International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(2)**(2023), 71–78
Available Online: http://ijmaa.in

Unique Fixed Point Theorem for Weakly Digital Metric Spaces Involving Auxiliary Functions

Abid Khan[1,*]

[1] *Department of Mathematics, Amity University Madhya Pradesh, Gwalior, India*

Abstract

In this paper, we prove unique fixed point theorem for pairs of weakly self-mapping in digital metric space. Our findings expand upon and enhance numerous previously published findings. We give an example to support our basic theorem and its corollaries.

Keywords: Digital metric space; fixed point theorem; self-mappings; weakly mappings.

1. Introduction

For the first time, Rosenfield [12] used digital topology as a tool to analyse digital photographs. In addition to creating the digital versions of topological ideas, Boxer [2] later investigated continuous digital functions. Ege and Karaca [4] established the well-known Banach Contraction Principle for digital images, as well as the relative and reduced Lefschetz fixed point theorem. They also suggested the idea of a digital metric space. One of the generalisations of metric space and digital topology is digital metric space. A growing field of general topology and functional analysis called "digital topology" examines the characteristics of 2D and 3D digital images. The topological concept was provided in digital form by L. Boxer [2]. To show that fixed point outcomes exist for digital metric spaces, many researchers tried to generalize new contractive mappings [8, 9, 14, 15, 17]. Inspired and driven by the aforementioned work, we derive a digital metric space via auxiliary functions to generalize results based on weakly digital metric space. An example is given in the support of our main result.

2. Definitions and Preliminaries

Definition 2.1. *[7] For a digital metric space (X, d, ρ), if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a Cauchy sequence, there is $M \in \mathbb{N}$ such that for all $n, m > M$, we have $x_n = x_m$.*

*Corresponding author (abid69304@gmail.com)

Definition 2.2. *[7] A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $L \in X$ if for all $\in > 0$, there is $M \in \mathbb{N}$ such that $d(x_n, L) < \epsilon$ for all $n > M$.*

Definition 2.3. *[7] A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $L \in X$ if for all $\in > 0$, there is $M \in \mathbb{N}$ such that $x_n = L$ for all $n > M$ i.e., $x_n = x_{n+1} = x_{n+2} = \cdots = L$.*

Definition 2.4. *[5] A digital metric space (X, d, ρ) is complete if any Cauchy sequence $\{x_n\}$ converges to a point L of (X, d, ρ).*

Definition 2.5. *[7] A digital metric space (X, d, ρ) is complete.*

Definition 2.6. *[5] Let (X, d, ρ) be a digital metric space and $T : (X, d, \rho) \to (X, d, \rho)$ be a self-map. If there exists $\lambda \in [0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$.*

Definition 2.7. *[6] Let $X \subseteq Z^n$ and (X, d, ρ) be a digital metric space. Then there does not exist a sequence $\{x_n\}$ of distinct elements in X, such that $d(x_{m+1}, x_m) < d(x_m, x_{m-1})$ for $m = 1, 2, 3, \ldots$.*

Proposition 2.8. *[7] Every digital contraction map $T : (X, d, \rho) \to (X, d, \rho)$ is digitally continuous.*

Definition 2.9. *[5] Suppose that (X, d, ρ) is a digital metric space and $P, Q : X \to X$, and be two self-maps defined on X. Then P and Q are compatible if $d(PQx, Qpx) \leq d(Px, Qx)$ for all $x \in X$.*

3. Main Result

In this section, we shall prove a unique fixed point theorem for pairs of self-mappings via auxiliary functions in the setting of complete weakly digital metric space. Where $\Psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\Psi(\rho) = 0$ if and only if $\rho = 0$. $\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that $\Psi(\rho) = 0$ if and only if $\rho = 0$.

