International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **11(2)**(2023), 119–124
Available Online: http://ijmaa.in

Paranormed Hyers-Ulam-Rassias Stability of Quartic Functional Equation

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Abstract

The goal of this paper is to study the Generalized Hyers - Ulam - Rassias (HUR) stability of quartic functional equation (QFE) $f_q(kr_1+(k-1)r_2)+f_q(kr_1-(k-1)r_2) = 2k^4 f_q(r_1)+2(k-1)^4 f_q(r_2) + 6k^2(k-1)^2[f_q(r_1+r_2)+f_q(r_1-r_2)] - 12k^2(k-1)^2[f_q(r_1)+f_q(r_2)]$ in Paranormed Spaces (P. Spaces) using direct method.

Keywords: Generalised Hyers - Ulam - Rassias (HUR) stability; paranormed Space; Quartic Functional Equation.

2020 Mathematics Subject Classification: 39B82, 46AXX, 46BXX.

1. Introduction

First of all, let us recollect the chronicle in the stability theory for functional equations (FEs). The stability problem for the FEs about the stability of group homomorphisms was started by Ulam [19]. The Ulam′s question was to an extent solved by Hyers [5]. Subsequently, Hyers' result was extended by several mathematicians like Aoki [1], Th.M.Rassias [15], Găvruta [4] and more.

The QFE was first introduced by Rassias [14], who solved its Ulam stability problem. Later, Lee et al. [10] remodified Rassias' QFE and obtained its general solution. Numerous mathematicians have extensively studied the stability problems of various QFE in a variety of spaces, including intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces, orthogonal spaces and many other (see [7, 8, 11, 13, 16]).

In this paper, we prove the Generalised HUR stability of the QFE

$$\begin{aligned} f_q(kr_1+(k-1)r_2)+f_q(kr_1-(k-1)r_2) &= 2k^4 f_q(r_1)+2(k-1)^4 f_q(r_2) \\ &\quad + 6k^2(k-1)^2[f_q(r_1+r_2)+f_q(r_1-r_2)] - 12k^2(k-1)^2[f_q(r_1)+f_q(r_2)] \end{aligned} \tag{1}$$

in P. spaces using direct method. Fast [2] and Steinhaus [18] independently established the idea of statistical convergence for real number sequences, and different generalisations and implementations

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of this idea have since been studied by various mathematicians (see [3,6,12,17]). Kolk [9] defined this idea in normed spaces. We recall some fundamental information about Fréchet spaces.

Definition 1.1 ([20]). *Let X be a vector space. A paranorm $P(.) : X \to [0, \infty)$ is a function on X such that*

PN1 $P(0) = 0$;

PN2 $P(-x) = P(x)$;

PN3 $P(x + y) \leq P(x) + P(y)$ *(triangle inequality);*

PN4 *If* $\{t_n\}$ *is a sequence of scalars with* $t_n \to t$ *and* $\{x_n\} \subset X$ *with* $P(x_n - x) \to 0$, *then* $P(t_n x_n - tx) \to 0$ *(continuity of multiplication).*

Definition 1.2 ([20]). *The pair $(X, P(.))$ is called a paranormed space if $P(.)$ is a paranorm on X. The paranorm is called total if, in addition, we have*

PN5 P(x) = 0 implies x= 0.

Definition 1.3 ([20]). *A Fréchet space is a total and complete paranormed space.*

2. Generalized HUR Stability of the QFE in Paranormed Spaces

In this section, we prove the Generalised HUR Stability of the QFE in P. Spaces using direct method. Throughout this section, let (V, P) be a Fréchet space and $(W, ||.||)$ be a Banach space. For a mapping $f_q : V \to W$, define $\hat{D}f_q : V^2 \to W$ as

$$\begin{aligned}\hat{D}f_q(r_1, r_2) &= f_q(kr_1 + (k-1)r_2) + f_q(kr_1 - (k-1)r_2) - 2k^4 f_q(r_1) - 2(k-1)^4 f_q(r_2) \\ &\quad - 6k^2(k-1)^2[f_q(r_1 + r_2) + f_q(r_1 - r_2)] + 12k^2(k-1)^2[f_q(r_1) + f_q(r_2)]\end{aligned}$$

for all $r_1, r_2 \in V$.

