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Finite-Dimensional Approximation of Modified Laverentiv Method for Nonlinear Ill-posed Operators

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Abstract: In [7], the authors have suggested a modified iterative Laverentiv method for solving nonlinear ill-posed equations involving monotone operators. In this paper, we propose the finite-dimensional approximation of the scheme considered in [7]. We have shown the regularization parameter and the error estimates are optimal order. Finally, we also present some numerical results to verify the theoretical estimates.

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Keywords: Nonlinear Ill-posed Problems, Regularization, Iterative Method, Parameter Choice Rule.

1. Introduction

Our aim is to solve the nonlinear monotone operator equation in the real Hilbert space H of the form

$$G(u) = v, \tag{1}$$

where $G : D(G) \subset H \to H$. We are assuming that, instead of the exact data v, we have only the noisy data $\tilde{v}$, with $\|\tilde{v} - v\| < \delta, \delta > 0$. Therefore in (1), we only have

$$G(u) = \tilde{v} \tag{2}$$

with $\|v - \tilde{v}\| < \delta$.

Under the suitable assumptions on G, the operator equation (1) may have a unique solution but it may not be stable with respect to perturbation of the data. In such cases, we have to apply regularization [1, 3, 14] methods to obtain a stable approximate solution. We assume that there is a solution $u^\dagger$ of (1) and the operator G is locally Fréchet differentiable. For monotone operators, the Lavrentiev method is widely using for a stable approximate solution [6–8, 13, 15]. In this paper, we are considering the finite dimensional approximation of the modified Lavrentiev scheme in [7] of the form

$$\tilde{u}_{n+1,h} = u_0 + (P_h K P_h + \beta I)^{-1} P_h \left(\tilde{v} - G(u_{n,h}) + K P_h(\tilde{u}_{n,h} - u_0)\right), \tag{3}$$

where $u_0 = \tilde{u_0} = \tilde{u}_{h,0}, K = G'(u_0)$ and $\beta > 0$ is the regularization parameter.

This paper is organized as follows. In Section 2, we discuss the convergence and convergence analysis of the scheme by using a priori parameter choice rule, convergence analysis by using a posteriori choice parameter rule is given in Section 3. In Section 4, we present some numerical examples of the scheme and conclusions are given in Section 5.

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2. Error Estimates

In our analysis, we have taken the following assumptions.

Assumption I: There exists a constant $k_0 > 0$ and $h(u, u_0, v) \in H$ satisfying $(G'(u) - G'(u_0))v = G'(u_0)h(u, u_0, v)$ with $\|h(u, u_0, v)\| \leq k_0\|v\|\|u - u_0\|\, \forall u, v$ in $B_{\frac{r}{2}}(u_0)$ and $\frac{k_0 r}{2}(1 + \gamma_h) < 1$, where $\gamma_h = \frac{\|K(I-P_h\|}{\beta}$.
Assumption II: $u^\dagger - u_0 = Ku$ for some $u \in H$.

Lemma 2.1 ([8, 9]). *If G is Fréchet differentiable in $B_{\frac{r}{2}}(u_0)$, then by Assumption I, $\forall u, v \in B_{\frac{r}{2}}(u_0)$, we have*

$$\|(K + \beta I)^{-1}K(G(u) - G(v) - K(u - v))\| \leq \frac{k_0 r}{2}\|u - v\|. \tag{4}$$

We use the following lemma, to prove all the iterations belong to the ball $B_{\frac{r}{2}}(u_0)$.

Lemma 2.2 ([2, 5]). *Suppose there is a sequence c_n of non-negative real numbers satisfying*

$$c_{n+1} \leq a_1 + a_2 c_n + a_3 c_n^2, a_1, a_2, a_3 \geq 0.$$

Let $c = \frac{2a_1}{1-a_1+\sqrt{(1-a_1)^2-4a_1a_3}}$ and $\bar{c} = \frac{1-a_2+\sqrt{(1-a_2)^2-4a_1a_3}}{2a_3}$. If $a_2 + 2\sqrt{a_1 a_3} < 1$ and $c_0 \leq \bar{c}$, then $c_n \leq \max\{c_0, c\}$.

Lemma 2.3. *Let $u^\dagger \in B_{r_0}(u_0) \subset B_{\frac{r}{2}}(u_0) \subset D(G)$,where $r_0 = (1 - \frac{k_0 r}{2}(1 + \gamma_h))\frac{r}{4}$. Suppose $\frac{\delta}{\beta} \leq (1 - (1 + \gamma_h)\frac{k_0 r}{2})\|u^\dagger - u_0\|$, then $\tilde{u}_{n+1} \in B_{\frac{r}{2}}(u_0), \forall n$.*

Proof.

$$\begin{aligned}
\tilde{u}_{n+1,h} - u_0 &= (P_h K P_h + \beta I)^{-1}\left(P_h(\tilde{v} - G(\tilde{u_{n,h}})) + KP_h(\tilde{u}_{n,h} - u_0)\right) \\
&= (P_h K P_h + \beta I)^{-1}\left(P_h(G(u_0) - G(\tilde{u}_{n,h}) - KP_h(u_0 - \tilde{u}_{n,h}) + \tilde{v} - G(u_0))\right) \\
&= (P_h K P_h + \beta I)^{-1}P_h\left(G(u_0) - G(\tilde{u}_{n,h}) - KP_h(u_0 - \tilde{u}_{n,h})\right) + (P_h K P_h + \beta I)^{-1}P_h(\tilde{v} - G(u_0))
\end{aligned}$$