Theorem 3.1. *Let (X, d, ρ) is a complete digital metric space, let N be a nonempty closed subset of X. Let $P, Q : N \to N$ and $G, H : N \to X$ be mappings satisfying $Q(N) \subset H(N)$ and for every $x, y \in X$,*

$$\Psi(d(Px, Qy)) \geq \varphi(d_{G,H}(x, y)) + \frac{1}{2}\Psi(d_{G,H}(x, y)) + \varphi(d_{G,H}(x, y)) \tag{1}$$

For all $x, y \in X$, where

$$d_{G,H}(x, y) = \max \left\{ \begin{array}{c} d(x, y), d(Gx, Hy), d(Gx, Px), d(Hy, Qy), \\ \frac{1}{3} d((Gx, Qy) + (Hy, Px)) \end{array} \right\}$$
$$d_{P,Q}(x, y) = \max \left\{ \begin{array}{c} d(x, y), d(Gx, Hy), d(Gx, Px), d(Hy, Qy), \\ \frac{1}{4} d((Gx, Qy) + (Hy, Px)) \end{array} \right\} \tag{2}$$

Then $\{P, G\}$ and $\{Q, H\}$ have a unique point of coincidence in X. Moreover, if $\{P, G\}$ and $\{Q, H\}$ are self-mappings, then P, Q, G and H have a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $Q(N) \subset G(N)$ and $P(N) \subset H(N)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_{2n-1} = P_{x_{2n-2}} = H_{x_{2n-1}}, \quad y_{2n} = Q_{x_{2n-1}} = G_{x_{2n}}, \quad n = 1, 2, 3, 4, \ldots$$

Suppose that $y_{n_0} = y_{n_{0+1}}$ for some n_0, then the sequence $\{y_n\}$ is constant for $n \geq n_0$. Indeed, let $n_0 = 2k$. Then $y_{2k} = y_{2k+1}$ and it follows from (1) that

$$\begin{aligned} \Psi\left(\left(y_{2k+1}, y_{2k+2}\right)\right) &= \Psi(Px_{2k}, Qx_{2k+1}) \\ &\leq \Psi\left(d_{G,H}\left(x_{2k}, x_{2k+1}\right)\right) - \varphi\left(d_{G,H}\left(x_{2k}, x_{2k+1}\right)\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} d_{G,H}\left(x_{2k}, x_{2k+1}\right) &= \max \left\{ \begin{array}{c} d\left(y_{2k}, y_{2k+1}\right), d\left(y_{2k}, Px_{2k},\right), d\left(y_{2k+1}, Qx_{2k+1}\right), \\ \frac{1}{3} d\left(\left(y_{2k}, Qx_{2k+1}\right) + d\left(y_{2k+1}, Px_{2k}\right)\right) \end{array} \right\} \\ &= \max \left\{ d\left(y_{2k+1}, y_{2k+2}\right), \frac{1}{3} d\left(\left(y_{2k}, y_{2k+2}\right)\right) \right\} \\ &= \max \left\{ d(y_{2k+1}, y_{2k+2}), \frac{1}{3} d\left(y_{2k}, y_{2k+2}\right) \right\} \\ &= d(y_{2k+1}, y_{2k+2}). \end{aligned}$$

By (3), we get

$$\begin{aligned} &= \max \left\{ \begin{array}{c} d\left(y_{2k}, y_{2k+1}\right), d\left(y_{2k}, Px_{2k},\right), d\left(y_{2k+1}, Qx_{2k+1}\right), \\ \frac{1}{4} d\left(\left(y_{2k}, Qx_{2k+1}\right) + \left(y_{2k+1}, Px_{2k}\right)\right) \end{array} \right\} \\ &= \max \left\{ d\left(y_{2k+1}, y_{2k+2}\right), \frac{1}{4} d\left(\left(y_{2k}, y_{2k+2}\right)\right) \right\} \\ &= d(y_{2k+1}, y_{2k+2}). \end{aligned}$$

By equations (3), and (4), we have

$$\Psi(y_{2k+1}, y_{2k+2}) \leq \Psi(y_{2k+1}, y_{2k+2}) - \varphi(y_{2k+1}, y_{2k+2})$$

And so $\varphi(y_{2k+1}, y_{2k+2}) \leq 0$ and $y_{2k+1} = y_{2k+2}$.