Theorem 2.1. *Let $j \in \{1, -1\}$ be fixed and $\eta : V^2 \to [0, \infty)$ be a function such that*

$$\lim_{b \to \infty} \frac{1}{k^{4bj}} \eta(k^{bj} r_1, k^{bj} r_2) = 0 \tag{2}$$

for all $r_1, r_2 \in V$. Let $f_q : V \to W$ be a mapping which maps zero to zero and satisfies

$$P(\hat{D}f_q(r_1, r_2)) \leq \eta(r_1, r_2) \tag{3}$$

for all $r_1, r_2 \in V$. Then there exists a unique quartic mapping $Q_n : V^2 \to W$ such that

$$P(f_q(r_1) - Q_n(r_1)) \leq \frac{1}{2k^4} \sum_{t=\frac{1-j}{2}}^{\infty} \frac{1}{k^{4tj}} \eta\left(k^{tj} r_1, 0\right) \tag{4}$$

for all $r_1 \in V$. The mapping $Q_n(r_1)$ is defined as

$$P\Big(\lim_{b\to\infty}\frac{f_q(k^{bj}r_1)}{k^{4bj}} - Q(r_1)\Big) \to 0 \tag{5}$$

for all $r_1 \in V$.

Proof. Let $j = 1$. Replace (r_1, r_2) by $(r_1, 0)$ in (3) to get

$$P(2f_q(kr_1) - 2k^4 f_q(r_1)) \le \eta(r_1, 0),$$

for all $r_1 \in V$. Or we can say,

$$P(f_q(r_1) - \frac{1}{k^4}f_q(kr_1)) \le \frac{1}{2k^4}\eta(r_1, 0) \tag{6}$$

After replacing r_1 by kr_1, and dividing by k^4 in (6) we get,

$$P\left(\frac{1}{k^4}f_q(kr_1) - \frac{1}{k^8}f_q(k^2r_1)\right) \le \frac{1}{2k^8}\eta(kr_1, 0) \tag{7}$$

for all $r_1 \in V$. Hence (6) and (7), we obtain

$$\begin{aligned} P\left(f_q(r_1) - \frac{1}{k^8}f_q(k^2r_1)\right) &\le \left(P\left(f_q(r_1) - \frac{1}{k^4}f_q(kr_1)\right) + P\left(\frac{1}{k^4}f_q(kr_1) - \frac{1}{k^8}f_q(k^2r_1)\right)\right) \\ &\le \frac{1}{2k^4}\eta(r_1, 0) + \frac{1}{2k^8}\eta(kr_1, 0) \end{aligned} \tag{8}$$

for all $r_1 \in V$. Hence using induction, we can say

$$\begin{aligned} P\Big(f_q(r_1) - \frac{1}{k^{4b}}f_q\big(k^b r_1\big)\Big) &\le \sum_{t=0}^{b-1}\frac{1}{2k^{4(t+1)}}\eta(k^t r_1, 0) \\ &\le \frac{1}{2k^4}\sum_{t=0}^{b-1}\frac{1}{k^{4t}}\eta(k^t r_1, 0), \end{aligned} \tag{9}$$

for all $r_1 \in V$. For some, nonnegative integers a and b, replace r_1 by $k^a r_1$ and divide by k^{4a} in (9), to obtain

$$\begin{aligned} P\Big(\frac{1}{k^{4a}}f_q\big(k^a r_1\big) - \frac{1}{k^{4(a+b)}}f_q\big(k^{(a+b)}r_1\big)\Big) &= \frac{1}{k^{4a}}P\Big(f_q\big(k^a r_1\big) - \frac{1}{k^{4b}}f_q\big(k^{(a+b)}r_1\big)\Big) \\ &\le \frac{1}{2k^4}\sum_{t=0}^{\infty}\frac{1}{k^{4(t+a)}}\eta(k^{(t+a)}r_1, 0), \end{aligned} \tag{10}$$

for all $r_1 \in V$. It follows from (10) that the sequence $\left\{\frac{1}{k^{4b}}f_q\big(k^b r_1\big)\right\}$ is a Cauchy for all $r_1 \in V$. Since W is complete, the sequence $\left\{\frac{1}{k^{4b}}f_q\big(k^b r_1\big)\right\}$ converges. So, one can define the mapping $Q_n : V \to W$ by