Let

$$\begin{aligned}
A &= (P_h KPh + \beta P_h)^{-1}P_h(\tilde{v} - G(u_0)) \\
\|A\| &= \|\tilde{u}_{1,h} - u_0\|.
\end{aligned}$$

Now consider,

$$\begin{aligned}
G(u_0) - G(\tilde{u}_{n,h}) - KP_h(u_0 - \tilde{u}_{n,h}) &= \int_0^1 (G^{'}(\tilde{u}_{n,h} + t(P_h(u_0 - \tilde{u}_{n,h})(P_h(u_0 - \tilde{u}_{n,h})) - K(P_h(u_0 - \tilde{u}_{n,h})))dt \\
&= K\phi(z, u_0, P_h(u_0 - \tilde{u}_{n,h})), z = \tilde{u}_{n,h} + t(P_h(u_0 - \tilde{u}_{n,h}); \\
P_h\left(G(u_0) - G(\tilde{u}_{n,h}) - KP_h(u_0 - \tilde{u}_{n,h})\right) &= P_h K h(z, u_0, P_h(u_0 - \tilde{u}_{n,h})) \\
&= P_h K(I - P_h + P_h)h(z, u_0, P_h(u_0 - \tilde{u}_{n,h})) \\
&= P_h K P_h h(z, u_0, P_h(u_0 - \tilde{u}_{n,h})) + P_h K(I - P_h)h(z, u_0, P_h(u_0 - \tilde{u}_{n,h})).
\end{aligned}$$

$$\|(P_h K P_h + \beta I)^{-1}G(u_0) - G(\tilde{u}_{n,h}) - KP_h(u_0 - \tilde{u}_{n,h})\| \leq \frac{k_0 r}{2}(1 + \gamma_h)\|\tilde{u}_{n,h} - u_0\| \tag{5}$$

Therefore

$$\|\tilde{u}_{n+1} - u_0\| \leq \frac{k_0 r}{2}(1 + \gamma_h)\|\tilde{u}_n - u_0\| + \|\tilde{u}_{1,h} - u_0\| \tag{6}$$

This is of the form

$$c_{n+1} \leq a_1 + a_2 c_n + a_3 c_n^2 \tag{7}$$

with $c_n = \|\tilde{u}_n - u_0\|, a_1 = \|\tilde{u}_{1,h} - u_0\|, a_2 = \frac{k_0 r}{2}(1+\gamma_h), a_3 = 0$. We have, $a_2 + 2\sqrt{a_1 a_3} = \frac{k_0 r}{2}(1+\gamma_h) < 1$. Therefore, by Lemma 2.2,

$$\begin{aligned} \|\tilde{u}_{n+1,h} - u_0\| &\leq \max\left\{\frac{\|\tilde{u}_{1,h} - u_0\|}{1 - \frac{k_0 r}{2}(1+\gamma_h)}, 0\right\} \\ \|\tilde{u}_{n+1,h} - u_0\| &\leq \frac{\|\tilde{u}_{1,h} - u_0\|}{1 - \frac{k_0 r}{2}(1+\gamma_h)}. \end{aligned}$$

Now consider,

$$\begin{aligned} \tilde{u}_{1,h} - u_0 &= (P_h K P_h + \beta I)^{-1} P_h(\tilde{v} - G(u_0)) \\ &= (P_h K P_h + \beta I)^{-1} P_h(v - G(u_0) - K P_h(u^\dagger - u_0) + K P_h(u^\dagger - u_0) + \tilde{v} - v) \\ &\leq \frac{k_0 r}{2}(1+\gamma_h)\|u^\dagger - u_0\| + \frac{\delta}{\beta} + \|u^\dagger - u_0\|. \end{aligned}$$

Since $\frac{\delta}{\beta} \leq (1 - (1+\gamma_h)\frac{k_0 r}{2})\|u^\dagger - u_0\|$, we have,

$$\begin{aligned} \frac{\|\tilde{u}_{1,h} - u_0\|}{1 - \frac{k_0 r}{2}(1+\gamma_h)} &\leq \frac{(2+\gamma_h)\|u^\dagger - u_0\|}{1 - \frac{k_0 r}{2}(1+\gamma_h)} \\ &\leq \frac{r}{2}. \\ \tilde{u}_{n+1} &\in B_{\frac{r}{2}}(u_0). \end{aligned} \tag{8}$$

□

Theorem 2.4. *Under Assumption I,*

$$\|\tilde{u}_{n+1} - \tilde{u}_n\| \leq \left(\frac{k_0 r}{2}(1+\gamma_h)\right)^n \|\tilde{u}_1 - u_0\|, \forall n. \tag{9}$$

Proof. From (3)

$$\begin{aligned} \tilde{u}_{n+1,h} &= u_0 + (P_h K P_h + \beta I)^{-1} P_h\left(\tilde{v} - G(\tilde{u}_{n,h}) + K P_h(\tilde{u}_{n,h} - u_0)\right) \\ \tilde{u}_{n,h} &= u_0 + (P_h K P_h + \beta I)^{-1} P_h\left(\tilde{v} - G(\tilde{u}_{n-1,h}) + K P_h(\tilde{u}_{n-1,h} - u_0)\right) \\ \tilde{u}_{n+1,h} - \tilde{u}_{n,h} &= (P_h K P_h + \beta I)^{-1} P_h\left(G(\tilde{u}_{n-1,h}) - G(\tilde{u}_{n,h}) - K P_h(\tilde{u}_{n-1,h} - \tilde{u}_{n,h})\right) \end{aligned}$$

