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Perfect 2-colorings of Shadow Graphs and Total Graphs of Cycles

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Abstract

We study the perfect 2-colorings (also known as the equitable partitions into two parts) of the shadow graphs and total graphs of cycle graphs. In particular, we determine all the admissible parameter matrices of perfect 2-colorings of these graphs.

Keywords: perfect colorings; equitable partitions; graph colorings; shadow graphs; total graphs; cycle graphs.
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1. Introduction

As a generalization of the concept of completely regular codes, given by Delsarte [14], perfect k-coloring provides a bridge among algebraic combinatorics, graph theory (distance regular graphs), and coding theory (perfect codes). Its connection to regular codes are discussed in a survey paper on completely regular codes [12]. This concept of perfect k-coloring is also known as *equitable partition* [17].

Definition 1.1. A *perfect k-coloring* of a graph $G(V,E)$ with matrix $M = (m_{ij})_{i,j\in\{1,2,\ldots,k\}}$ is a surjective map $\mathcal{C} : V(G) \to \{1,2,\ldots,k\}$ such that for each vertex v where $\mathcal{C}(v) = i$, the number of adjacent vertices with color j is equal to m_{ij}. Here, $V(G)$ is the vertex set of G. The $k \times k$ matrix M is called the *parameter matrix*.

This mapping or coloring of vertices of G with k colors forms a partition of the vertex set of G into k parts $P_1, P_2, \ldots, P_k$ such that, for all $i, j \in \{1,2,\ldots,k\}$, every vertex in P_i is adjacent to the same number of vertices, namely m_{ij} vertices, of P_j. In a perfect 2-colorings of graph G where the color 1 is yellow (assigned to the vertices in P_1), and the color 2 is red (assigned to the vertices in P_2), the parameter matrix is of the form

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

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and has the following meaning: *Every yellow vertex is adjacent to precisely m_{11} yellow vertices and m_{12} red vertices. Similarly, every red vertex is adjacent to precisely m_{21} yellow vertices and m_{22} red vertices.*

Definition 1.2. A parameter matrix $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is called *admissible* for a graph G whenever there exists a perfect 2-coloring of G.

The perfect 2-colorings with given parameter matrix of some Johnson graphs, including $J(6,3)$ and $J(7,3)$ [9], $J(8,3)$ and $J(8,4)$ [10] and, $J(n,3)$ where n is an odd number [16], are already determined. Several constructions for perfect 2-colorings of $J(2n,n)$ and $J(2n,3)$ are also presented in [10]. Perfect 2-colorings of the following graphs are also studied: the Platonic graphs [3] consisting of tetrahedral graph, cubical graph, octahedral graph, dodecahedral graph, and icosahedral graph; cubic (3-regular) graphs of order less than or equal to 10 [5]; transitive cubic graphs [8] which includes prism and crossed prism graphs, Mobius ladders and chordal cycles; hypercube graphs [15]; quartic (4-regular) graphs with order at most 8 [18]; toroidal grid graphs [23] or generalized prism graphs [2]; generalized Petersen graph $GP(n,2)$ where $n \geq 5$ [4], and $GP(n,3)$ [19]; infinite circulant graphs whose set of distances constitutes the segment of naturals $[1,n]$ [22]; grassmann graph of planes [13]; and hamming graphs [11].

Perfect 3-colorings of the following graphs are investigated: Johnson Graph $J(6,3)$ [1], platonic graphs [7], cubic graphs of order 10 [6], prism graphs and Mobius ladders [20], and 6-regular graphs of order 9 [21].

In this work, we consider finite, undirected and simple connected k-regular graph G. A graph G is *k-regular* if each vertex has the same number of adjacent vertices; that is every vertex has the same degree k. In particular, we determine the perfect 2-colorings of shadow graphs and total graphs of cycles. These graphs are 4-regular graphs.

Definition 1.3. Let u be a vertex in graph G. The *open neighbourhood set* $N(u)$ is the set of all vertices adjacent to u in G.

