh}\right) G^l_k + \Gamma^{*i}_{mh} \Gamma^{*m}_{kj} - h/k.$$

This curvature tensor is positively homogeneous of degree zero in $y_i$ and skew-symmetric in its last two lower indices. The curvature tensor $K^i_{jkh}$ satisfies the following relation

$$a)\;\; K^i_{jkh} y^h = K^i_{jk} \;\text{ and }\; b)\;\; K^i_{jkh} g^{jk} = K^i_h. \tag{15}$$

Ricci tensor $K_{jk}$, the curvature vector and the curvature scalar of the curvature tensor $K^i_{jkh}$ are given by

$$a)\;\; K^i_{jki} = K_{jk},\;\; b)\;\; K^i_{ik} = K_k \;\text{ and }\; c)\;\; K^i_i = K. \tag{16}$$

## 2. A Generalized $\mathcal{B}R-$ Recurrent Space

Let us consider a Finsler space $F_n$ in which Cartan's third curvature tensor $R^i_{jkh}$ satisfies the generalized recurrence property with respect to Berwald's connection parameters $G^i_{kh}$, i.e. characterized by the following condition

$$\mathcal{B}_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right), \quad R^i_{jkh} \neq 0, \tag{17}$$

where $\mathcal{B}_m$ is Berwald's covariant differential operator with respect to $x^m$, $\lambda_m$ and $\mu_m$ are called recurrence vectors.

**Definition 2.1.** *A Finsler space $F_n$ in which Cartan's third curvature tensor $R^i_{jkh}$ satisfies the condition (17), where $\lambda_m$ and $\mu_m$ are non-zero covariant vectors field. Such space and the tensor which satisfy the condition (17) will celled a generalized $\mathcal{B}R$-recurrent space and a generalized $\mathcal{B}-$recurrenttensor, respectively and denoted them briefly by $G\,(\mathcal{B}R)-RF_n$ and $G\mathcal{B}-R$, respectively.*

Let us consider a $G(\mathcal{B}R) - RF_n$ which characterized by the condition (17). Transvecting the condition (17) by $y^j$, using (11), (9), (3b) and (1a), we get

$$\mathcal{B}_m H^i_{kh} \;=\; \lambda_m H^i_{kh} + \mu_m \left(y^i g_{kh} - \delta^i_k y_h\right). \tag{18}$$

Further, transvecting the condition (18) by $y^k$, using (12), (9), (1a) and (3b), we get

$$\mathcal{B}_m H^i_h = \;\lambda_m H^i_h. \tag{19}$$

Contracting the indices i and h in the condition (18), using (13d), (1a) and (3a), we get

$$\mathcal{B}_m H_k = \;\lambda_m H_k. \tag{20}$$

# 3. Necessary and Sufficient Condition

In this section, we shall obtain the necessary and sufficient condition for some tensors to be generalized recurrent in a $G(\mathcal{B}R) - RF_n$ which characterized by the condition (17). Let us consider a $G(\mathcal{B}R) - RF_n$ which characterized by the condition (17). Differentiating (18) partially with respect to $y^j$, using (13a), (3c), (4) and (3e), we get

$$\dot{\partial}_j \left(\mathcal{B}_m H^i_{kh}\right) = \left(\dot{\partial}_j \lambda_m\right) H^i_{kh} + \lambda_m H^i_{jkh} + \left(\dot{\partial}_j \mu_m\right) \left(y^i g_{kh} - \delta^i_k y_h\right) + \mu_m \left(\delta^i_j g_{kh} + 2y^i C_{khj} - \delta^i_k g_{jh}\right). \tag{21}$$

Using the commutation formula exhibited by (8) for $(H^i_{kh})$ in (21) and using (13a), we get

$$\begin{aligned}\mathcal{B}_m H^i_{jkh} + H^r_{kh} G^i_{mjr} - H^i_{rh} G^r_{mjk} - H^i_{kr} G^r_{mjh} &= \left(\dot{\partial}_j \lambda_m\right) H^i_{kh} + \lambda_m H^i_{jkh} + \left(\dot{\partial}_j \mu_m\right) \left(y^i g_{kh} - \delta^i_k y_h\right) \\ &\quad + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) + 2y^i \mu_m C_{jkh}.\end{aligned} \tag{22}$$