Similarly, if $n_0 = 2k + 1$, then one easily obtains that $y_{2k+2} = y_{2k+3}$ and the sequence $\{y_n\}$ is constant. Therefore, $\{P, G\}$ and $\{Q, H\}$ have a point of coincidence in X. Now, suppose that $(y_n, y_{n+1}) > 0$ for each n. We shall show that for each $n = 0, 1, \ldots,$

$$(y_{n+1}, y_{n+2}) \leq d_{G,H}\left(x_n, x_{n+1}\right) = (y_n, y_{n+1}) \tag{4}$$

Using (4), we obtain that

$$\begin{aligned}\Psi(y_{2n+1}, y_{2n+2}) &= \Psi(P_{x2n}, Q_{2n+1}) \\ &\leq \Psi(d_{G,H}(x_{2n}, x_{2n+1})) - \varphi(d_{G,H}(x_{2n}, x_{2n+1})) \\ &< \Psi(d_{G,H}(x_{2n}, x_{2n+1})).\end{aligned} \tag{5}$$

On the other hand, the control function Ψ is no decreasing. Then

$$\Psi(y_{2n+1}, y_{2n+2}) \leq (d_{G,H}(x_{2n}, x_{2n+1})) \tag{6}$$

Moreover, we have

$$\begin{aligned}d_{G,H}(x_{2n}, x_{2n+1}) &= \max\left\{(y_{2n}, y_{2n+1}), (y_{2n}, Px_{2n},), (y_{2n+1}, Qx_{2n+1}), \frac{1}{3}((y_{2n}, Qx_{2n+1}) + (y_{2n+1}, Px_{2n}))\right\} \\ &= \max\left\{(y_{2n}, y_{2n+1}), (y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2}), \frac{1}{3}(y_{2n}, y_{2n+2})\right\} \\ &\leq \max\left\{(y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2}), \frac{1}{3}(y_{2n}, y_{2n+1}) + (y_{2n+1}, y_{2n+2})\right\} \\ &\leq \max\{(y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2})\}\end{aligned}$$

Similarly, we have

$$\begin{aligned}d_{G,H}(x_{2n}, x_{2n+1}) &= \max\left\{(y_{2n}, y_{2n+1}), (y_{2n}, Px_{2n},), (y_{2n+1}, Qx_{2n+1}), \frac{1}{4}((y_{2n}, Qx_{2n+1}) + (y_{2n+1}, Px_{2n}))\right\} \\ &= \max\left\{(y_{2n}, y_{2n+1}), (y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2}), \frac{1}{4}(y_{2n}, y_{2n+2})\right\} \\ &\leq \max\left\{(y_{2n}, y_{2n+1}), (y_{2n+1}, y_{2n+2}), \frac{1}{4}(y_{2n}, y_{2n+1}) + (y_{2n+1}, y_{2n+2})\right\}\end{aligned}$$

If $(y_{2n+1}, y_{2n+2}) \geq (y_{2n}, y_{2n+1})$, then by using the last inequality and (5), we have $d_{G,H}(x_{2n}, x_{2n+1}) = (y_{2n+1}, y_{2n+2})$ and (6) implies that

$$\begin{aligned}\Psi(y_{2n+1}, y_{2n+2}) &= \Psi(d_{G,H}(Px_{2n}, Qx_{2n+1})) \\ &\leq \Psi(y_{2n+1}, y_{2n+2}) - \varphi(y_{2n+1}, y_{2n+2})\end{aligned}$$

