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Strong Convergence of Common Solution of Variational Inequality and Fixed Point of an Asymptotically Pseudocontractive Mapping

Research Article

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Abstract: Variational inequality provides techniques for solving a variety of applied problems in Science and Engineering. The fixed point formulation of any variational inequality problem is not only useful for existence of solution of the variational inequality problem, but it also provides the facility to develop algorithms for approximation of solution of variational inequality problem.In this paper, we propose an iterative algorithm for solving a variational inequality problem over the set of common fixed point of asymptotically pseudocontractive mapping in a real Hilbert space. We have proved some strong convergence results. Our result improves and extends many corresponding results in the recent literature.

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Keywords: Strong convergence, asymptotically pseudocontractive mapping, nonexpansive mapping, variational inequality.

1. Introduction

Let H be a real Hilbert space. Let C be asubset of H and F(T) denote the set of fixed points of mapping T. We recall here some basic definitions:

Definition 1.1. *Let C be a nonempty closed convex subset of E.A mapping $f : C \to C$ is said to be a strict contraction if there exists a constant $\lambda \in (0, 1)$ satisfying $\|f(x) - f(y)\| \leq \lambda \|x - y\|$ for all $x, y \in C$.*

Definition 1.2. *A mapping $T : C \to C$ is said to be an asymptotically non-expansive if there exists a sequence $\{k_n\}$ with $k_n \to 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$, $\forall\ x, y \in C$. An asymptotically nonexpansive mapping contains strict contraction, nonexpansive mapping as a special case.*

Definition 1.3. *A mapping $T : C \to C$ is said to be an asymptotically pseudocontractive mapping in Banach spaces if there exists a sequence $\{k_n\}$ with $k_n \to 1$ and $j(x - y) \in J(x - y)$ for which the following inequality holds*

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2, \ \forall\ x, y \in C,\ n \geq 1$$

Definition 1.4. *A mapping $T : C \to C$ is said to be uniform L-Lipschitzian if there exists some $L > 0$ such that*

$$\|T^n x - T^n y\| \leq L \|x - y\|, \ \forall\ x, y \in C,\ \forall\ n \geq 1$$

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It is easy to see that if T is an asymptotically nonexpansive mapping, then it is both asymptotically pseudocontractive and uniformly L-Lipschitzian but the converse is not true in general.

A lot of research has been done by many authors in order to investigated the viscosity iterative algorithms to find the common element of the set of fixed point of pseudocontractive mappings and the set of solution of variational inequality problem. In 2000, Moudafi [1] introduced the viscosity iterative algorithm for strong convergence of nonexpansive mappings in real Hilbert space. Later, in 2004, Xu [2] extended results of Moudafi [1] and introduced the following viscosity technique for nonexpansive mapping in a uniformly smooth Banach Space :

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 0$$

Where f is a contraction and $\{\alpha n\}$ is a sequence in $\{0,1\}$. The implicit midpoint rule is a powerful method for solving ordinary differential equations; see [8,9] and the references therein. Recently in 2015, Xu [4] applied the viscosity technique to the implicit midpoint rule for a nonexpansive mapping. They introduced the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0 \tag{1}$$

They proved that the sequence generated by Equation (1) converges strongly to a fixed point of T, which also solves the following variational inequality in Hilbert space:

$$\langle (I-f)q, x-q\rangle \geq 0, \quad x \in F(T) \tag{2}$$

In 2017, Luo [7] proved strong convergence for strict pseudocontractive mapping with some appropriate conditions on parameters by using the above Equation (1), the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces which also solves the following variational inequality problem

$$\langle (I-f)q, x-q\rangle \geq 0, \quad x \in F(T)$$

Recently, Yan [10] extended the result of Luo [7] from nonexpansive mapping to asymptotically nonexpansive mapping and introduced the following generalized viscosity implicit rule for asymptotically nonexpansive mapping in Hilbert space

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0$$

which is also the solution of the variational inequality

$$\langle (I-f)q, y-q\rangle \geq 0, \quad for\ all\ y \in F(T)$$

In this paper, we propose an iterative algorithm and prove strong convergence of an asymptotically pseudocontractive mapping in a reflexive smooth Banach space E which also solves a certain variational inequality. Our algorithm improves some conditions of above iterative algorithm and extends result of Yan [10] from asymptotically nonexpansive mapping to asymptotically pseudocontractive mapping in a real Hilbert space. Also, our result improves and extends many recent results.

