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Quasi Class Q(N), Quasi class Q*(N) and m-quasi k-paranormal Composite Multiplication Operators on the Complex Hilbert Space

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Abstract

In this paper, the condition under which composite multiplication operators on $L^2(\mu)$-space become m-Quasi-k-Paranomal, Quasi Class Q(N) and Quasi Class Q*(N) operators have been obtained in terms of radon-nikodym derivative f_0.

Keywords: composite multiplication operator; conditional expectation; m-Quasi-k-paranormal; Quasi Class Q(N); Quasi Class Q*(N).

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1. Introduction

Let (X, Σ, μ) be a σ-finite measure space. Then a mapping T from X into X is said to be a measurable transformation if $T^{-1}(E) \in \Sigma$ for every $E \in \Sigma$. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(E)) = 0$ whenever $\mu(E) = 0$. If T is non-singular then the measure μT^{-1} defined as $\mu T^{-1}(E) = \mu(T^{-1}(E))$ for every E in Σ, is an absolutely continuous measure on Σ with respect to μ. Since μ is a σ-finite measure, then by the Radon-Nikodym theorem, there exists a non-negative function f_0 in $L^1(\mu)$ such that $\mu T^{-1}(E) = \int_E f_0 d\mu$ for every $E \in \Sigma$. The function f_0 is called the Radon-Nikodym derivative of μT^{-1} with respect to μ.

Every non-singular measurable transformation T from X into itself induces a linear transformation C_T on $L^p(\mu)$ defined as $C_T f = f \circ T$ for every f in $L^p(\mu)$. In case C_T is continuous from $L^p(\mu)$ into itself, then it is called a composition operator on $L^p(\mu)$ induced by T. We restrict our study of the composition operators on $L^2(\mu)$ which has Hilbert space structure. If u is an essentially bounded complex-valued measurable function on X, then the mapping M_u on $L^2(\mu)$ defined by $M_u f = u \cdot f$, is a continuous operator with range in $L^2(\mu)$. The operator M_u is known as the multiplication operator induced by u. A composite multiplication operator is linear transformation acting on a set of complex valued Σ

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measurable functions f of the form

$$M_{u,T}(f) = C_T M_u(f) = u \circ Tf \circ T$$

Where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T. In the study considered is the using conditional expectation of composite multiplication operator on L^2-spaces. For each $f \in L^p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A gf d\mu = \int_A gE(f) d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of $L^p(\mu)$, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \geq 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [1–4].

Definition 1.1 (Para normal operator). *Let H be a Complex Hilbert Space. An operator T on H is called paranormal operator if* $\|Tx\|^2 \leq \|T^2x\| \|x\|$ *for all* $x \in H$.

Definition 1.2 (Quasi-paranormal operator). *Let H be a Complex Hilbert Space. An operator T on H is called Quasi-paranormal operator if* $\|T^2x\|^2 \leq \|T^3x\| \|Tx\|$ *for all* $x \in H$.

Definition 1.3 (k-Quasi paranormal operator). *Let H be a Complex Hilbert Space. An operator T on H is called Quasi-normal operator if* $\|T^{k+1}x\|^2 \leq \|T^{k+2}x\| \|T^kx\|$ *for all* $x \in H$.

Definition 1.4 (m-Quasi k-paranomal operator). *Let H be a Complex Hilbert Space. An operator T on H is called m-Quasi k-paranormal operator if* $\|T^{m+k+1}x\| \|T^mx\|^k \geq \|T^{m+1}x\|^{k+1}$ *for some positive integer m, k and for all* $x \in H$.

Definition 1.5 (Class Q(N)). *Let H be a Complex Hilbert Space. An operator T on H is called Class Q(N) if* $N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2$ *for all* $x \in H$ *ie., if* $T^{*^2}T^2 - NT^*T + I \geq 0$.

Definition 1.6 (Class Q*(N)). *Let H be a Complex Hilbert Space. An operator T on H is called Class Q*(N) if* $N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$ *for all* $x \in H$ *ie., if* $T^{*^2}T^2 - NTT^* + I \geq 0$.