Therefore by assumption (A-I),

$$\begin{aligned} \|\tilde{u}_{n+1,h} - \tilde{u}_{n,h}\| &\leq \frac{k_0 r}{2}(1+\gamma_h)\|\tilde{u}_{n,h} - \tilde{u}_{n-1,h}\| \\ &\leq \left(\frac{k_0 r}{2}(1+\gamma_h)\right)^n \|\tilde{u}_{1,h} - u_0\|. \end{aligned}$$

□

Theorem 2.5. *If (3) satisfied Assumption-I and Assumption II then $\tilde{u}_{n+1} \to u^\dagger$ as $\delta \to 0$. For the choice of $\beta \sim (\gamma_h + \delta)^{1/2}$, and the iteration number n, $\left(\frac{k_0 r}{2}(1+\gamma_h)\right)^n \leq (\delta + \gamma_h)^{1/2}$ we will get*

$$\|\tilde{u}_{n+1,h} - u^\dagger\| = O(\delta + \gamma_h)^{\frac{1}{2}}. \tag{10}$$

Proof.

$$\begin{aligned}
\tilde{u}_{n+1,h} - u^\dagger &= u_0 - u^\dagger + (P_hKP_h + \beta I)^{-1} P_h\left(\tilde{v} - G(\tilde{u}_{n,h})\right) + KP_h(\tilde{u}_{n,h} - u_0)) \\
&= (P_hKP_hh + \beta I)^{-1}\Big((P_hKP_h + \beta I)(u_0 - u^\dagger)\Big) + (P_hKP_hh + \beta I)^{-1} P_h\left(\tilde{v} - G(\tilde{u}_{n,h}) + KP_h(\tilde{u}_{n,h} - u_0)\right) \\
&= (P_hKP_h + \beta I)^{-1} P_h\Big(\tilde{v} - G(\tilde{u}_{n,h}) - KP_h(u^\dagger - \tilde{u}_{n,h})) + \beta(u^\dagger - u_0)\Big) \\
&= (P_hKP_h + \beta I)^{-1}\Big(P_h(v - G(u)_{n,h}) - K(u^\dagger - \tilde{u}_{n,h}) + \tilde{v} - v\Big) \\
&+ (P_hKP_h + \beta I)^{-1}(K(u^\dagger - \tilde{u}_{n,h}) - KP_h(u^\dagger - \tilde{u}_{n,h})) + (P_hKP_h + \beta I)^{-1}\Big(\beta(u^\dagger - u_0)\Big).
\end{aligned}$$

Consider,

$$\begin{aligned}
B &= \beta(P_hKPh + \beta I)^{-1}(u^\dagger - u_0) \\
&= \beta(P_hKPh + \beta I)^{-1}K\left(I - P_h + P_h\right)\hat{u} \\
&= \beta(P_hKPh + \beta I)^{-1}\left(K(I - P_h) + (I - P_h + P_h)KP_h\right)\hat{u} \\
&= \beta(P_hKPh + \beta I)^{-1}P_hKP_h\hat{u} + \beta(P_hKPh + \beta I)^{-1}(K(I - P_h) + (I - P_h)K)\hat{u} \\
\|B\| &\leq \beta\|\hat{u}\| + \|(K(I - P_h))\hat{u}\| + \|((I - P_h)K)\hat{u}\|.
\end{aligned}$$

Let,

$$\begin{aligned}
E &= (P_hKP_h + \beta I)^{-1}(K(u^\dagger - \tilde{u}_{n,h}) - KP_h(u^\dagger - \tilde{u}_{n,h})) \\
&= (P_hKP_h + \beta I)^{-1}\Big(K(I - P_h)(u^\dagger - \tilde{u}_{n,h})\Big) \\
&= (P_hKP_h + \beta I)^{-1}\Big(K(I - P_h)(u^\dagger - \tilde{u}_{n,h})\Big) \\
\|E\| &\leq \|\frac{K(I - P_h)}{\beta}\|\|(u^\dagger - \tilde{u}_{n,h})\| \\
&\leq \|\frac{K(I - P_h)}{\beta}\|r. \qquad (11)
\end{aligned}$$

Let $\zeta_h = \frac{\|K(I-P_h)\|r}{\beta}$, and $\psi_h = \|(K(I - P_h))u\| + \|((I - P_h)K)u\|$. Therefore,

$$\begin{aligned}
\|\tilde{u}_{n+1,h} - u^\dagger\| &\leq \frac{k_0r}{2}(1 + \gamma_h)\|\tilde{u}_{n,h} - u^\dagger\| + \frac{\delta}{\beta} + \beta\|u\| + \zeta_h + \psi_h \\
(1 - \frac{k_0r}{2}(1 + \gamma_h))\|\tilde{u}_{n+1,h} - u^\dagger\| &\leq \|\tilde{u}_{n,h} - \tilde{u}_{n+1,h}\| + \frac{\delta}{\beta} + \beta\|u\| + \zeta_h + \psi_h \\
&\leq \left(\frac{k_0r}{2}(1 + \gamma_h)\right)^n\|\tilde{u}_{1,h} - u_0\| + \frac{\delta}{\beta} + \beta\|u\| + \zeta_h + \psi_h \\
&\leq \left(\frac{k_0r}{2}(1 + \gamma_h)\right)^n\|\tilde{u}_{1,h} - u_0\| + \frac{\delta}{\beta} + \beta\|u\| + \frac{\|K(I - P_h)\|r}{\beta} + \psi_h \\
&\leq \left(\frac{k_0r}{2}(1 + \gamma_h)\right)^n\|\tilde{u}_{1,h} - u_0\| + \frac{\delta + r\beta\gamma_h}{\beta} + \beta\|u\| + \psi_h.
\end{aligned}$$