Thus, shows that

$$\mathcal{B}_m H^i_{jkh} = \lambda_m H^i_{jkh} + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right)$$

if and only if

$$H^r_{kh} G^i_{mjr} - H^i_{rh} G^r_{mjk} - H^i_{kr} G^r_{mjh} - \left(\dot{\partial}_j \lambda_m\right) H^i_{kh} - \left(\dot{\partial}_j \mu_m\right) \left(y^i g_{kh} - \delta^i_k y_h\right) - 2y^i \mu_m C_{jkh} = 0. \tag{23}$$

Thus, we conclude

**Theorem 3.1.** *In $G(\mathcal{B}\mathrm{R}) - \mathrm{RF_n}$, Berwald curvature tensor $H^i_{jkh}$ is a generalized recurrent tensor if and only if (23) holds good.*

Transvecting (22) by $g_{ip}$, using (13c), (10), (3d) and (1a), we get

$$\mathcal{B}_m H_{jpkh} = \lambda_m H_{jpkh} + \mu_m (g_{jp} g_{kh} - g_{kp} g_{jh})$$

if and only if

$$\begin{aligned} g_{ip}\left[H^r_{kh} G^i_{mjr} - H^i_{rh} G^r_{mjk} - H^i_{kr} G^r_{mjh}\right] &= g_{ip}\left[\left(\dot{\partial}_j \lambda_m\right) H^i_{kh} + \left(\dot{\partial}_j \mu_m\right) \left(y^i g_{kh} - \delta^i_k y_h\right)\right] \\ &\quad + 2y_p \mu_m C_{jkh} + 2H^i_{jkh} y^t \mathcal{B}_t C_{ipm}.\end{aligned} \tag{24}$$

Thus, we conclude

**Theorem 3.2.** *In $G(\mathcal{B}\mathrm{R}) - \mathrm{RF_n}$, the associate curvature tensor $H_{jpkh}$ is generalized recurrent if and only if (24) holds good.*

The curvature tensor $R^i_{jkh}$ and $K^i_{jkh}$ are connected by the formula [6]

$$R^i_{jkh} = K^i_{jkh} + C^i_{js} K^s_{rkh} y^r. \tag{25}$$

Using (11) in (25), we get

$$R^i_{jkh} = K^i_{jkh} + C^i_{jr} H^r_{kh}. \tag{26}$$

Taking the covariant derivative for (26) with respect to $x^j$ in the sense of Berwald, we get

$$\mathcal{B}_m R^i_{jkh} = \mathcal{B}_m K^i_{jkh} + \left(\mathcal{B}_m C^i_{jr}\right) H^r_{kh} + C^i_{jr}(\mathcal{B}_m H^r_{kh}). \tag{27}$$

Using the conditions (17) and (18) in (27), we get

$$\lambda_m R^i_{jkh} + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) = \mathcal{B}_m K^i_{jkh} + \left(\mathcal{B}_m C^i_{jr}\right) H^r_{kh} + C^i_{jr}\left[\lambda_m H^r_{kh} + \mu_m \left(y^r g_{kh} - \delta^r_k y_h\right)\right]. \tag{28}$$

Using (26) in (28), we get

$$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right) - \left(\mathcal{B}_m C^i_{jr}\right) H^r_{kh} - \mu_m C^i_{jr}\left(y^r g_{kh} - \delta^r_k y_h\right). \tag{29}$$

This shows that

$$\mathcal{B}_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m \left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right)$$

if and only if

$$\left(\mathcal{B}_m C^i_{jr}\right) H^r_{kh} + \mu_m C^i_{jr}\left(y^r g_{kh} - \delta^r_k y_h\right) = 0. \tag{30}$$