Definition 1.7 (Quasi Class Q(N)). *Let H be a Complex Hilbert Space. An operator T on H is called Quasi Class Q(N) if* $N\|T^2x\|^2 \leq \|T^3x\|^2 + \|Tx\|^2$ *for all* $x \in H$ *ie., if* $T^{*^3}T^3 - NT^{*2}T^2 + T^*T \geq 0$.

Definition 1.8 (Quasi Class Q*(N)). *Let H be a Complex Hilbert Space. An operator T on H is called Quasi Class Q(N) if* $N\|T^2x\|^2 \leq \|T^3x\|^2 + \|Tx\|^2$ *for all* $x \in H$ *ie., if* $T^{*^3}T^3 - NT^{*2}T^2 + T^*T \geq 0$.

1.1 Related Work in the Field

The study of weighted composition operators on L^2 spaces was initiated by R. K. Singh and D. C. Kumar [5]. During the last thirty years, several authors have studied the properties of various classes of weighted composition operator. Boundedness of the composition operators in $L^p(\Sigma)$, $(1 \leq p < \infty)$ spaces, where the measure spaces are σ-finite, appeared already in [6]. Also boundedness of weighted operators on $C(X, E)$ has been studied in [7]. Recently S. Senthil, P. Thangaraju, M. Nithya, B. Surya devi and D. C. Kumar, have proved several theorems on n-normal, n-quasi-normal, k-paranormal, and (n, k) paranormal of composite multiplication operators on L^2 spaces [8–12]. In this paper we investigate composite multiplication operators on $L^2(\mu)$-space become Quasi-P-Normal operators and n-Power class Q operator have been obtained in terms of radon-nikodym derivative f_0.

2. Characterization on Composite Multiplication of Quasi Class Q and Quasi Class Q* Operators on L^2-Space

Proposition 2.1. *Let the composite multiplication operator $M_{u,T} \in B(L^2(\mu))$. Then for $u \geq 0$*

(i) $M^*_{u,T} M_{u,T} f = u^2 f_0 f$

(ii) $M_{u,T} M^*_{u,T} f = u^2 \circ T \cdot f_0 \circ T \cdot E(f)$

(iii) $M^n_{u,T}(f) = (C_T M_u)^n(f) = u_n(f \circ T^n), u_n = u \circ T.u \circ T^2.u \circ T^3 \dots u \circ T^n$

(iv) $M^*_{u,T} f = u f_0 \cdot E(f) \circ T^{-1}$

(v) $M^{*^n}_{u,T} f = u f_0 \cdot E(u f_0) \circ T^{-(n-1)} \cdot E(f) \circ T^{-n}$, *where* $E(u f_0) \circ T^{-(n-1)} = E(u f_0) \circ T^{-1} \cdot E(u f_0) \circ T^{-2} \dots E(u f_0) \circ T^{-(n-1)}$.

Theorem 2.2. *Let the $M_{u,T}$ be a composite multiplication operator on$L^2(\mu)$. Then $M_{u,T}$ is Quasi class Q composite multiplication operator if and only if*

$$hu^2 E(hu^2) \circ T^{-1} E(hu^2) \circ T^{-2} - 2hu^2 E(hu^2) \circ T^{-1} + hu^2 \geq 0$$

Proof. Now we consider:

$$\begin{aligned}
M^{*3}_{u,T} M^3_{u,T} f &= M^{*3}_{u,T} M^2_{u,T}(u \circ T f \circ T) \\
&= M^{*3}_{u,T} M_{u,T} u \circ T(u \circ T f \circ T) \circ T \\
&= M^{*3}_{u,T} u \circ T(u \circ T u \circ T^2 f \circ T^2) \circ T \\
&= M^{*2}_{u,T} hu E\left(u \circ T u \circ T^2 u \circ T^3 f \circ T^3\right) \circ T^{-1} \\
&= M^*_{u,T} hu^2 u \circ T E\left(hu^2\right) \circ T^{-1} f \circ T \\
&= hu E\left(hu^2 u \circ T E\left(hu^2\right) \circ T^{-1} f \circ T\right) \circ T^{-1}
\end{aligned}$$

$$= hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2}f$$

Consider:

$$\begin{aligned}
M^{*2}_{u,T}M^2_{u,T}f &= M^{*2}_{u,T}M_{u,T}(u \circ Tf \circ T) \\
&= M^{*2}_{u,T}u \circ T(u \circ Tf \circ T) \circ T \\
&= M^{*2}_{u,T}u \circ Tu \circ T^2 f \circ T^2 \\
&= M^{*}_{u,T}huE(u \circ Tu \circ T^2 f \circ T^2) \circ T^{-1} \\
&= M^{*}_{u,T}(hu^2 u \circ Tf \circ T) \\
&= huE(hu^2 u \circ Tf \circ T) \circ T^{-1} \\
&= hu^2E(hu^2) \circ T^{-1}f
\end{aligned}$$