2. Preliminaries

Let C be a nonempty closed convex subset of H.We shall make use of the following Lemmas.

Lemma 2.1 ([11]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0$$

Where $\{\alpha_n\}$ is a sequence in (0,1) and δ_n is a sequence in R such that

1. $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
2. $\lim\limits_{n\to\infty} \sup \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.2 ([3]). *Let T be an asymptotically nonexpansive mapping defined on a nonempty bounded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \to z$ weakly and $Tx_n - x_n \to 0$, then $z \in F(T)$.*

Lemma 2.3 ([9]). *Let $\boldsymbol{E}$ be a reflexive smooth Banach space with a weakly sequential continuous duality mapping J. Let C be a nonempty bounded and closed convex subset of $\boldsymbol{E}$ and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotical pseudocontraction. Then I-T is demiclosed at zero, where I is the identical mapping, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.*

3. Main Result

Theorem 3.1. *Let H be a real Hilbert space .Let C be a nonempty bounded and closed convex subset of H and $T : C \to C$ be a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ such that $F(T) \neq \phi$ and $f : C \to C$ be a contraction with coefficient $\alpha \in (0,1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) \tag{3}$$

Where $\{\alpha_n\}$ is a real sequence in [0,1] satisfying the following conditions:

(i). $\lim\limits_{n\to\infty} \alpha_n = 0,\ \sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(ii). $\lim\limits_{n\to\infty} \frac{k_n}{\alpha_n} = 0$;

(iii). $\sum\limits_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iv). $\sum\limits_{n=0}^{\infty} \sup\limits_{x\in C'} \left|T^{n+1}x - T^n x\right| < \infty$, *where C' is a closed convex subset of C that contains sequence $\{x_n\}$.*

Then $\{x_n\}$ converges strongly to a fixed point q of the asymptotically pseudocontractive mapping T, which is also the solution of the variational inequality $\langle (I-f)\, q, y-q\rangle \geq 0$, for all $y \in F(T)$.

Proof. We divide the proof into five steps:

Step 1: First, we show that $\{x_n\}$ is bounded. Indeed, take $p \in F(T)$ arbitrarily, since $\lim\limits_{n\to\infty} \frac{k_n}{\alpha_n} = 0$, then there exists $N \in \mathbf{N}$ such that for all $n \geq N$, $\frac{k_n}{\alpha_n} \leq \left(\frac{1-\alpha}{2}\right)$. Choose a constant $M_1 > 0$ sufficiently large such that

$$\|x_N - p\| \leq M_1, \|f(p) - p\| \leq \frac{1-\alpha}{2} M_1$$

We proceed by induction to show that $\|x_n - p\| \leq M_1$, $\forall\ n \geq 1$. Assume $\|x_n - p\| \leq M_1$, for some $n \geq N$. We show that $\|x_{n+1} - p\| \leq M_1$. We observe

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\| \\
&= \left\|\alpha_n [f(x_n) - p] + (1-\alpha_n)[T^n\left(\frac{x_n + x_{n+1}}{2}\right) - p]\right\| \\
&\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\left\|T^n\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\| \\
&\leq \alpha_n\alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)L\left\|\left(\frac{x_n + x_{n+1}}{2}\right) - p\right\| \\
&\leq \alpha_n\alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + \frac{L(1-\alpha_n)}{2}\|x_n - p\| + \frac{L(1-\alpha_n)}{2}\|x_{n+1} - p\|
\end{aligned}$$