Definition 1.4. The *shadow graph* $D_2(G)$ of G is obtained by taking two copies of G, namely G and G', and joining each vertex u_i in G to the neighbours of corresponding vertex v_i in G'.

Shown in Figure 1a is the shadow graph $D_2(C_{12})$, constructed from the cycle graph C_{12} with vertices $u_1, u_2, \ldots, u_{12}$. Let $v_1, v_2, \ldots, v_{12}$, be the vertices of C'_{12}. In the construction of $D_2(C_{12})$, the new vertex corresponding to u_3 is v_3. Since v_3 is adjacent to v_2 and v_4 in C'_{12}, then u_3 is also adjacent to v_2 and u_4 in $D_2(C_{12})$. The shadow graph $D_2(C_{12})$ is shown in Figure 1a.

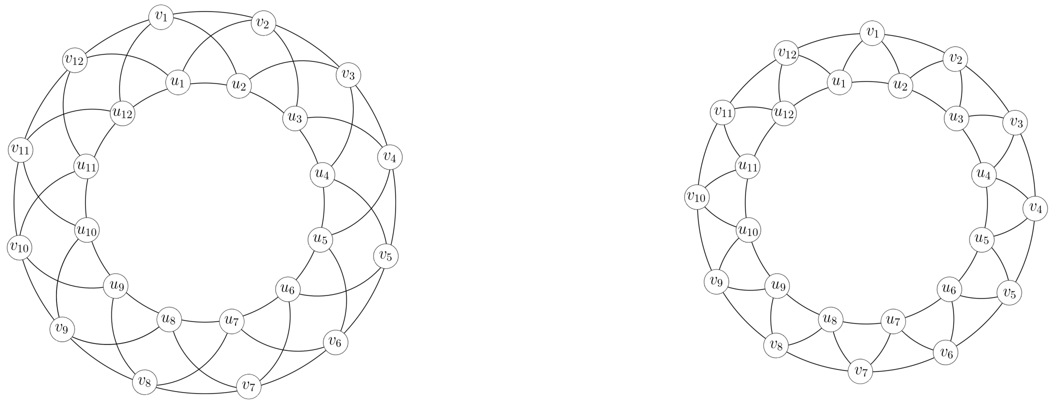


Figure 1: The (a) shadow graph $D_2(C_{12})$ and (b) total graph $T(C_{12})$

Definition 1.5. *The total graph $T(G)$ of graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.*

As an example, refer to the the shadow graph $T(C_{12})$ constructed from the cycle graph C_{12} shown in Figure 1b.

2. Main Results

In this section we present our results on perfect 2-colorings of shadow graphs and total graphs of cycles. We start with presenting existing results on when two parameter matrices result to equivalent colorings under reassignment of colors.

Remark 2.1. *[8]* Two matrices of perfect 2-colorings are called *equivalent* whenever one can be obtained from the other by a permutation of rows and columns corresponding to a reassignment of colors.

Given a 4-regular graph G, consider the parameter matrix $M = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$ that corresponds to a perfect 2-coloring of G. This matrix M means that every yellow vertex of G is adjacent to 4 red vertices, while every red vertex of G is adjacent to 1 yellow vertex and 3 red vertices. Now, reassign yellow vertices with color red, and vise versa. Then, every yellow vertex of G is adjacent to 3 yellow vertices and 1 red vertex, while every red vertex of G is adjacent to 4 yellow vertices. This reassignment of colors is represented by the parameter matrix $M' = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, which can be obtained by a permutation of rows and columns of M.

Now we determine all admissible parameter matrices of perfect 2-colorings of the shadow graphs and total graphs of cycle graphs. Since the graphs are connected, then we cannot have $m_{12} = 0$ or $m_{21} = 0$ respectively. Otherwise, all adjacent vertices of a white (respectively red) vertex are assigned with color white (respectively red).

Without loss of generality, we may assume that every matrix M admissible for some 4-regular graph satisfies $1 \leq m_{12} \leq m_{21} \leq 4$. Naturally, $m_{11} + m_{12} = 4$ and $m_{21} + m_{22} = 4$.