Assume that $M_{u,T}$ is Quasi Class Q. Then $f \in L^2(\mu)$

$$\langle (M^{*3}_{u,T}M^3_{u,T} - 2M^{*2}_{u,T}M^2_{u,T} + M^{*}_{u,T}M_{u,T})f, f\rangle \geq 0$$

Let $f = \mu_A$along with $\mu(A) < \infty$. Then

$$\begin{aligned}
& \langle (M^{*3}_{u,T}M^3_{u,T} - 2M^{*2}_{u,T}M^2_{u,T} + M^{*}_{u,T}M_{u,T})\psi_A, \psi_A\rangle \geq 0 \\
\Leftrightarrow \quad & \Big\langle (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2hu^2E(hu^2) \circ T^{-1} + hu^2)\psi_A, \psi_A \Big\rangle \geq 0 \\
\Leftrightarrow \quad & \int (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2hu^2E(hu^2) \circ T^{-1} + hu^2)\psi_A d\mu \geq 0 \\
\Leftrightarrow \quad & \int_A (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2hu^2E(hu^2) \circ T^{-1} + hu^2) d\mu \geq 0 \\
\Leftrightarrow \quad & hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2hu^2E(hu^2) \circ T^{-1} + hu^2 \geq 0
\end{aligned}$$

□

Corollary 2.3. *The composition operator C_T on $B(L^2(\mu))$ is Quasi-Class Q if and only if $hE(h) \circ T^{-1}E(h) \circ T^{-2} - 2hE(h) \circ T^{-1} + h \geq 0$ almost everywhere.*

Proof. The proof is obtained from Theorem 2.2 by putting $u = 1$. □

Theorem 2.4. *Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M_{u,T}$ is Quasi class Q* composite multiplication operator if and only if*

$$hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2h^2u^4 + hu^2 \geq 0$$

Proof. Now we consider:

$$\begin{aligned} M_{u,T}^{*3}M_{u,T}^{3}f &= M_{u,T}^{*3}M_{u,T}^{2}(u \circ Tf \circ T) \\ &= M_{u,T}^{*3}M_{u,T}u \circ T(u \circ Tf \circ T) \circ T \\ &= M_{u,T}^{*3}u \circ T(u \circ Tu \circ T^2 f \circ T^2) \circ T \\ &= M_{u,T}^{*2}huE\left(u \circ Tu \circ T^2u \circ T^3 f \circ T^3\right) \circ T^{-1} \\ &= M_{u,T}^{*}hu^2u \circ TE\left(hu^2\right) \circ T^{-1}f \circ T \\ &= huE\left(hu^2u \circ TE\left(hu^2\right) \circ T^{-1}f \circ T\right) \circ T^{-1} \\ &= hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2}f \end{aligned}$$

Consider:

$$\begin{aligned} (M_{u,T}^{*}M_{u,T})^2 f &= M_{u,T}^{*}M_{u,T}(M_{u,T}^{*}M_{u,T})f \\ &= M_{u,T}^{*}M_{u,T}M_{u,T}^{*}u \circ Tf \circ T \\ &= M_{u,T}^{*}M_{u,T}huE(u \circ Tf \circ T) \circ T^{-1} \\ &= M_{u,T}^{*}M_{u,T}hu^2 f \\ &= M_{u,T}^{*}u \circ T(hu^2 f) \circ T \\ &= huE(u \circ Th \circ Tu^2 \circ Tf \circ T) \circ T^{-1} \\ &= h^2u^4 f \end{aligned}$$