It follows that

$$\begin{aligned}
\left[1 - \frac{L(1-\alpha_n)}{2}\right]\|x_{n+1} - p\| &\leq \left[\alpha_n\alpha + \frac{L(1-\alpha_n)}{2}\right]\|x_n - p\| + \alpha_n \|f(p) - p\| \\
\|x_{n+1} - p\| &\leq \frac{2\alpha_n\alpha + L(1-\alpha_n)}{2 - L(1-\alpha_n)}\|x_n - p\| + \frac{2\alpha_n}{2 - L(1-\alpha_n)}\|f(p) - p\| \\
&= [1 - \frac{2\alpha_n(1-\alpha) + 2L(1-\alpha_n) - L\alpha_n}{2 - L(1-\alpha_n)}]\|x_n - p\| + \frac{2\alpha_n}{2 - L(1-\alpha_n)}\|f(p) - p\| \\
&\leq \left[1 - \frac{\alpha_n(1-\alpha)}{2 - L(1-\alpha_n)}\right]\|x_n - p\| + \frac{\alpha_n(1-\alpha)}{2 - L(1-\alpha_n)}\frac{\|f(p) - p\|}{1-\alpha} \\
&\leq Max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha}\right\} \\
&\leq M_1 \qquad (4)
\end{aligned}$$

This implies that $\{x_n\}$ is bounded. It turns out that $f(x_n)$, $T^n\left(\frac{x_n+x_{n+1}}{2}\right)$ are also bounded.

Step 2: Next, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. It follows from Equation (3) that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\|\alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) - \alpha_{n-1}f(x_{n-1}) + (1-\alpha_{n-1})T^{n-1}\left(\frac{x_{n-1} + x_n}{2}\right)\right\| \\
&= \Bigg\|\alpha_n\left(f(x_n) - f(x_{n-1})\right) + (\alpha_n - \alpha_{n-1})f(x_{n-1}) + (1-\alpha_n)\left[T^n\left(\frac{x_n + x_{n+1}}{2}\right) - T^n\left(\frac{x_{n-1} + x_n}{2}\right)\right] \\
&+ (1-\alpha_n)T^n\left(\frac{x_{n-1} + x_n}{2}\right) - (1-\alpha_{n-1})T^n\left(\frac{x_{n-1} + x_n}{2}\right) \\
&+ (1-\alpha_{n-1})\left[T^n\left(\frac{x_{n-1} + x_n}{2}\right) - T^{n-1}\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Bigg\| \\
&= \Bigg\|\alpha_n\left(f(x_n) - f(x_{n-1})\right) + (1-\alpha_n)\left[T^n\left(\frac{x_n + x_{n+1}}{2}\right) - T^n\left(\frac{x_{n-1} + x_n}{2}\right)\right] \\
&+ (\alpha_n - \alpha_{n-1})\left[f(x_{n-1}) - T^n\left(\frac{x_{n-1} + x_n}{2}\right)\right] + (1-\alpha_{n-1})\left[T^n\left(\frac{x_{n-1} + x_n}{2}\right) - T^{n-1}\left(\frac{x_{n-1} + x_n}{2}\right)\right]\Bigg\| \\
&= \alpha\alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)L\left\|\frac{x_{n+1} + x_n}{2} - \frac{x_{n-1} + x_n}{2}\right\| + |\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1}) - T^n\left(\frac{x_{n-1} + x_n}{2}\right)\right\| \\
&+ \sup_{x\in c'}\left\|T^n x - T^{n-1}x\right\| \\
&\leq \frac{2\alpha\alpha_n + (1-\alpha_n)L}{2}\|x_n - x_{n-1}\| + \frac{(1-\alpha_n)L}{2}\|x_{n+1} - x_n\| + |\alpha_n - \alpha_{n-1}|M_2 + \sup_{x\in c'}\left\|T^n x - T^{n-1}x\right\|
\end{aligned}$$

where M_2 is a constant such that

$$M_2 = \sup_{n\geq 0}\left\|f(x_{n-1}) - T^n\left(\frac{x_{n-1} + x_n}{2}\right)\right\|$$

It follows that

$$\frac{2 - (1-\alpha_n)L}{2}\|x_{n+1} - x_n\| \leq \frac{2\alpha\alpha_n + (1-\alpha_n)L}{2}\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_2 + \sup_{x\in c'}\left\|T^n x - T^{n-1}x\right\|$$