Let $C^{'} = \max\{1, r\beta\}$, then

$$\|\tilde{u}_{n+1,h} - u^\dagger\| \leq \left(\frac{k_0r}{2}(1 + \gamma_h)\right)^n\|\tilde{u}_{1,h} - u_0\| + \frac{C^{'}(\delta + \gamma_h)}{\beta} + \beta\|u\| + \psi_h$$

Therefore the choice of the β, n, and the choice of $\psi_h \sim (\delta + \gamma_h)^{1/2}$ we will get

$$\|u^\dagger - \tilde{u}_{n+1,h}\| = O(\delta + \gamma_h)^{1/2}. \qquad (12)$$

□

3. Error Estimates with an a Posteriori Parameter Choice

The stopping rule

In this section, we consider an a posteriori stopping rule to terminate the iteration which depends on the noisy level and the discretization error. We terminate the iteration if there is an integer N satisfies,

$$\|P_h(G(\tilde{u}_N) - \tilde{v})\| \leq C(\delta + \gamma_h) < \|(P_h\,(\tilde{v} - G(\tilde{u}_{n,h}))\,\|, \quad 0 \leq n \leq N-1, \tag{13}$$

where $C > 1$.

Parameter choice rule

We choose β by solving,

$$\|\beta^2(P_hKP_h + \beta I)^{-1}(P_h(\tilde{v} - G(u_0)))\| = C(\delta + \gamma_h). \tag{14}$$

Next we show that by using above parameter choice rule, the parameter obtained this method is of the $O(\delta^{1/2})$.

Proposition 3.1. *Assume there is a constant $M > 0$ such that $\beta \leq M$, and suppose that the parameter β is computed by (14), then*

$$\beta \sim (\delta + \gamma_h)^{1/2}. \tag{15}$$

Proof.

$$\begin{aligned}
\beta^2\|(P_hKP_h + \beta I)^{-1}P_h(\tilde{v} - G(u_0))\| &= C(\delta + \gamma_h) \\
\beta^2\|\tilde{u}_{1,h} - u_0\| &= C(\delta + \gamma_h) \\
C(\delta + \gamma_h) &= \beta^2\|\tilde{u}_{1,h} - u_0\| \leq \|u^\dagger - u_0\| \leq \beta^2\frac{r}{2} \\
\frac{2C}{r}(\delta + \gamma_h) &\leq \beta^2.
\end{aligned}$$

Therefore,

$$\beta \geq C_1^{'}(\delta + \gamma_h)^{1/2}, C_1^{'} = \left(\frac{2C}{r}\right)^{1/2}. \tag{16}$$

Again we have,

$$\begin{aligned}
\beta^2\|\tilde{u}_{1,h} - u_0\| &= C(\delta + \gamma_h) \\
\beta^2 &= \frac{C(\delta + \gamma_h)}{\|\tilde{u}_{1,h} - u_0\|}
\end{aligned}$$

Now consider,

$$\begin{aligned}
\|\tilde{u}_{1,h} - u_0\| &= \|(P_hKP_h + \beta I)^{-1}P_h(\tilde{v} - G(u_0))\| \\
&\geq \frac{\|P_h(\tilde{v} - G(u_0))\|}{\|P_hKP_h + \beta I\|} \\
\frac{1}{\|\tilde{u}_{1,h} - u_0\|} &\leq \frac{\|P_hKP_h + \beta I\|}{\|P_h(\tilde{v} - G(\tilde{u}_0))\|} \\
&\leq \frac{\|P_hKP_h\| + \beta}{\|P_h(\tilde{v} - G(u_0))\|} \\
&\leq \frac{\|P_hKP_h\| + M}{\|P_h(\tilde{v} - G(u_0))\|}
\end{aligned}$$

$$\beta^2 = \frac{C(\delta+\gamma_h)}{\|\tilde{u}_{1,h}-u_0\|} \leq \frac{C(\delta+\gamma_h)(\|P_hKP_h\|+M)}{\|P_h(\tilde{v}-G(u_0))\|}.$$

Therefore,

$$\beta \leq C_2^{'}(\delta+\gamma_h)^{1/2}, C_2' = \left(\frac{C(\|P_hKP_h\|+M)}{\|P_h(\tilde{v}-G(\tilde{u}_0))\|}\right)^{1/2}. \tag{17}$$

From (16) and (17) we get,

$$\beta \sim (\delta+\gamma_h)^{1/2}.$$

□

Lemma 3.2. *Let $p_0 \geq 0$ and $p_0 \leq p_0^{'}$, for some $p_0^{'}$. Suppose that the sequence has η_n has the following property*

$$p_0 - \tau\eta_n \leq \eta_{n+1} \leq p_0 + \tau\eta_n, n = 0, 1, 2..... \tag{18}$$