Lemma 2.2. *Only the following ten matrices are the parameter matrices of a perfect 2-coloring of a 4-regular graph.*

$$M_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad M_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \quad M_3 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \quad M_4 = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix} \quad M_5 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \quad M_7 = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} \quad M_8 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad M_9 = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} \quad M_{10} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$$

Lemma 2.3. *[8]* Let $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ be an admissible matrix for graph G then

- the ratio of the numbers of yellow and red vertices of G is $\dfrac{m_{21}}{m_{12}}$;
- the order of G, denoted by $|G|$, is divisible by $\dfrac{m_{12}+m_{21}}{gcd(m_{12}, m_{21})}$, where $gcd(m_{12}, m_{21})$ is the greatest common divisor of m_{12} and m_{21}.

Theorem 2.4. The admissible matrices of the shadow graph $D_2(C_n)$ of cycle graph C_n are exhausted by the list:

(i) $M_1, M_2, M_4, M_6, M_8, M_9$ are not admissible for any n;

(ii) M_3 is admissible for any n divisible by 4;

(iii) M_5 is admissible for any n;

(iv) M_7 is admissible for any n divisible by 3;

(v) M_{10} is admissible for any n divisible by 2.

Proof. (i) Consider vertex $u_1 \in V(D_2(C_n))$, and assign it with color 1 (yellow). The parameter matrix $M_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ suggests that 3 adjacent vertices of u_1 be assigned with color 1 (yellow) and the remaining adjacent vertex of u_1 be assigned with color 2 (red). We name the red vertex as v_2 (See Figure 2a). We have $N(u_2) = N(v_2)$ and $\mathcal{C}(u_2) \neq \mathcal{C}(v_2)$. We arrive at a contradiction to the parameter matrix M_1 since we cannot have the vertices in the neighbourhood of u_2 and v_2 be assigned with 3 yellow and 1 red, and at the same time 1 yellow and 3 red.

The following cases are analogous to the proof above. We start with assigning eithwe color 1 (yellow) or color 2 (red) to vertex $u \in V(D_2(C_n))$. Then we assign colors to the adjacent vertices of u based on the given parameter matrix, and we look for contradictions.

In the cases of $M_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ and $M_4 = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, we consider vertex u_1 and assign it with color yellow. Since $N(u_2) = N(v_2)$ and $\mathcal{C}(v_2) \neq \mathcal{C}(u_2)$ (Figure 2a) we arrive at a contradiction to the parameter matrix M_2 and M_4 respectively. For $M_6 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ and $M_8 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, consider vertex u_1, and assign it with color red. We have $N(u_2) = N(v_2)$ and $\mathcal{C}(u_2) \neq \mathcal{C}(v_2)$ (Figure 2b). We arrive at a contradiction to the parameter matrix M_6 and M_8.

(ii) The contrapositive of Lemma 2.3 suggests that if $|V(D_2(C_n))| = 2n$ is not divisible by 4, then $M_3 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$, is not an admissible matrix for $D_2(C_n)$.

We consider the following assignment of colors:

For all i, $\mathcal{C}(u_i) = 1$ (yellow), and

$$\mathcal{C}(v_i) = \begin{cases} 1 \text{ (red)} & \text{if } i \equiv 1,2 \pmod 4, 1 \leq i \leq n \\ 2 \text{ (yellow)} & \text{if } i \equiv 3,0 \pmod 4, 1 \leq i \leq n \end{cases}$$

(See Figure 3a as an example). This holds when n is divisible by 4.