$P\Big(\lim_{b\to\infty}\Big\{\frac{1}{k^{4b}}f_q(k^b r_1)\Big\} - Q_n(r_1)\Big) \to 0$ for all $r_1 \in V$. By (PN4), we get

$$P\Big(\lim_{b\to\infty}\Big\{\frac{t_b}{k^{4b}}f_q(k^b r_1)\Big\} - tQ_n(r_1)\Big) \to 0.$$

Assuming $b \to \infty$ in (9), we see that (4) holds, for all $r_1 \in V$. To show that $Q_n(r_1)$ satisfies (1), replace (r_1, r_2) by $(k^b r_1, k^b r_2)$ and divide by k^{4b} in (3), we get

$$\frac{1}{k^{4b}}P(\hat{D}f_q(k^b r_1, k^b r_2)) \leq \frac{1}{k^{4b}}\eta(k^b r_1, k^b r_2),$$

for all $r_1, r_2 \in V$. Letting $b \to \infty$ in above inequality and using definition of $Q_n(r_1)$, we see that

$$\begin{aligned} P(Q_n(kr_1 + (k-1)r_2) + Q_n(kr_1 - (k-1)r_2) - 2k^4 Q_n(r_1) - 2(k-1)^4 Q_n(r_2) \\ - 6k^2(k-1)^2[Q_n(r_1+r_2) + Q_n(r_1-r_2)] + 12k^2(k-1)^2[Q_n(r_1) + Q_n(r_2)]) = 0, \end{aligned}$$

for all $r_1, r_2 \in V$. Using (PN5) in above equation, we have

$$\begin{aligned} Q_n(kr_1 + (k-1)r_2) + Q_n(kr_1 - (k-1)r_2) = 2k^4 Q_n(r_1) + 2(k-1)^4 Q_n(r_2) \\ + 6k^2(k-1)^2[Q_n(r_1+r_2) + Q_n(r_1-r_2)] - 12k^2(k-1)^2[Q_n(r_1) + Q_n(r_2)], \end{aligned}$$

for all $r_1, r_2 \in V$. Hence $Q_n : V \to W$ satisfies (1), for all $r_1, r_2 \in V$. Now let $Q_n' : V \to W$ be another quartic mapping satisfying (1). Then we have

$$\begin{aligned} P(Q_n(r_1) - Q_n'(r_1)) &= \frac{1}{k^{4b}}(P(Q_n(k^b r_1) - Q_n'(k^b r_1))) \\ &\leq \frac{1}{k^{4b}}(P(Q_n(k^b r_1) - f(k^b r_1)) + P(f(k^b r_1) - Q_n'(k^b r_1))) \\ &\leq \frac{1}{2k^4}\sum_{t=0}^{\infty}\frac{1}{2k^{4(t+b)}}\eta(k^{(t+b)}r_1, 0) \to 0 \quad as \;\; b \to \infty \end{aligned}$$

for all $r_1 \in V$. So, we conclude that $Q_n(r_1) = Q_n'(r_1)$ for all $r_1 \in V$. Thus $Q_n : V \to W$ is unique quartic mapping as desired. We can demonstrate the result in the same manner for $j = -1$. This completes the proof. □

Corollary 2.2. *Let Λ and τ be real numbers and $\hat{D}f_q : V \to W$ be a function such that*

$$P(\hat{D}f_q(r_1, r_2)) \leq \begin{cases} \Lambda, \\ \Lambda(P(r_1)^\tau + P(r_2)^\tau), \quad \tau \neq 4 \end{cases} \tag{11}$$

for all $r_1, r_2 \in V$. Then there exists a unique quartic mapping $Q_n : V^2 \to W$ such that

$$P(f_q(r_1) - Q_n(r_1)) \leq \begin{cases} \frac{\Lambda}{2|k^4-1|}, \\ \frac{\Lambda}{2|k^4-k^\tau|}P(r_1)^\tau, \end{cases} \tag{12}$$

for all $r_1 \in V$.

Proof. Taking

$$\eta(r_1, r_2) = \begin{cases} \Lambda, \\ \Lambda(P(r_1)^\tau + P(r_2)^\tau), \quad \tau \neq 4 \end{cases}$$

for all $r_1, r_2 \in V$ in Theorem 2.1, we get the desired result. □

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