If $\tau < 1$ and $\eta_0 \leq \frac{1}{1-\tau}p_0^{'}$, then for all n,

$$\eta_n \leq \frac{1}{1-\tau}p_0^{'}. \tag{19}$$

Proof. Proof is by induction. When $k = 0$, it is obvious. Assume the result is true for $n = j$, to prove the result is true for $n = j+1$, we have

$$\begin{aligned} \eta_{j+1} &\leq p_0 + \tau\eta_j \\ &\leq p_0^{'} + \tau\eta_j \\ &\leq p_0^{'} + \tau\frac{1}{1-\tau}p_0^{'} \\ &= \frac{p_0^{'}}{1-\tau}. \end{aligned}$$

Hence proved. □

Lemma 3.3. *Let $p_0 = \|\beta(P_hKP_h+\beta I)^{-1}(u^\dagger-u_0)+(P_hKP_h+\beta I)^{-1}(\tilde{v}-v)+K(u^\dagger-\tilde{u}_{n,h})-KP_h(u^\dagger-\tilde{u}_{n,h})+K(u^\dagger-\tilde{u}_{n,h})-KP_h(u^\dagger-\tilde{u}_{n,h})\|$ then $\forall n$,*

$$p_0 - \frac{k_0r}{2}\|\tilde{u}_{n,h}-u^\dagger\| \leq \|\tilde{u}_{n+1,h}-u^\dagger\| + p_0 + \frac{k_0r}{2}\|\tilde{u}_{n,h}-u^\dagger\|. \tag{20}$$

Proof.

$$\begin{aligned} \tilde{u}_{n+1,h}-u^\dagger &= (P_hKP_h+\beta I)^{-1}\Big(P_h(v-G(\tilde{u}_{n,h}))-K(u^\dagger-\tilde{u}_{n,h})+\tilde{v}-v)\Big) \\ &+ (P_hKP_h+\beta I)^{-1}(K(u^\dagger-\tilde{u}_{n,h})-KP_h(u^\dagger-\tilde{u}_{n,h}))-(P_hKP_h+\beta I)^{-1}\Big(P_h(\beta(u_0-u^\dagger))\Big) \\ &= (P_hKP_h+\beta I)^{-1}(P_h(G(u^\dagger)-G(\tilde{u}_{n,h})-K(u^\dagger-\tilde{u}_{n,h}))+C+D+G. \end{aligned}$$

Where,

$$\begin{aligned} C &= -(P_hKP_h+\beta I)^{-1}(\beta(u_0-u^\dagger)) \\ D &= (P_hKP_h+\beta I)^{-1}(\tilde{v}-v) \\ G &= (P_hKP_h+\beta I)^{-1}(K(u^\dagger-\tilde{u}_{n,h})-KP_h(u^\dagger-\tilde{u}_{n,h})). \end{aligned}$$

Therefore,

$$\begin{aligned}
\|\tilde{u}_{n+1} - u^\dagger - (C+D+G)\| &= \|(P_hKP_h + \beta I)^{-1}\left(P_h(G(u^\dagger) - G(\tilde{u}_{n,h}) - K(u^\dagger - \tilde{u}_{n,h})\right)\| \\
|\|\tilde{u}_{n+1,h} - u^\dagger\| - p_0| &\le \|(P_hKP_h + \beta I)^{-1}\left(P_h(G(u^\dagger) - G(\tilde{u}_{n,h}) - K(u^\dagger - \tilde{u}_{n,h}))\right)\| \\
&\le \frac{k_0 r}{2}(1+\gamma_h)\|\tilde{u}_{n,h} - u^\dagger\|. \\
p_0 - \frac{k_0 r}{2}(1+\gamma_h)\|\tilde{u}_{n,h} - u^\dagger\| &\le \|\tilde{u}_{n+1,h} - u^\dagger\| + p_0 + \frac{k_0 r}{2}(1+\gamma_h)\|\tilde{u}_{n,h} - u^\dagger\|. \qquad (21)
\end{aligned}$$

□

Theorem 3.4. *Under Assumption A-I, Assumption II and the choice of regularization parameter, we have*

$$\|\tilde{u}_{n+1,h} - u^\dagger\| = O(\delta + \gamma_h)^{1/2}. \qquad (22)$$

Proof. From the Proposition 3.1, $C_1^{'}(\delta+\gamma_h)^{1/2}) \le \beta \le C_2^{'}(\delta+\gamma_h)^{1/2})$. Therefore,

$$\frac{\delta+\gamma_h}{\beta} \sim (\delta+\gamma_h)^{1/2}. \qquad (23)$$

Now compare equation (20) with (18), $p_0 = C+D+G, \tau = \frac{k_0 r}{2}(1+\gamma_h) < 1$, and $\eta_n = \|\tilde{u}_n - u^\dagger\|$. Clearly

$$\begin{aligned}
p_0 &= \|(P_hKP_h + \beta I)^{-1}(\tilde{v} - v) - (P_hKP_h + \beta I)^{-1}(\beta(u_0 - u^\dagger)) + K(u^\dagger - \tilde{u}_{n,h}) - KP_h(u^\dagger - \tilde{u}_{n,h})\| \\
&\le \frac{\delta}{\beta} + \beta\|\hat{u}\| + \frac{\|K(I-P_h)\|r}{\beta} + \|K(I-P_h)\hat{u}\| + \|((I-P_h)K)\hat{u}\|.
\end{aligned}$$