We have $N(u_j) = \{u_{j-1}, u_{j+1}, v_{j-1}, v_{j+1}\} = N(v_j)$. Consider u_j such that $j \equiv 1,0 \pmod 4$, then $\mathcal{C}(u_{j-1}) = \mathcal{C}(u_{j+1}) = \mathcal{C}(v_{j-1}) = 1$ (yellow) and $\mathcal{C}(v_{j+1}) = 2$ (red). When we have u_j such that $j \equiv 2,3 \pmod 4$, then $\mathcal{C}(u_{j-1}) = \mathcal{C}(u_{j+1}) = \mathcal{C}(v_{j+1}) = 1$ (yellow) and $\mathcal{C}(v_{j-1}) = 2$ (red). Now, consider v_j such that $j \equiv 1,0 \pmod 4$, then $\mathcal{C}(u_{j-1}) = \mathcal{C}(u_{j+1}) = \mathcal{C}(v_{j-1}) = 1$ (yellow) and $\mathcal{C}(v_{j+1}) = 2$ (red). Finally, when v_j such that $j \equiv 2,3 \pmod 4$, we have $\mathcal{C}(u_{j-1}) = \mathcal{C}(u_{j+1}) = \mathcal{C}(v_{j+1}) = 1$ (yellow) and $\mathcal{C}(v_{j-1}) = 2$ (red). Thus, the mapping agrees with parameter matrix M_3.

(iii) With $M_5 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, we have the following assignment of colors: For all i, $\mathcal{C}(u_i) = 1$ (yellow), $\mathcal{C}(v_i) = 2$ (red). (See Figure 3b). This holds for any positive integer n. It can be shown that this mapping agrees with matrix M_5.

(iv) By Lemma 2.3, matrix $M_7 = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$, is not an admissible matrix for $D_2(C_n)$ when $2n$ is not divisible by 3. We consider the following assignment of colors:

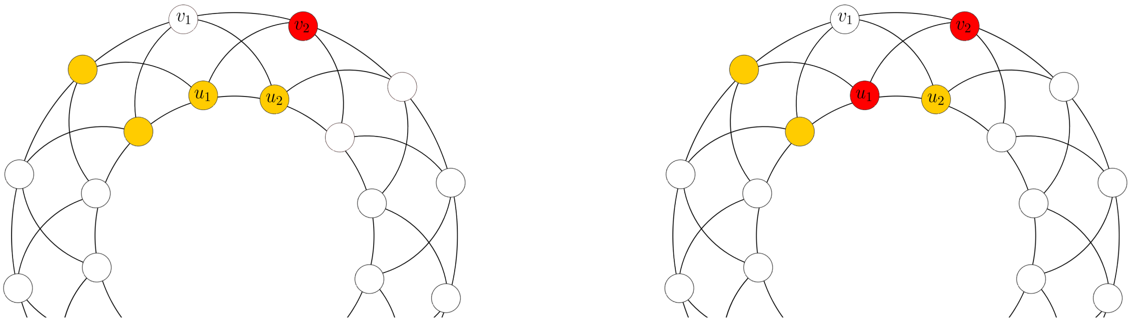
$$\mathcal{C}(u_i) = \mathcal{C}(v_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \not\equiv 0 \pmod 3, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 0 \pmod 3, 1 \leq i \leq n \end{cases}$$

(See Figure 3c as an example). This holds when n is divisible by 3. This mapping agrees with matrix M_7.

(v) With $M_{10} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$, we consider the following assignment of colors:

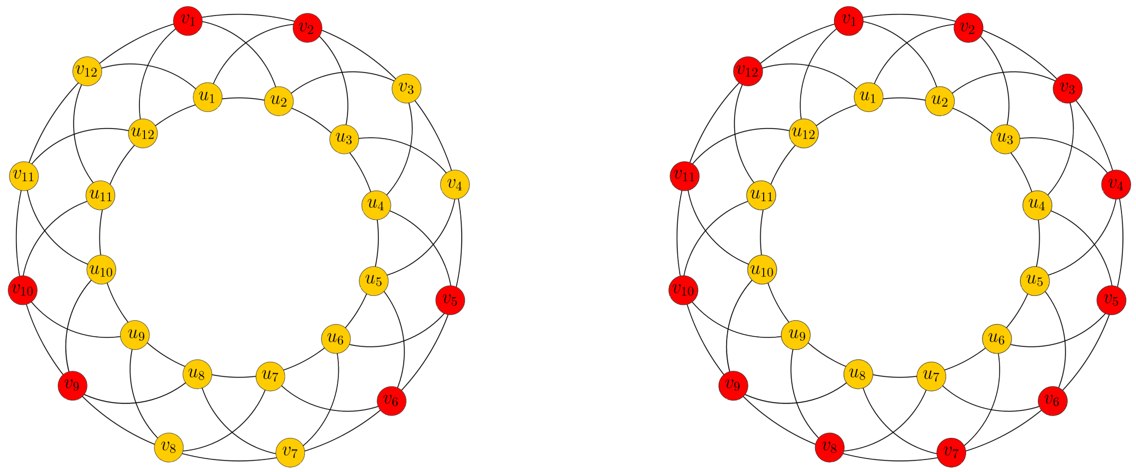
$$\mathcal{C}(u_i) = \mathcal{C}(v_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 1 \pmod 2, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 0 \pmod 2, 1 \leq i \leq n \end{cases}$$