Thus, we conclude

**Theorem 3.3.** *In $G(\mathcal{B}R) - RF_n$, Cartan's fourth curvature tensor $K^i_{jkh}$ is a generalized recurrent if and only if (30) holds.*

Contracting the indices i and j in (22), using (14c), (3g), (3d) and (5a), we get

$$\mathcal{B}_m \left(H_{kh} - H_{hk}\right) = \lambda_m \left(H_{kh} - H_{hk}\right) + (n-1)\,\mu_m g_{kh}.$$

if and only if

$$H^r_{kh} G^i_{mir} - H^i_{rh} G^r_{mik} - H^i_{kr} G^r_{mih} - \left(\dot{\partial}_i \lambda_m\right) H^i_{kh} - \left(\dot{\partial}_i \mu_m\right)\left(y^i g_{kh} - \delta^i_k y_h\right) = 0. \tag{31}$$

Thus, we conclude

**Theorem 3.4.** *In $G(\mathcal{B}R) - RF_n$, the tensor $(H_{kh} - H_{hk})$ is non vanishing if and only if (31) holds good.*

Contracting the indices i and j in (22), Putting $(H^i_{ikh} = H_{kh})$ and using (3g), (3d) and (5a), we get

$$\mathcal{B}_m H_{kh} = \lambda_m H_{kh} + (n-1)\mu_m g_{kh}. \tag{32}$$

if and only if

$$H^r_{kh} G^i_{mir} - H^i_{rh} G^r_{mik} - H^i_{kr} G^r_{mih} - \left(\dot{\partial}_i \lambda_m\right) H^i_{kh} - \left(\dot{\partial}_i \mu_m\right)\left(y^i g_{kh} - \delta^i_k y_h\right) = 0. \tag{33}$$

Thus, we conclude

**Theorem 3.5.** *In $G(\mathcal{B}R) - RF_n$, H-Ricci tensor $H_{kh}$ is non-vanishing if and only if (33) holds good.*

Transvecting (29) by $y^h$, using (15a), (9), (1a), (12), (5b), (1b) and (5d), we get

$$\mathcal{B}_m K^i_{jk} = \lambda_m K^i_{jk} + \mu_m \left(\delta^i_j y_k - \delta^i_k y_j\right) + \left(\mathcal{B}_m C^i_{jr}\right) H^r_k - \mu_m C^i_{jk} F^2. \tag{34}$$

This shows that

$$\mathcal{B}_m K^i_{jk} = \lambda_m K^i_{jk}$$

if and only if

$$\mu_m \left(\delta^i_j y_k - \delta^i_k y_j\right) + \left(\mathcal{B}_m C^i_{jr}\right) H^r_k - \mu_m C^i_{jk} F^2 = 0. \tag{35}$$

Thus, we conclude

**Theorem 3.6.** *In $G(\mathcal{B}R)-RF_n$, the torsion tensor $K^i_{jk}$ behaves as recurrent if and only if (35) holds.*

Transvecting (29) by $g^{jk}$ and using (15b), (3f), (3g), (3h), (5c) and (5b), we get

$$\mathcal{B}_m K^i_h = \lambda_m K^i_h + \left(\mathcal{B}_m C^i_{jr}\right) H^r_{kh} g^{jk} - \mu_m y_h C^i + K^i_{jkh}(\mathcal{B}_m g^{jk}). \tag{36}$$

This shows that

$$\mathcal{B}_m K^i_h = \lambda_m K^i_h$$

if and only if

$$\left(\mathcal{B}_m C^i_{jr}\right) H^r_{kh} g^{jk} - \mu_m y_h C^i + K^i_{jkh}\left(\mathcal{B}_m g^{jk}\right) = 0. \tag{37}$$

Thus, we conclude

**Theorem 3.7.** *In $G(\mathcal{B}R)-RF_n$, the deviation tensor $K^i_h$ behaves as recurrent if and only if (37) holds.*