This is only possible when $\varphi(y_{2n+1}, y_{2n+2}) = 0$ it is contradiction. Hence $(y_{2n+1}, y_{2n+2}) \leq (y_{2n}, y_{2n+1})$, and $d_{G,H}(x_{2n}, x_{2n+1}) \leq (y_{2n}, y_{2n+1})$. In a similar way, one can obtain that

$$(y_{2n+3}, y_{2n+2}) \leq d_{G,H}(x_{2n+2}, x_{2n+1}) = (y_{2n+2}, y_{2n+1}).$$

So (6) holds for each $n \in N$. It follows that the sequence $\{d(y_n, y_{n+1})\}$ is nondecreasing and the limit

$$\lim_{n\to\infty} (y_n, y_{n+1}) = \lim_{n\to\infty} d_{G,H}(x_n, x_{n+1})$$

exists. We denote this limit by l^*, we have $l^* \geq 0$. Suppose that $l^* > 0$. Then

$$\Psi(y_{n+1}, y_{n+2}) \leq \Psi(d_{G,H}(x_n, x_{n+1})) - \varphi(d_{G,H}(x_n, x_{n+1})).$$

Passing to the (upper) limit when $n \to \infty$, we get

$$\Psi(l^*) \leq \Psi(l^*) - \lim_{n\to\infty} \inf \varphi(d_{G,H}(x_n, x_{n+1})) \leq \Psi(l^*) - \varphi(l^*), \tag{7}$$

i.e., $\varphi(l^*) \leq 0$. Using the properties of control functions, we get that $l^* = 0$, which is a contradiction. Hence we have $\lim\limits_{n\to\infty} (y_n, y_{n+1}) = 0$. Now we show that $\{y_n\}$ is a Cauchy sequence in X. It is enough to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose the contrary. Then, for some $\epsilon > 0$, there exist subsequence $\left\{y_{2n(k)}\right\}$ and $\left\{y_{2m(k)}\right\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest index satisfying $n(k) > m(k)$ and $\left(y_{n(k)}, y_{m(k)}\right) \geq \epsilon$. In particular, $\left(y_{n(k)-2}, y_{m(k)}\right) < \epsilon$. Using the triangle inequality and the known relation $|d(x, z - d(x, z)| \leq d(x, z)$, we obtain that

$$\begin{aligned}
\lim_{k\to\infty} (y_{2n(k)}, y_{2m(k)}) &= \lim_{k\to\infty} \left(y_{2n(k)}, y_{2m(k)-1}\right) = \lim_{k\to\infty} \left(y_{2n(k)+1}, y_{2m(k)}\right) \\
&= \lim_{k\to\infty} \left(y_{2n(k)+1}, y_{2m(k)-1}\right) = \epsilon
\end{aligned} \tag{8}$$

By using the previous limits, we get that

$$\lim_{k\to\infty} d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right) = \epsilon.$$

Indeed,

$$\begin{aligned}
d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right) = \max\Big\{&\left(y_{2n(k)}, y_{2m(k)-1}\right), \left(y_{2n(k)}, y_{2m(k)+1}\right), \left(y_{2m(k)-1}, y_{2m(k)}\right), \\
&\frac{1}{3}\left((y_{2n(k)}, y_{2m(k)}) + (y_{2n(k)}, y_{2m(k)-1})\right)\Big\} \to max\left\{\epsilon, \frac{1}{3}(\epsilon+\epsilon)\right\} = \epsilon.
\end{aligned}$$

Again we have,

$$\begin{aligned}
d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right) = \max\Big\{&\left(y_{2n(k)}, y_{2m(k)-1}\right), \left(y_{2n(k)}, y_{2m(k)+1}\right), \left(y_{2m(k)-1}, y_{2m(k)}\right), \\
&\frac{1}{4}\left((y_{2n(k)}, y_{2m(k)}) + (y_{2n(k)}, y_{2m(k)-1})\right)\Big\} \to \max\left\{\epsilon, \frac{1}{4}(\epsilon+\epsilon)\right\} = \epsilon.
\end{aligned}$$