Take $p_0^{'} = \frac{\delta}{\beta} + \beta\|\hat{u}\| + \frac{\|K(I-P_h)\|r}{\beta} + \|K(I-P_h)\hat{u}\| + \|((I-P_h)K)\hat{u}\|$. Therefore by Lemma 3.2,

$$\begin{aligned}
\|\tilde{u}_{n+1} - u^\dagger\| &\le \frac{1}{1-\frac{k_0 r}{2}(1+\gamma_h)}\left(\frac{\delta}{\beta} + \beta\|\hat{u}\| + \frac{\|K(I-P_h)\|r}{\beta} + \psi_h\right) \\
&= \frac{1}{1-\frac{k_0 r}{2}(1+\gamma_h)}\left(\frac{(\delta+\beta r\gamma_h)}{\beta} + \beta\|\hat{u}\| + \psi_h\right) \\
&\le \frac{1}{1-\frac{k_0 r}{2}(1+\gamma_h)}\left(\frac{C'(\delta+\gamma_h)}{\beta} + \beta\|\hat{u}\| + \psi_h\right), C^{'} = \max\{1, \beta r\} \\
&= \frac{1}{1-\frac{k_0 r}{2}(1+\gamma_h)}\left(C^{'}O(\delta+\gamma_h)^{1/2} + \|\hat{u}\|O(\delta+\gamma_h)^{1/2} + \psi_h\right) \\
&\le \frac{1}{1-\frac{k_0 r}{2}(1+\gamma_h)}C_3^{'}\left(2(\delta+\gamma_h)^{1/2} + \psi_h\right), C_3^{'} = \max\{C^{'}, \|\hat{u}\|, 1\} \\
&= O\left((\delta+\gamma_h)^{1/2} + \psi_h\right).
\end{aligned}$$

Therefore, by the choice of $\psi_h \sim (\delta+\gamma_h)^{1/2}$

$$\|\tilde{u}_N - u^\dagger\| = O\left(\delta+\gamma_h\right)^{1/2}. \qquad (24)$$

□

4. Numerical Illustrations

For the numerical implementation, we use the Haar basis and the projection considered in [10].

Example 4.1. *Take* $H = L^2[0,1], G : H \to H$ *defined by [7, 8, 12]*

$$G(u) := (\tan^{-1} u)^3 + \int_0^1 e^{-|x-y|} u(y)dy. \tag{25}$$

For the data

$$v(x) = \begin{cases} 0 & \frac{1}{3} \leq u \leq \frac{2}{3} \\ 2 + (\tan^{-1} 1)^3 - e^{u-1} - e^{-u} & \textit{otherwise} \end{cases}$$

the solution is

$$u(x) = \begin{cases} 0 & \frac{1}{3} \leq u \leq \frac{2}{3} \\ 1 & \textit{otherwise.} \end{cases}$$

For our analysis, we use $0.1\%, 1\%,$ *and* 10% *data errors, regularization parameter* β *is computed using equation (14). The computational results are given in Table 1. The exact and computed solutions corresponding to 0.1%, 1%, and* 10% *with different sizes of matrices are given in Figure 1, Figure 2, Figure 3, Figure 4, Figure 5, and Figure 6 respectively.*