Contracting the indices i and h in (29) and using (16a), (3d), (5b) and (5d), we get

$$\mathcal{B}_m K_{jk} = \lambda_m K_{jk}$$

if and only if

$$\left(\mathcal{B}_m C^i_{jr}\right) H^r_{ki} + \mu_m C^i_{jk} y_h = 0. \tag{38}$$

Thus, we conclude

**Theorem 3.8.** *In $G(\mathcal{B}R)-RF_n$, K-Ricci tensor $K_{jk}$ behaves as recurrent if and only if (38) holds.*

Contracting the indices i and j in (34), using (16b), (3g), (3a) and (5e), we get

$$\mathcal{B}_m K_k = \lambda_m K_k + (n-1)\,\mu_m y_k + (\mathcal{B}_m C_r)\, H^r_k - \mu_m C_k F^2.$$

This shows that

$$\mathcal{B}_m K_k = \lambda_m K_k \tag{39}$$

if and only if

$$(n-1)\,\mu_m y_k + (\mathcal{B}_m C_r)\, H^r_k - \mu_m C_k F^2 = 0. \tag{40}$$

Thus, we conclude

**Theorem 3.9.** *In $G(\mathcal{B}R)-RF_n$, The curvature vector $K_k$ behaves as recurrent if and only if (40) holds.*

Contracting the indices i and h in (36), using (16c), (5f) and (16a), we get

$$\mathcal{B}_m K = \lambda_m K + \left(\mathcal{B}_m C^i_{jr}\right) H^r_{ki} g^{jk} + K_{jk}(\mathcal{B}_m g^{jk}).$$

This shows that

$$\mathcal{B}_m K = \lambda_m K$$

if and only if

$$\left(\mathcal{B}_m C^i_{jr}\right) H^r_{ki} g^{jk} + K_{jk}(\mathcal{B}_m g^{jk}) = 0. \tag{41}$$

Thus, we conclude

**Theorem 3.10.** *In $G(\mathcal{B}R)-RF_n$, The curvature scalar $K$ behaves as recurrent if and only if (41) holds.*

We know that [6]

$$R_j = K_j + C^i_{jr}H^r_i. \tag{42}$$

Taking the covariant derivative for (42) with respect to $x^m$ in the sense of Berwald, we get

$$\mathcal{B}_m R_j = \mathcal{B}_m K_j + (\mathcal{B}_m C^i_{jr})H^r_i + C^i_{jr}(\mathcal{B}_m H^r_i). \tag{43}$$

Using (39) and (19) in (43), we get

$$\mathcal{B}_m R_j = \lambda_m(K_j + C^i_{jr}H^r_i) + (\mathcal{B}_m C^i_{jr})H^r_i. \tag{44}$$

Using (42) in (44), we get

$$\mathcal{B}_m R_j = \lambda_m R_j + (\mathcal{B}_m C^i_{jr})H^r_i.$$

This shows that

$$\mathcal{B}_m R_j = \lambda_m R_j$$

if and only if

$$(\mathcal{B}_m C^i_{jr})H^r_i = 0. \tag{45}$$

Thus, we conclude

**Theorem 3.11.** *In $G(\mathcal{B}R)-RF_n$, The curvature vector $R_j$ behaves as recurrent if and only if (45) holds [provided (40) holds].*

Differentiating the condition (20) partially with respect to $y^m$, we get

$$\dot{\partial}_h\left(\mathcal{B}_m H_k\right) = \left(\dot{\partial}_h \lambda_m\right) H_k + \lambda_m(\dot{\partial}_h H_k). \tag{46}$$

Using the commutation formula exhibited by (8) for $(H_k)$ in (46), in view of (13e), we get

$$\mathcal{B}_m H_{kh} - H_r G^r_{mhk} = \left(\dot{\partial}_h \lambda_m\right) H_k + \lambda_m H_{kh}. \tag{47}$$

Using (32) in (47), we get

$$(n-1)\,\mu_m g_{kh} - H_r G^r_{mhk} = \left(\dot{\partial}_h \lambda_m\right) H_k. \tag{48}$$