In this case, the vertices in $N(u_i) = N(v_i)$ and are assigned with the same color. (See Figure 3d). This mapping, which agrees with matrix M_{10}, holds when n is an even number. When n is odd, we arrive at a contradiction to M_{10}.

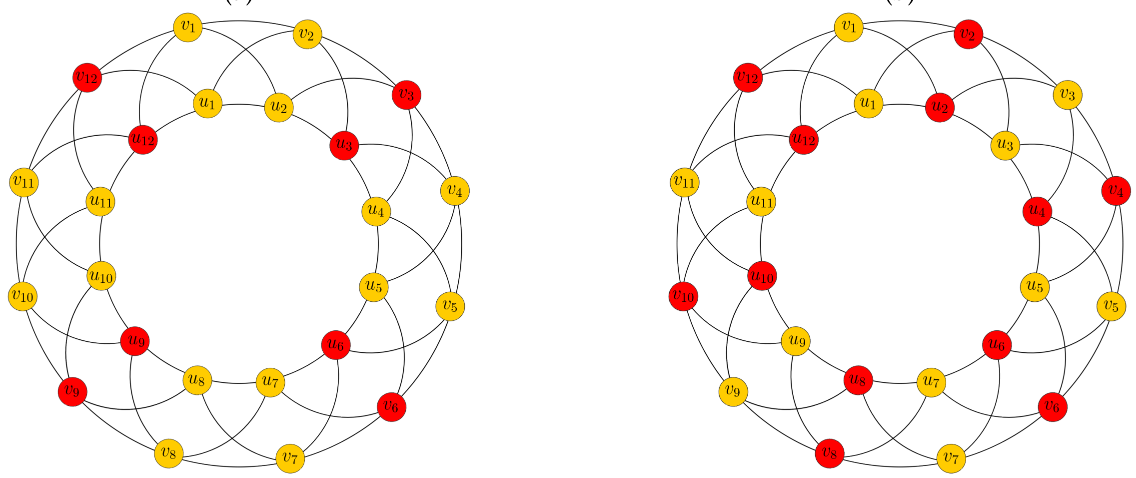


(a) (b)

Figure 2: Initial assignment of colors to the vertices of $D_2(C_n)$ resulting to non-perfect 2-colorings



(a) (b)



(c) (d)

Figure 3: Perfect 2-coloring of $D_2(C_{12})$ with parameter matrix (a) $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

$\square$

Theorem 2.5. The admissible matrices of the total graph $T(C_n)$ of cycle graph C_n are exhausted by the list:

(i) $M_1, M_2, M_3, M_9, M_{10}$ are not admissible for any n;

(ii) M_4 is admissible for any n divisible by 5;

(iii) M_5 is admissible for any n;

(iv) M_6 is admissible for any n divisible by 5;

(v) M_7 is admissible for any n divisible by 3;

(vi) M_8 is admissible for any n divisible by 2.

Proof. (i) As in the proof of the previous theorem, we start with assigning either color 1 (yellow) or color 2 (red) to vertex $u \in V(T(C_n))$. Then we assign colors to the adjacent vertices of u based on the given parameter matrix, and we look for contradictions.