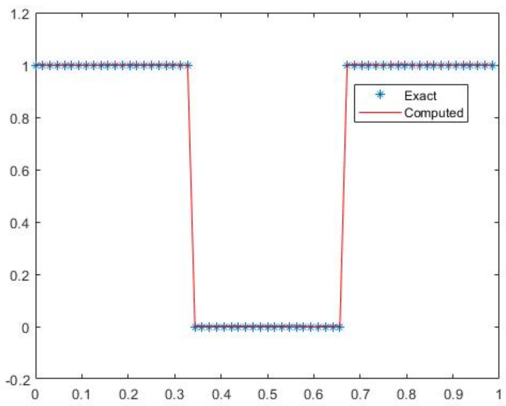

Figure 1. **Soln. when** $\delta = 0.1\%$, **Size of matrix is 64**

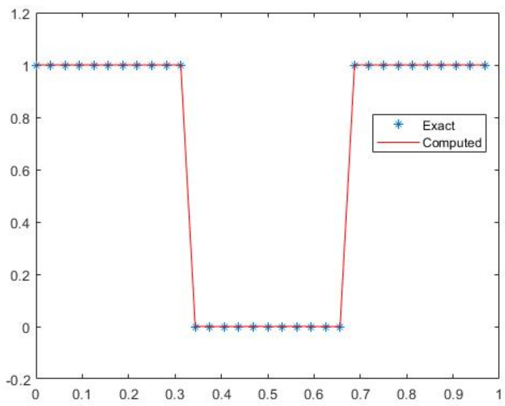

Figure 2. **Soln. when** $\delta = 0.1\%$, **Size of matrix is 32**

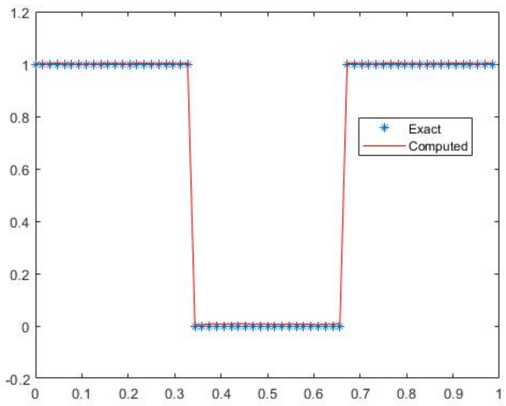

Figure 3. **Soln. when** $\delta = 1\%$, **Size of matrix is 64**

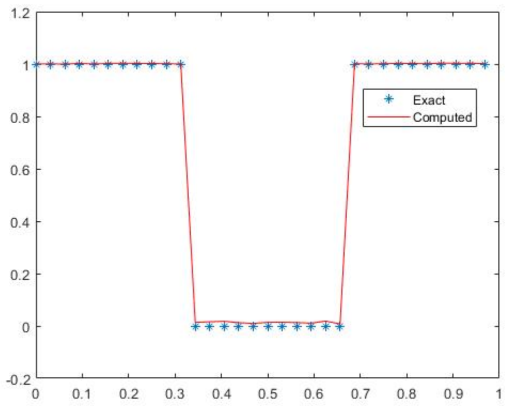

Figure 4. **Soln. when** $\delta = 1\%$, **Size of matrix is 32**

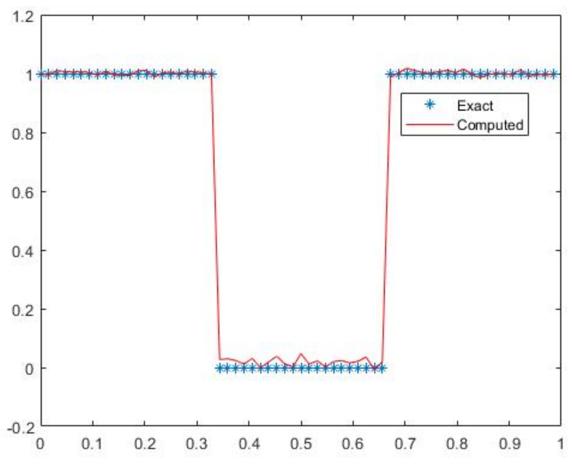

Figure 5. Soln. when $\delta = 10, \%$ Size of matrix is 64

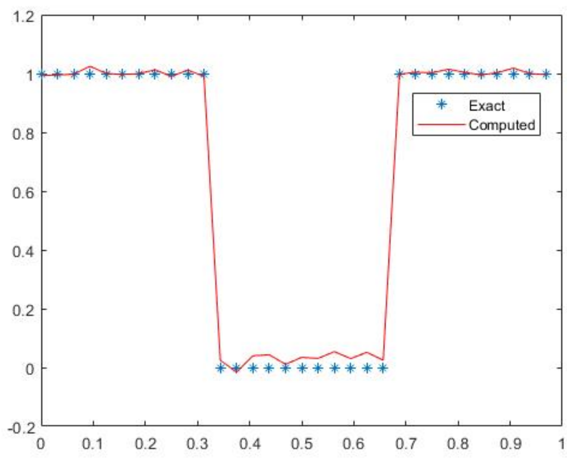

Figure 6. Soln. when $\delta = 10\%$, Size of matrix is 32

Table 1. Computational Results of Example 1

δ	$m = 2^h$	n	β	Error $= \frac{\|u_n - E\|}{\|E\|}$	Error$/(\delta + \gamma_h)^{1/2}$
0.001	64	14	0.0000044	0.000174	0.00055
	32	13	0.0000089	0.000369	0.00117
	16	12	0.0000179	0.0004	0.00126
0.01	16	6	0.0159	0.0416	0.1543
	32	7	0.0091	0.0218	0.1072
	64	9	0.0049	0.0151	0.0942
0.1	16	5	0.0835	0.0427	0.1060
	32	5	0.0631	0.0327	0.0903
	64	7	0.0403	0.0270	0.0794

Example 4.2. *Let $H = H^1[0,1]$ and $G : H \to H$ defined by [7, 8, 12],*

$$G(u) := \int_0^1 e^{-|x-y|} u(y) dy + u^3. \tag{26}$$

Take $u(x) = 2 + x - x^3$, then $v(x) = 12e^{u-1} - 7e^{-x} + 4 - 10x - 2x^3 + (2 + x - x^3)^3$. We consider 1% and 10% data error for computation. The computational results are giver in Table 2. The solutions corresponding to 1%, and 10% with different sizes of matrices are given in Figure 7, Figure 8, Figure 9, Figure 10, Figure 11 and Figure 12 respectively.

Table 2. Computational Results of Example 2

δ	$m = 2^h$	n	β	Error $= \frac{\|u_n - E\|}{\|E\|}$	Error$/(\delta + \gamma_h)^{1/2}$
0.001	16	6	0.0047	0.0416	0.1649
	32	7	0.0025	0.0216	0.1201
	64	9	0.0012	0.0148	0.1147
0.01	16	6	0.0159	0.0416	0.1543
	32	7	0.0091	0.0218	0.1072
	64	9	0.0049	0.0151	0.0942
0.1	16	5	0.0835	0.0427	0.1060
	32	5	0.0631	0.0327	0.0903
	64	7	0.0403	0.0270	0.0794