Applying (7), we obtain

$$\Psi\left(y_{2n(k)+1}, y_{2m(k)}\right) = \Psi\left(P_{x_{2n(k)}}, Q_{x_{2n(k)-1}}\right)$$
$$\leq \Psi(d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right)) - \varphi(d_{G,H}\left(x_{2n(k)}, x_{2m(k)-1}\right)).$$

Passing to the limit $k \to \infty$, we obtain that $\Psi(\epsilon) \leq \Psi(\epsilon) - \varphi(\epsilon)$, which is contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in the complete metric (X, d). so there exists $u \in X$ such that $\lim_{n\to\infty} y_n = u$. To prove the uniqueness property of u, suppose that u' is another point of coincidence of G and P, that is

$$u' = Gv' = Pv' \tag{9}$$

For some $v' \in N$. By (4), we have

$$\Psi\left(u', u\right) = \Psi\left(Pv', Qu\right) \leq \Psi\left(d_{G,H}\left(v', u\right)\right) - \varphi\left(d_{G,H}\left(v', u\right)\right)$$

Where

$$d_{G,H}\left(v', u\right) = \max\left\{\left(u', u\right), \frac{1}{3}\left(d_{G,H}\left(v', u\right)\right) - \varphi\left(d_{G,H}\left(v', u\right)\right)\right\}$$
$$d_{P,Q}\left(v', u\right) = \max\left\{\left(u', u\right), \frac{1}{4}\left(d_{G,H}\left(v', u\right)\right) - \varphi\left(d_{G,H}\left(v', u\right)\right)\right\}$$

It follows from (9) that $u' = u$. Therefore, u is the unique point of coincidence of $\{P, G\}$ and $\{Q, H\}$. Now, if $\{P, G\}$ and $\{Q, H\}$ are weakly compatible, then by (8) and (9), we have $Pu = P\left(Gv\right) = G\left(Pv\right) = Gu = w_1$ and $Qu = Q\left(Hu\right) = H\left(Qu\right) = Hu = w_2$. by (4), we have

$$\Psi\left(w_1, w_2\right) = \Psi\left(Pu, Qu\right) \leq \Psi\left(d_{G,H}\left(u, u\right)\right) - \varphi\left(d_{G,H}\left(u, u\right)\right),$$

Where

$$d_{G,H}\left(u, u\right) = \max\left\{\left(w_1, w_2\right), \frac{1}{3}\left(w_1, w_2\right) + \left(w_1, w_2\right)\right\}$$
$$d_{P,Q}\left(u, u\right) = \max\left\{\left(w_1, w_2\right), \frac{1}{4}\left(w_1, w_2\right) + \left(w_1, w_2\right)\right\}$$

It follows that $w_1 = w_2$, that is,

$$Pu = Gu = Qu = Hu \tag{10}$$

By (4) and (10), we have

$$\Psi\left(Pu, Qu\right) \leq \Psi\left(d_{G,H}\left(u, u\right)\right) - \varphi\left(d_{G,H}\left(u, u\right)\right),$$

Where

$$d_{G,H}(u,u) = \max\left\{(Gv,Hu),(Gv,Pv),(Hu,Qu),\frac{1}{3}(Gv,Qu)+(Pv,Qu)\right\}$$
$$d_{P,Q}(u,u) = \max\left\{(Gv,Hu),(Gv,Pv),(Hu,Qu),\frac{1}{4}(Gv,Qu)+(Pv,Qu)\right\}$$

Therefore, we deduce that $Pv = Qu$, that is, $u = Qu$. It follows from (10) that

$$u = Pu = Gu = Qu = Hu.$$

Then u is the unique common fixed point of P, G, H and Q. □

Example 3.2. *Let (X, d, ρ) is a complete digital metric space, let $X = [4, 40]$ and d be the usual metric on X. Define $P, Q, G, H : X \to X$ as follows: $PX = 4$ for each X;*