Consider $u_1 \in V(T(C_n))$ and let $\mathcal{C}(u_1) = 1$ (yellow). Note that if $w \in N(u_1)$, then $2 \leq |N(u_1) \cap N(w)| \leq 3$. In the case of $M_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, there exist $w \in N(u_1)$ such that $\mathcal{C}(w) = 2$ (red). But since $2 \leq |N(u_1) \cap N(w)| \leq 3$, then at least two vertices adjacent to w are colored yellow which is a contradiction to parameter matrix M_1. (See Figure 4a when $w = u_2$).

For $M_2 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, any vertex u colored red will have two adjacent vertices, say w and w', assigned with color yellow, and the remaining two vertices assigned with color red. Either w or w' will be adjacent to two red vertices which is a contradiction to matrix M_2. (See Figure 4b as an example).

For $M_3 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$, any vertex u colored red will have three adjacent vertices, say w, w' and w'', assigned with color yellow, and the remaining vertex assigned with color red. Either w, w' or w'' will be adjacent to two red vertices which is a contradiction to matrix M_3. (See Figure 4c as an example).

For $M_9 = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$, any vertex u colored yellow will have an adjacent vertex assigned with color yellow, and the remaining three vertex assigned with color red. There exist $w \in N(u)$ with $\mathcal{C}(w) = 2$ (red) that is adjacent to $w' \in N(u)$ with $\mathcal{C}(w') = 2$; a contradiction to matrix M_9. (See Figure 4d as an example).

For $M_{10} = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$, any vertex u colored yellow will have adjacent vertices assigned with color red. Hence, any $w \in N(u), \mathcal{C}(w) = 2$ (red) is adjacent to $w' \in N(u)$ with $\mathcal{C}(w') = 2$; a contradiction to matrix M_{10}. (See Figure 4e as an example).