Assume that $M_{u,T}$ is Quasi Class Q*. Then $f \in L^2(\mu)$

$$\left\langle (M_{u,T}^{*3}M_{u,T}^{3} - 2(M_{u,T}^{*}M_{u,T})^2 + M_{u,T}^{*}M_{u,T})f, f \right\rangle \geq 0$$

Let $f = \mu_A$along with $\mu(A) < \infty$. Then

$$\begin{aligned} & \left\langle (M_{u,T}^{*3}M_{u,T}^{3} - 2(M_{u,T}^{*}M_{u,T})^2 + M_{u,T}^{*}M_{u,T})\psi_A, \psi_A \right\rangle \geq 0 \\ \Leftrightarrow \quad & \left\langle (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2h^2u^4 + hu^2)\psi_A, \psi_A \right\rangle \geq 0 \\ \Leftrightarrow \quad & \int (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2h^2u^4 + hu^2)\psi_A d\mu \geq 0 \\ \Leftrightarrow \quad & \int_A (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2h^2u^4 + hu^2) d\mu \geq 0 \\ \Leftrightarrow \quad & hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - 2h^2u^4 + hu^2 \geq 0 \end{aligned}$$

□

Corollary 2.5. *The composition operator C_T on $B(L^2(\mu))$ is Quasi-Class Q* if and only if $hE(h) \circ T^{-1}E(h) \circ T^{-2} - 2h^2 + h \geq 0$ almost everywhere.*

Proof. The proof is obtained from Theorem 2.4 by putting $u = 1$. □

3. Characterization on Composite Multiplication of Quasi Class Q (N) and Quasi Class Q* (N) Operators on L^2-Space

Theorem 3.1. *Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M_{u,T}$ is Quasi class Q(N) composite multiplication operator if and only if $hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - Nhu^2E(hu^2) \circ T^{-1} + hu^2 \geq 0$ for fixed real number $N \geq 0$.*

Proof. Now we consider:

$$\begin{aligned}
M^{*3}_{u,T}M^{3}_{u,T}f &= M^{*3}_{u,T}M^{2}_{u,T}(u \circ Tf \circ T) \\
&= M^{*3}_{u,T}M_{u,T}u \circ T(u \circ Tf \circ T) \circ T \\
&= M^{*3}_{u,T}u \circ T(u \circ Tu \circ T^2 f \circ T^2) \circ T \\
&= M^{*2}_{u,T}huE\left(u \circ Tu \circ T^2 u \circ T^3 f \circ T^3\right) \circ T^{-1} \\
&= M^{*}_{u,T}hu^2 u \circ TE\left(hu^2\right) \circ T^{-1} f \circ T \\
&= huE\left(hu^2 u \circ TE\left(hu^2\right) \circ T^{-1} f \circ T\right) \circ T^{-1} \\
&= hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} f
\end{aligned}$$

Consider:

$$\begin{aligned}
M^{*2}_{u,T}M^{2}_{u,T}f &= M^{*2}_{u,T}M_{u,T}(u \circ Tf \circ T) \\
&= M^{*2}_{u,T}u \circ T(u \circ Tf \circ T) \circ T \\
&= M^{*2}_{u,T}u \circ Tu \circ T^2 f \circ T^2 \\
&= M^{*}_{u,T}huE(u \circ Tu \circ T^2 f \circ T^2) \circ T^{-1} \\
&= M^{*}_{u,T}(hu^2 u \circ Tf \circ T) \\
&= huE(hu^2 u \circ Tf \circ T) \circ T^{-1} \\
&= hu^2E(hu^2) \circ T^{-1} f
\end{aligned}$$

Assume that $M_{u,T}$ is Quasi Class Q(N). Then $f \in L^2(\mu)$

$$\left\langle (M^{*3}_{u,T}M^{3}_{u,T} - NM^{*2}_{u,T}M^{2}_{u,T} + M^{*}_{u,T}M_{u,T})f, f \right\rangle \geq 0$$