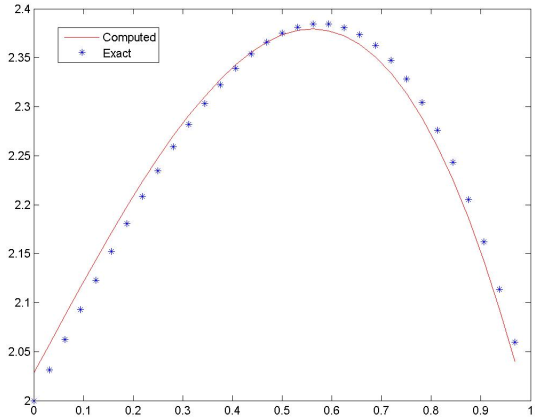

Figure 7. **Soln. when $\delta = 1\%$, Size of matrix is 32**

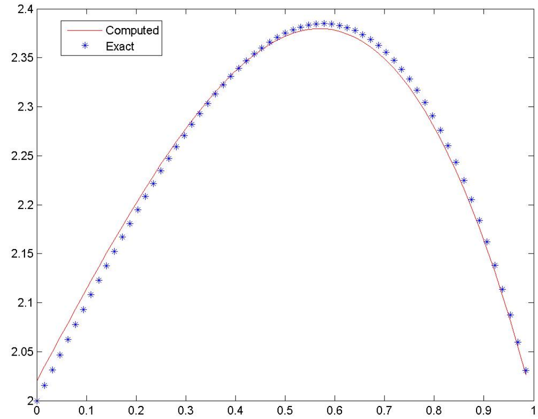

Figure 8. **Soln. when $\delta = 1\%$, Size of matrix is 64**

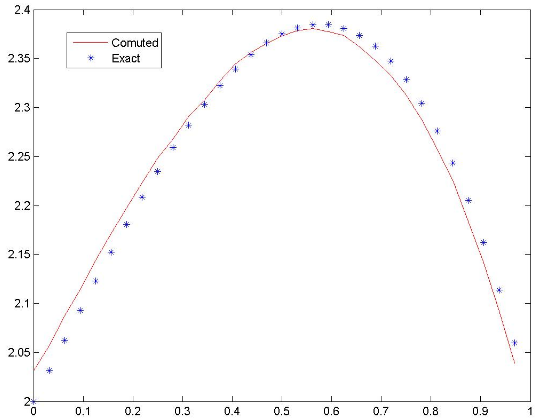

Figure 9. **Soln. when $\delta = 10\%$, Size of matrix is 32**

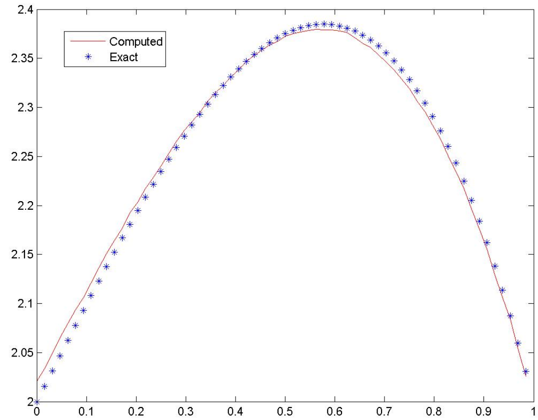

Figure 10. **Soln. when $\delta = 10\%$, Size of matrix is 64**

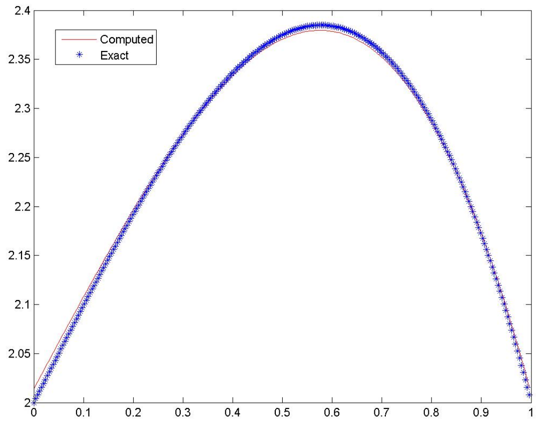

Figure 11. **Soln. when $\delta = 1\%$, Size of matrix is 128**

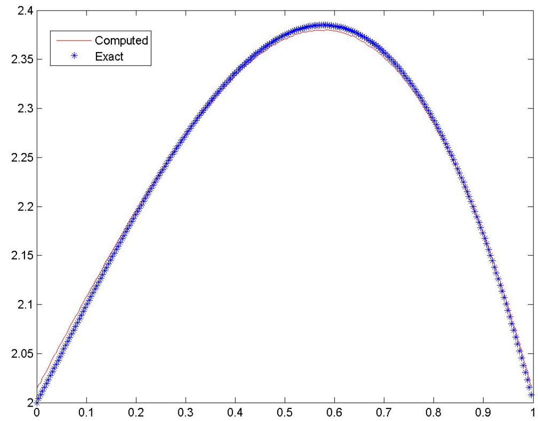

Figure 12. Soln. when $\delta = 10\%$, Size of matrix is 128

5. Conclusion

In this, we suggested the finite-dimensional approximation of an iterative scheme considered in [7] and a parameter choice rule for the regularization parameter β. We have proved that the suggested method achieves $O(\delta+\gamma_h)^{1/2}$. We have implemented this scheme for nonlinear integral equations with various solutions and is shown that all numerical results are consistent with the theoretical estimates.

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