This implies

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \frac{2\alpha\alpha_n+(1-\alpha_n)L}{2-(1-\alpha_n)L}\|x_n-x_{n-1}\|+\frac{2M_2}{2-(1-\alpha_n)L}|\alpha_n-\alpha_{n-1}|+\frac{2}{2-(1-\alpha_n)L}\sup_{x\in C'}\left\|T^n x-T^{n-1}x\right\| \\ &\le \left(1-\frac{2[1-\alpha\alpha_n-(1-\alpha_n)L}{2-(1-\alpha_n)L}\right)\|x_n-x_{n-1}\|+\frac{2M_2}{2-(1-\alpha_n)L}|\alpha_n-\alpha_{n-1}| \\ &+\frac{2}{2-(1-\alpha_n)L}\sup_{x\in C'}\left\|T^n x-T^{n-1}x\right\| \end{aligned} \tag{5}$$

Let $\gamma_n=2\frac{[1-\alpha\alpha_n-(1-\alpha_n)L]}{2-(1-\alpha_n)L}$. We note

$$\begin{aligned}\gamma_n &= \frac{2\left[(1-\alpha)\alpha_n+(1-\alpha_n)(1-L)\right]}{2-(1-\alpha_n)L} \\ &\ge \frac{2\left[(1-\alpha)\alpha_n+(1-\alpha_n)(1-L)\right]}{2-L} \\ &= \left[(1-\alpha)\alpha_n+\alpha_n L\right] \\ &\ge \alpha_n(1-\alpha)+L \ge \frac{1-\alpha}{2}\alpha_n\end{aligned}$$

By condition (i), we have $\sum_{n=0}^{\infty}\gamma_n=\infty$. Apply Lemma 2.1 to Equation (5), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0 \tag{6}$$

Step 3: Next, we prove that $\lim\limits_{n\to\infty}\|x_n-Tx_n\|=0$. In fact, we have

$$\left\|x_{n+1}-T^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\|=\alpha_n\left\|f(x_n)-T^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\to 0 \ \text{ as } \ \text{n}\to\infty \tag{7}$$

Moreover, we get

$$\begin{aligned}\|x_n-T^n x_n\| &= \left\|x_n-x_{n+1}+x_{n+1}-T^n\left(\frac{x_n+x_{n+1}}{2}\right)+T^n\left(\frac{x_n+x_{n+1}}{2}\right)-T^n x_n\right\| \\ &\le \|x_{n+1}-x_n\|+\left\|x_{n+1}-T^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+\left\|T^n\left(\frac{x_n+x_{n+1}}{2}\right)-T^n x_n\right\| \\ &\le \|x_{n+1}-x_n\|+\left\|x_{n+1}-T^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\|+L\|x_{n+1}-x_n\| \\ &= (L+1))\|x_{n+1}-x_n\|+\left\|x_{n+1}-T^n\left(\frac{x_n+x_{n+1}}{2}\right)\right\|\end{aligned}$$

Combining Equation (6) and Equation (7), we can obtain

$$\lim_{n\to\infty}\|x_n-T^n x_n\|=0$$

We notice

$$\begin{aligned}\|x_n-Tx_n\| &= \left\|x_n-T^n x_n+T^n x_n-T^{n+1}x_n+T^{n+1}x_n-Tx_n\right\| \\ &\le \|x_n-T^n x_n\|+\left\|T^n x_n-T^{n+1}x_n\right\|+L\|T^n x_n-x_n\| \\ &\le \|x_n-T^n x_n\|+\sup_{x\in C'}\left\|T^n x_n-T^{n+1}x_n\right\|+L\|T^n x_n-x_n\|\end{aligned}$$

By condition (iv) and Equation (**??**), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0 \tag{8}$$

Step 4: Next, we claim that

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n\rangle \leq 0 \tag{9}$$

where $q = P_{F(T)}f(q)$. Indeed, there exists a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle q - f(q), j(q - x_n)\rangle = \lim_{i\to\infty} \langle q - f(q), q - x_{ni}\rangle$$

Since $\{x_n\}$ is bounded , there exists a subsequence of $\{x_n\}$ which converges weakly to p. Without loss of generality, we may assume that $x_{ni} \to p$ weakly. From Equation (8) and Lemma 2.2, we have $p \in F(T)$. This together with the property of the metric projection implies that

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n\rangle = \lim_{i\to\infty} \langle q - f(q), q - x_{ni}\rangle = \langle q - f(q), q - p\rangle \leq 0.$$

Then Equation (9) holds.