Let $f = \mu_A$ along with $\mu(A) < \infty$. Then

$$\begin{aligned}
&\left\langle (M^{*3}_{u,T}M^{3}_{u,T} - NM^{*2}_{u,T}M^{2}_{u,T} + M^{*}_{u,T}M_{u,T})\psi_A, \psi_A \right\rangle \geq 0 \\
\Leftrightarrow \quad &\left\langle (hu^2E(hu^2) \circ T^{-1}E(hu^2) \circ T^{-2} - Nhu^2E(hu^2) \circ T^{-1} + hu^2)\psi_A, \psi_A \right\rangle \geq 0
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad \int (hu^2E(hu^2)\circ T^{-1}E(hu^2)\circ T^{-2} - Nhu^2E(hu^2)\circ T^{-1} + hu^2)\psi_A d\mu \geq 0\\
&\Leftrightarrow \quad \int_A (hu^2E(hu^2)\circ T^{-1}E(hu^2)\circ T^{-2} - Nhu^2E(hu^2)\circ T^{-1} + hu^2) d\mu \geq 0\\
&\Leftrightarrow \quad hu^2E(hu^2)\circ T^{-1}E(hu^2)\circ T^{-2} - Nhu^2E(hu^2)\circ T^{-1} + hu^2 \geq 0
\end{aligned}
$$

for fixed real number $N \geq 0$. □

Corollary 3.2. *The composition operator C_T on $B(L^2(\mu))$ is Quasi-Class Q(N) if and only if $hE(h)\circ T^{-1}E(h)\circ T^{-2} - NhE(h)\circ T^{-1} + h \geq 0$ almost everywhere.*

Proof. The proof is obtained from Theorem 3.1 by putting $u = 1$. □

Theorem 3.3. *Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M_{u,T}$ is Quasi class Q*(N) composite multiplication operator if and only if $hu^2E(hu^2)\circ T^{-1}E(hu^2)\circ T^{-2} - Nu\circ Tu^2\circ T^2h\circ T^2E(hu)\circ TE(f) + hu^2 \geq 0$ for fixed real number $N \geq 0$.*

Proof. Now we consider:

$$
\begin{aligned}
M^{*3}_{u,T}M^3_{u,T}f &= M^{*3}_{u,T}M^2_{u,T}(u\circ Tf\circ T)\\
&= M^{*3}_{u,T}M_{u,T}u\circ T(u\circ Tf\circ T)\circ T\\
&= M^{*3}_{u,T}u\circ T(u\circ Tu\circ T^2f\circ T^2)\circ T\\
&= M^{*2}_{u,T}huE\left(u\circ Tu\circ T^2u\circ T^3f\circ T^3\right)\circ T^{-1}\\
&= M^{*}_{u,T}hu^2u\circ TE\left(hu^2\right)\circ T^{-1}f\circ T\\
&= huE\left(hu^2u\circ TE\left(hu^2\right)\circ T^{-1}f\circ T\right)\circ T^{-1}\\
&= hu^2E(hu^2)\circ T^{-1}E(hu^2)\circ T^{-2}f
\end{aligned}
$$

Consider:

$$
\begin{aligned}
M^2_{u,T}M^{*2}_{u,T}f &= M^2_{u,T}M^*_{u,T}huE(f)\circ T^{-1}\\
&= M^2_{u,T}huE(huE(f)\circ T^{-1})\circ T^{-1}\\
&= M^2_{u,T}huE(hu)\circ T^{-1}E(f)\circ T^{-2}\\
&= M_{u,T}u\circ T(huE(hu)\circ T^{-1}E(f)\circ T^{-2})\circ T\\
&= M_{u,T}u^2\circ Th\circ TE(hu)E(f)\circ T^{-1}\\
&= u\circ T(u^2\circ Th\circ TE(hu)E(f)\circ T^{-1})\circ T\\
&= u\circ Tu^2\circ T^2h\circ T^2E(hu)\circ TE(f)
\end{aligned}
$$

Assume that $M_{u,T}$ is Quasi Class Q*(N). Then $f \in L^2(\mu)$

$$\left\langle (M^{*3}_{u,T}M^3_{u,T} - NM^2_{u,T}M^{*2}_{u,T} + M^*_{u,T}M_{u,T})f, f\right\rangle \geq 0$$