$$GX = X \;\; if \;\; x \le 16, \;\; and \;\; GX = 16 \;\; if \;\; 16 < x < 22, \;\; GX = \frac{x+18}{5} \;\; if \;\; 16 \le x \le 25$$
$$HX = 4 \;\; if \;\; x = 4 \;\; or \;\; 12 \;\; and \;\; GX = \frac{X+15}{5} \;\; if \;\; x > 25; \;\; 25 \;\; HX = 17 + X \;\; if \;\; 13 \le x \le 14$$
$$QX = 4 \;\; if \;\; x < 8 \;\; or \;\; x > 12, \;\; HX = 24 + X \;\; if \;\; 4 < x < 8, \;\; QX = 4 + x \;\; if \;\; 14 \le x \le 15$$
$$HX = 16 \;\; if \;\; 13 \le x \le 14;$$

Then P, Q, G and H satisfy all the conditions of the above theorem and have a unique common fixed point $x = 4$ being self mappings, all P, Q, G and H are weakly compatible mappings.

Corollary 3.3. *Let P and Q be self mappings of a complete digital metric space (X, d, ρ) into itself. Suppose $P(X) \subset Q(X)$. If there exists $\alpha \in (0, 1)$ and a positive integer k such that $d\left(P^k(x), P^k(y)\right) \le \alpha d(Q(x), Q(y))$ for all x and y in X, then P and Q have a unique common fixed point.*

References

[1] S. K. Barve, Q. A. Kabir and R. D. Daheriya, *Unique common fixed point theorem for weakly compatible mappings in digital metric space*, International journal of scientific Research and Reviews, 8(1)(2019), 2114-2121.

[2] L. Boxer, *Properties of Digital Homotopy*, Journal of Mathematical Imaging and vision, 22(2005), 19-26,

[3] L. Boxer, *Continuous Maps on Digital Simple Closed Curves*, Applied Ma-thematics, 1(2010), 377-386.

[4] O. Ege and I. Karaca, *Lefschetz Fixed Point Theorem for Digital Images*, Fixed Point Theory and Applications, 2013(2013), 13 pages.

[5] O. Ege and I. Karaca, *Banach fixed point theorem for digital images*, J. Nonlinear Sci. Appl., 8(2015), 237-245.

[6] B. Fisher and S. Sessa, *Common fixed points of two pairs of weakly commuting mappings*, Univ. of Novisad, Math. Ser., 16(1986), 45-59,

[7] S. E. Han, *Banach fixed point theorem from the viewpoint of digital topology*, J. Nonlinear Sci. Appl., 9(2015), 895-905.

[8] G. Jungck, *Commuting mappings and fixed point*, Amer. Math. Monthly, 83(1976), 261-263.

[9] G. Jungck, *Compatible mappings and common fixed points (2)*, International Journal of Mathematics and Mathematical Sciences, 11(1988), 285-288.

[10] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japon., 38(1993), 381-390.

[11] A. Rosenfeld, *Digital Topology*, The American Mathematical Monthly, 86(1979), 621-630.

[12] A. Rosenfeld, *Continuous functions on digital pictures*, Pattern Recognition Letters, 4(1986), 177-184.

[13] B. E. Rhoades and S. Sessa, *Common fixed point theorems for three mappings under a weak commutativity condition*, Indian. J. Pure. Appl. Math., 17(1986), 47-57.

[14] K. Sridevi, M. V. R. Kameswari and D. M. K. Kiran, *Fixed point theorems for digital contractive type mappings in digital metric space*, International Journal of Mathematics Trends and Technology, 48(3)(2017), 159-167.

[15] S. Sessa, R. N. Mukherjee and T. Som, *A common fixed point theorem for weakly commuting mappings*, Math. Japonica, 31(1986), 235-245.

[16] S. Sessa and B. Fisher, *Common fixed points of weakly commuting mappings*, Bull. Acad. Polon. Sci. Ser. Sci. Math., 35(1987), 341-349.

[17] S. Sessa, *On a weak commutativity condition of mappings in fixed point consideration*, Publ. Inst. Math. Soc., 32(1982), 149-153.