Step 5: Finally, we show that $x_n \to q$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \left\langle \alpha_n f(x_n) + (1-\alpha_n)T^n\left(\frac{x_n + x_{n+1}}{2}\right) - q, x_{n+1} - q\right\rangle \\
&= \left\langle \alpha_n(f(x_n) - q) + (1-\alpha_n)(T^n\left(\frac{x_n + x_{n+1}}{2}\right) - q), x_{n+1} - q\right\rangle \\
&= \alpha_n \langle f(x_n) - q, x_{n+1} - q\rangle + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle + (1-\alpha_n)\left\langle T^n\left(\frac{x_n + x_{n+1}}{2}\right) - q, x_{n+1} - q\right\rangle \\
&\leq \alpha\alpha_n \|x_n - q\| . \|x_{n+1} - q\| + (1-\alpha_n)L\left\|\frac{x_n - q}{2} + \frac{x_{n+1} - q}{2}\right\| . \|x_{n|+1} - q\| + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle \\
&\leq \frac{\alpha\alpha_n}{2}\|x_n - q\|^2 + \frac{\alpha\alpha_n}{2} . \|x_{n+1} - q\|^2 + \frac{(1-\alpha_n)L}{4}\|x_n - q\|^2 + \frac{(1-\alpha_n)L}{4}\|x_{n+1} - q\|^2 \\
&+ \frac{(1-\alpha_n)L}{2}\|x_{n+1} - q\|^2 + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle
\end{aligned}$$

which implies

$$\frac{4 - 2\alpha\alpha_n - 3(1-\alpha_n)L}{4}\|x_{n+1} - q\|^2 \leq \frac{2\alpha\alpha_n + (1-\alpha_n)L}{4}\|x_n - q\|^2 + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle$$

That is

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \frac{2\alpha\alpha_n + (1-\alpha_n)L}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)L}\|x_n - q\|^2 + \frac{4\alpha_n}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)L}\langle f(q) - q, x_{n+1} - q\rangle \\
&= 1 - \frac{4(\alpha_n L + 1 - \alpha\alpha_n - L)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)L}\|x_n - q\|^2 + \frac{4\alpha_n}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)L}\langle f(q) - q, x_{n+1} - q\rangle
\end{aligned} \tag{10}$$

Put $\gamma_n = \frac{4(\alpha_n L + 1 - \alpha\alpha_n - L)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)L}$, we have

$$\begin{aligned}
\gamma_n &= \frac{4[(\alpha_n - 1)(1 - L) + \alpha_n(1-\alpha)]}{1 - 2\alpha\alpha_n + 3(\alpha_n - 1)L + 3\alpha_n} \\
&\geq \frac{4[(\alpha_n - 1)(1 - L) + \alpha_n(1-\alpha)]}{1 + 3\alpha_n} \\
&\geq (1 - L)(\alpha_n - 1) + \alpha_n(1-\alpha) \\
&\geq \alpha_n(1-\alpha) - L \geq \frac{1-\alpha}{2}\alpha_n
\end{aligned}$$

Apply Lemma 2.1 to Equation (10), we obtain $x_n \to q$ as $n \to \infty$. This completes the proof. □

Since nonexpansive mapping are asymptotically nonexpansive and also asymptotically nonexpansive mapping are asymptotically pseudocontractive ,we obtain the results of Xu [4] and Yan [10]. We obtain the result as:

Theorem 3.2 ([10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$ such that $F(T) \neq \phi$ and $f : C \to C$ a strict contraction with coefficient $\alpha \in [0, 1]$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T^n \left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0$$

Where $\{\alpha_n\}$ is a real sequence in [0,1] satisfying the conditions:

(i). $\lim\limits_{n\to\infty} \alpha_n = 0$; $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;

(ii). $\lim\limits_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;

(iii). $\sum\limits_{n=1}^{\infty} \left|\alpha^{n+1} - \alpha^n\right| < \infty$;

(iv). $\sum\limits_{n=0}^{\infty} \sup\limits_{x\in C'} \left\|T^{n+1}x - T^n x\right\| < \infty$, *where C' is a closed convex subset of C that contains sequence $\{x_n\}$.*

Then $\{x_n\}$ converges strongly to a fixed point q of the asymptotically nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)q, y - q\rangle \geq 0, \quad for\ all\ y \in F(T)$$

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