Let $f = \mu_A$ along with $\mu(A) < \infty$. Then

$$
\begin{aligned}
& \langle (M^{*3}_{u,T} M^3_{u,T} - N M^2_{u,T} M^{*2}_{u,T} + M^*_{u,T} M_{u,T}) \psi_A, \psi_A \rangle \geq 0 \\
\Leftrightarrow \quad & \left\langle (hu^2 E(hu^2) \circ T^{-1} E(hu^2) \circ T^{-2} - Nu \circ Tu^2 \circ T^2 h \circ T^2 E(hu) \circ TE(f) + hu^2) \psi_A, \psi_A \right\rangle \geq 0 \\
\Leftrightarrow \quad & \int (hu^2 E(hu^2) \circ T^{-1} E(hu^2) \circ T^{-2} - Nu \circ Tu^2 \circ T^2 h \circ T^2 E(hu) \circ TE(f) + hu^2) \psi_A d\mu \geq 0 \\
\Leftrightarrow \quad & \int_A (hu^2 E(hu^2) \circ T^{-1} E(hu^2) \circ T^{-2} - Nu \circ Tu^2 \circ T^2 h \circ T^2 E(hu) \circ TE(f) + hu^2) d\mu \geq 0 \\
\Leftrightarrow \quad & hu^2 E(hu^2) \circ T^{-1} E(hu^2) \circ T^{-2} - Nu \circ Tu^2 \circ T^2 h \circ T^2 E(hu) \circ TE(f) + hu^2 \geq 0
\end{aligned}
$$

for fixed real number $N \geq 0$. □

Corollary 3.4. *The composition operator C_T on $B(L^2(\mu))$ is Quasi-Class Q*(N) if and only if $hE(h) \circ T^{-1} E(h) \circ T^{-2} - Nh \circ T^2 E(h) \circ TE(f) + h \geq 0$ almost everywhere.*

Proof. The proof is obtained from Theorem 3.3 by putting $u = 1$. □

4. Characterization on Composite Multiplication of M-quasi k-paranormal Operators on L^2-Space

In this chapter, m-Quasi k-paranormal Composite multiplication operators on a Hilbert Space are characterized.

Theorem 4.1. *Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M_{u,T}$ is m-Quasi k-paranormal composite multiplication operator if and only if $huE(hu) \circ T^{-(m+k)} E(u_{m+k+1}) f - (k+1)\mu^k huE(hu) \circ T^{-m} E(u_{m+1}) f + k\mu^{k+1} huE(hu) \circ T^{-(m-1)} E(u_m) f \geq 0$ almost everywhere.*

Proof. Suppose $M_{u,T}$ is a m-Quasi k-paranormal

$$
(k+1)\mu^k M^{*m+1}_{u,T} M^{m+1}_{u,T} - k\mu^{k+1} M^{*m}_{u,T} M^m_{u,T} \leq M^{*m+k+1}_{u,T} M^{m+k+1}_{u,T}
$$
$$
M^{*m+k+1}_{u,T} M^{m+k+1}_{u,T} - (k+1)\mu^k M^{*m+1}_{u,T} M^{m+1}_{u,T} + k\mu^{k+1} M^{*m}_{u,T} M^m_{u,T} \geq 0
$$

This implies that

$$
\left\langle (M^{*m+k+1}_{u,T} M^{m+k+1}_{u,T} - (k+1)\mu^k M^{*m+1}_{u,T} M^{m+1}_{u,T} + k\mu^{k+1} M^{*m}_{u,T} M^m_{u,T}) f, f \right\rangle \geq 0 \ \ \forall \ \ f \in L^2(\mu)
$$

Since

$$
M^{*m+k+1}_{u,T} M^{m+k+1}_{u,T} f = huE(hu) \circ T^{-(m+k)} E(u_{m+k+1}) f
$$
$$
M^{*m+1}_{u,T} M^{m+1}_{u,T} f = huE(hu) \circ T^{-m} E(u_{m+1}) f
$$

$$M_{u,T}^{*m}M_{u,T}^{m}f = huE(hu) \circ T^{-(m-1)}E(u_m)f$$

$$\int_E (huE(hu) \circ T^{-(m+k)}E(u_{m+k+1})f - (k+1)\mu^k huE(hu) \circ T^{-m}E(u_{m+1})f + k\mu^{k+1}huE(hu) \circ T^{-(m-1)}E(u_m)f)d\mu \geq 0$$

for every $E \in \Sigma$.