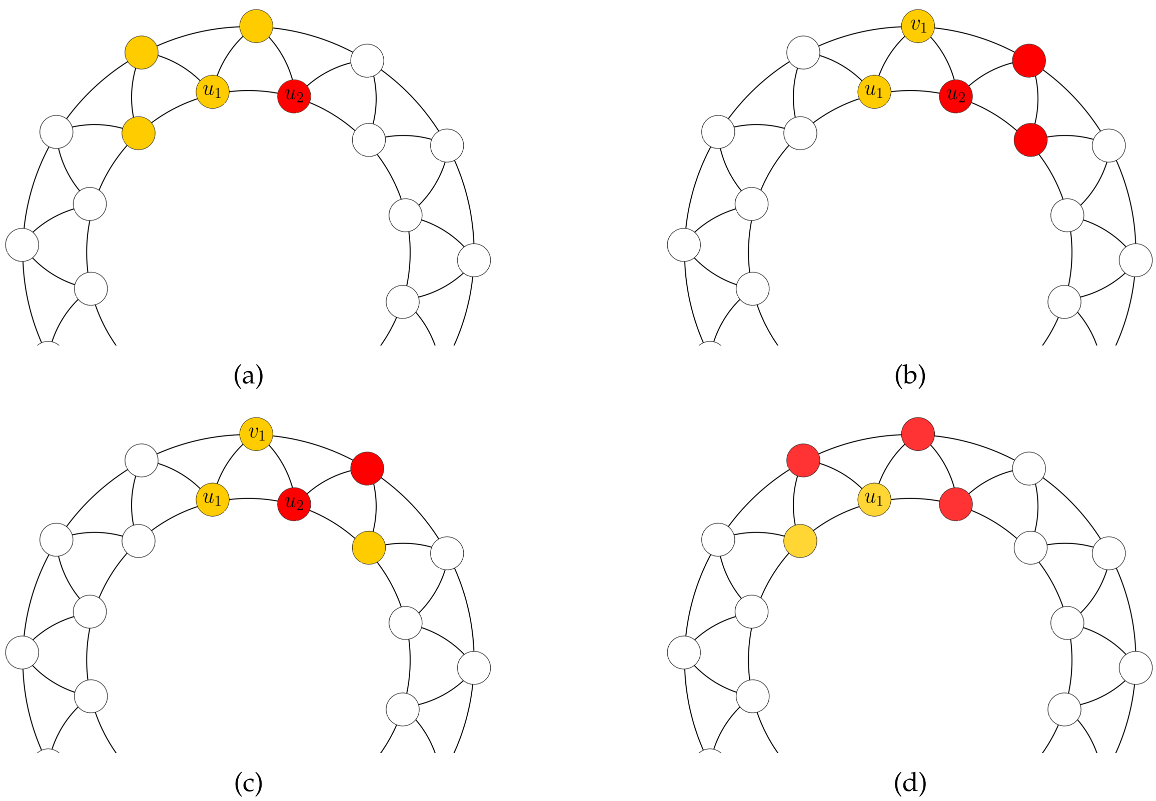


Figure 4: Initial assignment of colors to the vertices of $T(C_n)$ resulting to non-perfect 2-colorings

(ii) The contrapositive of Lemma 2.3 suggests that if $|V(T(C_n))| = 2n$ is not divisible by 5, then $M_4 = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$, is not an admissible matrix for $T(C_n)$.

We now consider the following assignment of colors:

$$\mathcal{C}(u_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 0,1,3,4 \pmod 5, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 2 \pmod 5, 1 \leq i \leq n \end{cases}$$

$$\mathcal{C}(v_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 0,1,2,3 \pmod 5, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 4 \pmod 5, 1 \leq i \leq n \end{cases}$$

This coloring, which agrees with matrix M_4, holds when n is divisible by 5. See Figure 5a for example.