Transvecting (48) by $y^k$, using (1a), (7) and (14b), we get

$$\mu_m y_h = \left(\dot{\partial}_h \lambda_m\right) H$$

or

$$\mu_m = \frac{\left(\dot{\partial}_h \lambda_m\right) H}{y_h}. \tag{49}$$

Thus, we conclude

**Theorem 3.12.** *In $G(\mathcal{B}R)-RF_n$, The covariant vector $\mu_m$ is independent of $y^i$ if and only if the covariant vector $\lambda_m$ is independent $y^i$ [provided (33) holds].*

Using (49) in (48), we get

$$\left(\dot{\partial}_h \lambda_m\right)\left[\frac{(n-1)\,Hg_{kh}}{y_h} - H_k\right] = H_r G^r_{mhk}. \tag{50}$$

Transvecting (50) by $y^m$ and using (7), we get

$$\left(\dot{\partial}_h \lambda_m\right) y^m \left[\frac{(n-1)\,Hg_{kh}}{y_h} - H_k\right] = 0$$

or

$$\left(\dot{\partial}_h \lambda - \lambda_h\right)\left[\frac{(n-1)\,Hg_{kh}}{y_h} - H_k\right] = 0, \tag{51}$$

where $\lambda = \lambda_h y^h$. The equation (51) implies at least one of the following condition

$$a)\;\; \lambda_h = \dot{\partial}_h \lambda, \quad b)\;\; H_k = \frac{(n-1)\,Hg_{kh}}{y_h}. \tag{52}$$

Thus, we conclude

**Theorem 3.13.** *In $G(\mathcal{B}R) - RF_n$, for which the covariant vector $\lambda_m$ is not independent of $y^i$, at least one of the conditions (52a) or (52b) holds [provided (33) holds].*

Suppose (52b) holds, then (50) implies

$$H_r G^r_{mhk} = 0$$

Since $n \neq 1$ and $H \neq 0$, we get

$$y_r G^r_{mhk} = 0.$$

Therefore the space is a Landsberg space. Thus, we conclude

**Theorem 3.14.** *The $G(\mathcal{B}R) - RF_n$ is aLandsberg space if the condition (52b) holds [provided (33) holds].*

# 4. Conclusion

(1). The space whose defined by the condition (17) is called generalized $\mathcal{B}R-$ recurrent Finsler space.

(2). In generalized $\mathcal{B}R-$recurrent Finsler space Berwald curvature tensor $H^i_{jkh}$, the associate curvature tensor $H_{jpkh}$ and Cartan's fourth curvature tensor $K^i_{jkh}$ are generalized recurrent if and only if (23), (24) and (30) hold, respectively.

(3). In generalized $\mathcal{B}R-$recurrent Finsler space, the tensor $(H_{kh} - H_{hk})$ and $H-$Ricci tensor $H_{kh}$ are non-vanishing if and only if (31) and (33) hold, respectively.

(4). In generalized $\mathcal{B}R-$recurrent Finsler space, the torsion tensor $K^i_{jk}$, the deviation tensor $K^i_h$, $K-$Ricci tensor $K_{jk}$, the curvature vector $K_k$ and the curvature scalar $K$ behave as recurrent if and only if (35), (37), (38), (40) and (41) hold, respectively.

(5). In generalized $\mathcal{B}R-$recurrent Finsler space, the curvature vector $R_j$ behaves as recurrent if and only if (45) holds [provided (40) holds].

(6). In generalized $\mathcal{B}R-$recurrent Finsler space The covariant vector $\mu_m$ is independent of $y^i$ if and only if the covariant vector $\lambda_m$ is independent $y^i$ [provided (33) holds].

(7). In generalized $\mathcal{B}R-$recurrent Finsler space the covariant vector $\lambda_m$ is not independent of $y^i$, at least one of the conditions (52a) or (52 b) holds [provided (33) holds].

(8). The generalized $\mathcal{B}R-$recurrent Finsler space is a Landsberg space if the condition (52b) holds [provided (33) holds].

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