$$\Leftrightarrow huE(hu) \circ T^{-(m+k)}E(u_{m+k+1})f - (k+1)\mu^k huE(hu) \circ T^{-m}E(u_{m+1})f + k\mu^{k+1}huE(hu) \circ T^{-(m-1)}E(u_m)f \geq 0$$

almost everywhere. □

Corollary 4.2. *The composition operator C_T on $B(L^2(\mu))$ is m-Quasi k-paranormal if and only if*

$$hE(h) \circ T^{-(m+k)} - (k+1)\mu^k hE(h) \circ T^{-m} + k\mu^{k+1}hE(h) \circ T^{-(m-1)} \geq 0$$

almost everywhere.

Proof. The proof is obtained from Theorem 4.1 by putting $u = 1$. □

Theorem 4.3. *Let the $M_{u,T}$ be a composite multiplication operator on $L^2(\mu)$. Then $M_{u,T}^*$ is m-Quasi k-paranormal composite multiplication operator if and only if $u_{m+k+1}(hu) \circ T^{m+k+1}E(hu) \circ TE(f) - (k+1)\mu^k u_{m+1}(hu) \circ T^{m+1}E(hu) \circ TE(f) + k\mu^{k+1}u_m(hu) \circ T^m E(hu) \circ TE(f) \geq 0$ almost everywhere.*

Proof. Suppose $M_{u,T}^*$ is a m-Quasi k-paranormal

$$(k+1)\mu^k M_{u,T}^{m+1}M_{u,T}^{*m+1} - k\mu^{k+1}M_{u,T}^{m}M_{u,T}^{*m} \leq M_{u,T}^{m+k+1}M_{u,T}^{*m+k+1}$$

$$M_{u,T}^{m+k+1}M_{u,T}^{*m+k+1} - (k+1)\mu^k M_{u,T}^{m+1}M_{u,T}^{*m+1} + k\mu^{k+1}M_{u,T}^{m}M_{u,T}^{*m} \geq 0$$

This implies that

$$\left\langle (M_{u,T}^{m+k+1}M_{u,T}^{*m+k+1} - (k+1)\mu^k M_{u,T}^{m+1}M_{u,T}^{*m+1} + k\mu^{k+1}M_{u,T}^{m}M_{u,T}^{*m})f, f \right\rangle \geq 0 \;\; \forall \;\; f \in L^2(\mu)$$

Since

$$M_{u,T}^{m+k+1}M_{u,T}^{*m+k+1}f = u_{m+k+1}(hu) \circ T^{m+k+1}E(hu) \circ TE(f)$$
$$M_{u,T}^{m+1}M_{u,T}^{*m+1}f = u_{m+1}(hu) \circ T^{m+1}E(hu) \circ TE(f)$$
$$M_{u,T}^{m}M_{u,T}^{*m}f = u_m(hu) \circ T^m E(hu) \circ TE(f)$$

$$\int_E (u_{m+k+1}(hu) \circ T^{m+k+1}E(hu) \circ TE(f) - (k+1)\mu^k u_{m+1}(hu) \circ T^{m+1}E(hu) \circ TE(f)$$
$$+ k\mu^{k+1} u_m(hu) \circ T^m E(hu) \circ TE(f))d\mu \geq 0$$

for every $E \in \Sigma$.

$$\Leftrightarrow \quad u_{m+k+1}(hu) \circ T^{m+k+1}E(hu) \circ TE(f) - (k+1)\mu^k u_{m+1}(hu) \circ T^{m+1}E(hu) \circ TE(f)$$
$$+ k\mu^{k+1} u_m(hu) \circ T^m E(hu) \circ TE(f) \geq 0$$

almost everywhere. □

Corollary 4.4. *The composition operator C_T^* on $B(L^2(\mu))$ is m-Quasi k-paranormal if and only if $(h) \circ T^{m+k+1}E(h) \circ TE(f) - (k+1)\mu^k(h) \circ T^{m+1}E(h) \circ TE(f) + k\mu^{k+1}(h) \circ T^m E(h) \circ TE(f) \geq 0$ almost everywhere.*

Proof. The proof is obtained from Theorem 4.3 by putting $u = 1$. □

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