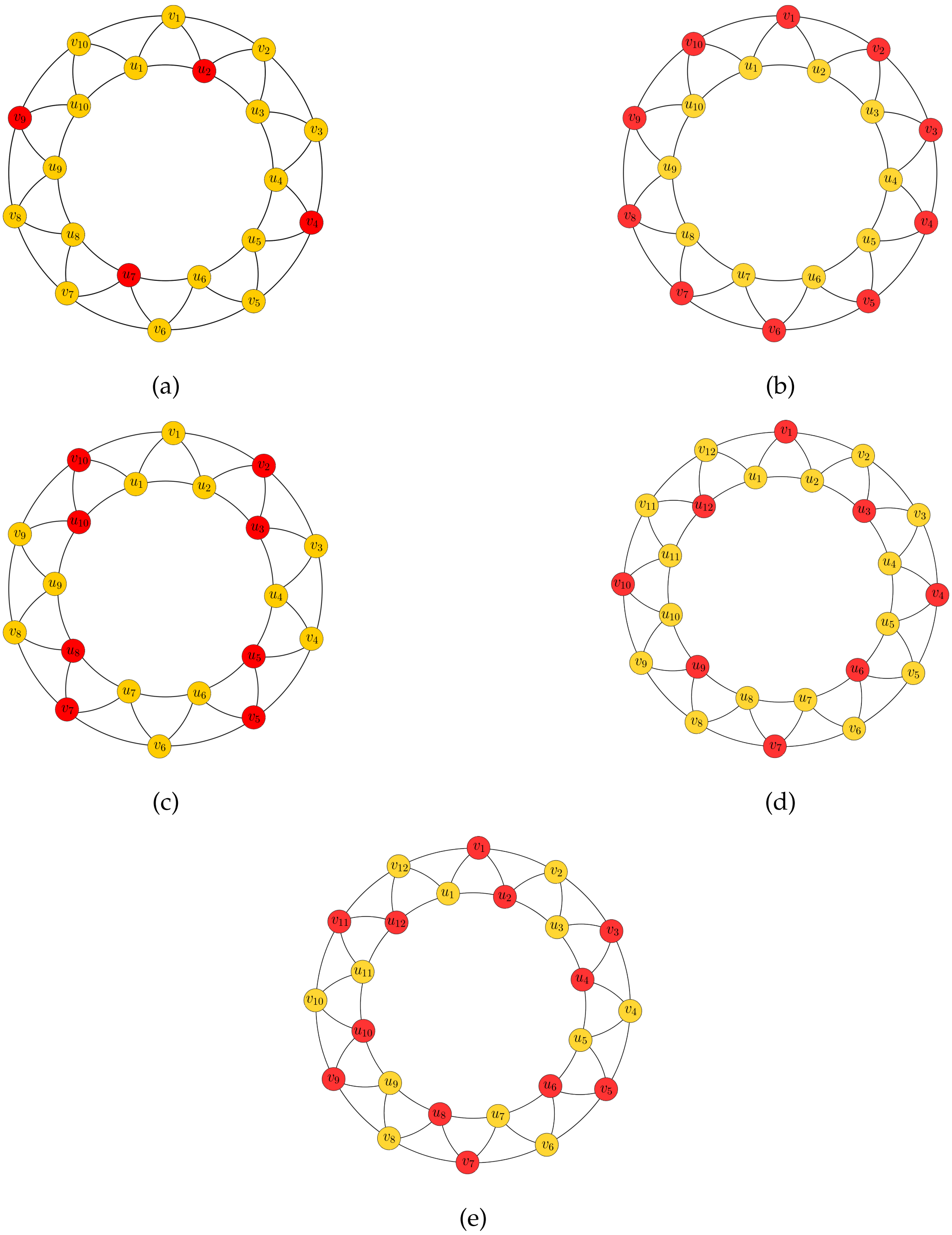


Figure 5: Perfect 2-coloring of $T(C_{10})$ or $T(C_{12})$ with parameter matrix (a) $\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

(iii) For $M_5 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, consider the following assignment of colors: $\mathcal{C}(u_i) = 1$ (yellow), $\mathcal{C}(v_i) = 2$ (red). (See Figure 5b). This coloring agrees with matrix M_5 and holds for any positive integer n.

(iv) By Lemma 2.3, matrix $M_6 = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$, is not an admissible matrix for $T(C_n)$ when $2n$ is not divisible by 5.

We now consider the following assignment of colors:

$$\mathcal{C}(u_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 1,2,4 \pmod 5, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 0,3 \pmod 5, 1 \leq i \leq n \end{cases}$$

$$\mathcal{C}(v_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 1,3,4 \pmod 5, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 0,2 \pmod 5, 1 \leq i \leq n \end{cases}$$

This coloring, which agrees with matrix M_6, holds when n is divisible by 5 .See Figure 5c) for example.

(v) By Lemma 2.3, matrix $M_7 = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$, is not an admissible matrix for $T(C_n)$ when $2n$ is not divisible by 3.

We now consider the following assignment of colors:

$$\mathcal{C}(u_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 1,2, \pmod 3, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 0 \pmod 3, 1 \leq i \leq n \end{cases}$$

$$\mathcal{C}(v_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 0,2 \pmod 3, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 1 \pmod 3, 1 \leq i \leq n \end{cases}$$

This coloring, which agrees with matrix M_7, holds when n is divisible by 3 .See Figure 5d) for example.

(vi) For $M_8 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, consider the following assignment of colors:

$$\mathcal{C}(u_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 1 \pmod 2, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 0 \pmod 2, 1 \leq i \leq n \end{cases}$$

$$\mathcal{C}(v_i) = \begin{cases} 1 \text{ (yellow)} & \text{if } i \equiv 0 \pmod 2, 1 \leq i \leq n \\ 2 \text{ (red)} & \text{if } i \equiv 1 \pmod 2, 1 \leq i \leq n \end{cases}$$

(See Figure 5e). This coloring agrees with matrix M_8, and holds for any n is an even number. When n is odd, we arrive at a contradiction to M_8. □

3. Summary

In this paper, we have determined all perfect 2-colorings of shadow graphs and total graphs of cycle graphs. Note that a total graph $T(C_n)$ of cycle C_n is an antiprism graph, graph corresponding to the skeleton of antiprisms. The 4-regular graphs of order $2n$ are clasess of quartic graphs which adds to the literature of determining perfect 2-colorings of quartic graphs discussed in [18].

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