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Analytical Study of the Generalization Hurwitz - Lerch Zeta Function of Two Variables Using Beta Function

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Abstract

Firstly, the Generalized Hurwitz-Lerch Zeta Function of two variables is defined using Beta Function. Then several integral representations and summation formula are investigated for this function. The function that we have introduced here has also been represented in term of generalised Hypergeometric Function ${}_pF_q$. To strengthen our main results, we have also considered some important special cases.

Keywords: Generalized Hurwitz-Lerch Zeta function; Gamma function; Beta function; Hypergeometric function; Binomial series; Eulerian integral.

1. Introduction

A class of Mathematical Functions that arise in the solution of various classical problems of mathematical physics are termed as Special Functions, for example some Special Functions arise in solving the equation of heat flow or wave propagation in cylindrical co-ordinates, and in many other such physical problems. Special functions have also applications in number theory, for example the Hypergeometric functions are useful in constructing conformal mapping of polygonal regions whose sides are circular areas. In the recent past, some applications are also seen in quantum mechanics and in the angular momentum theory for example Gegenbauer polynomials are used in the developments of four-dimensional spherical harmonics. Zeta function in one of the special functions that is widely used in number theory and is defined as [1]:

$$\Phi(s,a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s};\ \mathbb{R}(s) > 1 \tag{1}$$

For $a = 0$ the zeta function reduces to Riemann Zeta Function [1]:

$$\Phi(s) = \sum_{m=1}^{\infty} \frac{1}{m^s};\ \mathbb{R}(s) > 1 \tag{2}$$

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The Hurwitz-Lerch Zeta Function is defined as [2,3]:

$$\Phi(z,s,a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^s} \tag{3}$$

for all $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $s \in \mathbb{C}$ when $|z| < 1$ and $\mathbb{R}(s) > 1$ when $|z| = 1$. The Generalized Hurwitz-Lerch Zeta Function of two variables is introduced by Pathan and Dawan [4] as:

$$\Phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n} \frac{z^m t^n}{(m+n+a)^s} \tag{4}$$

for all $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$; $s, z, t \in \mathbb{C}$ and $a, \gamma, \gamma' \neq \{0, -1, -2, \dots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}(s + \gamma + \gamma' - \alpha - \alpha' - \beta - \beta') > 0$ when $|z| = 1$ and $|t| = 1$. Hence forth we will denote the Hurwitz-Lerch Zeta Function of two variables by H - L.Z.F. - 2V. Motivated by the above generalization and extensions of Zeta Function, in the present paper, we have further extended the H - L.Z.F. - 2V (given in (4) above) in terms of Beta Function and defined it as:

$$\Phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n B(\beta+m,\gamma-\beta)B(\beta'+n,\gamma'-\beta')}{m!n!B\,(\beta,\gamma-\beta)\,B(\beta',\gamma'-\beta')} \frac{z^m t^n}{(m+n+a)^s} \tag{5}$$

$(\alpha, \alpha', \beta, \beta' \in C; s, z, t \in \mathbb{C})$; $a, \gamma, \gamma' \neq \{0, -1, -2, \dots\}$ when $|z| < 1$ and $|t| < 1$; $\mathbb{R}(s + \gamma + \gamma' - \alpha - \alpha' - \beta - \beta') > 0$ when $|z| = 1$ and $|t| = 1$. Several Integral representations, Summation Formula, Differential Formula are obtained for our function introduced in (5).

2. Preliminaries

In this next section we mention some of the known formulae and results which we need in the proofs of our main results.

Definition 2.1 ([5]). *The Eulerian Integral is given as*

$$\frac{1}{(m+n+a)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(m+n+a)t} dt \quad (\min \ \mathbb{R}(a) > 0, m+n \in N_0) \tag{6}$$

Definition 2.2. *The integral representation of the Pochhammer symbol* $(\alpha)_m$ *is defined as:*

$$(\alpha)_m = \frac{1}{\Gamma\alpha} \int_0^{\infty} t^{\alpha+m-1} e^{-t} dt \tag{7}$$

Definition 2.3. *The Binomial Series*

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^{n-k} b^k \tag{8}$$

where, $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$.

Definition 2.4. *The following identity*

$$B(\beta, \gamma - \beta) = \frac{\gamma}{\beta} B(\beta + 1, \gamma - \beta) \tag{9}$$

$$(\alpha)_{m+1} = \alpha(\alpha)_m \tag{10}$$

Definition 2.5. *The generalized hypergeometric function $F(-)$ is given as [5]:*

$$pFq\left[\left(\beta_1, \ldots, \beta_p; \delta_1, \ldots, \delta_q\right); z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\beta_i)_n z^n}{\prod_{j=1}^{4} \left(\beta_j\right)_n n!} \tag{11}$$

where $p, q \in Z^+; b_j \neq 0, -1, -2\ldots.$

Definition 2.6 ([6]). *The following identity*

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A(l, k) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} A(l, k - l) \tag{12}$$

3. Main Results

3.1 Integral Representations

Theorem 3.1. *For*

(i) $\mathbb{R}(s) > 0$, $\mathbb{R}(a) > 0$ *provided* $|z| \leq 1$ *and* $|t| \leq 1$;

(ii) $\mathbb{R}(s) > 1$, *provided* $z = 1$ *and* $t = 1$ *then, the following integral representation for* $\phi_{\alpha,\alpha',\beta,\beta';,}\gamma'(z,t,s,a)$ *holds true:*

$$\begin{aligned} \phi_(\alpha, \alpha', \beta, \beta'; \gamma, \gamma')(z, t, s, a) = & \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} \\ & \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta + m, \gamma - \beta) B(\beta' + n, \gamma' - \beta')}{m! n! B(\beta, \gamma - \beta) B(\beta', \gamma' - \beta')} \\ & z^m t^n \left(e^{-x}\right)^m \left(e^{-x}\right)^n dx \end{aligned} \tag{13}$$

Proof. Using Eulerian Integral given in (6) on right - hand side of (5), we get:

$$\begin{aligned} \phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z, t, s, a) = & \sum_{m,n=0}^{\infty} \frac{(\alpha)_m \left(\alpha'\right)_n B(\beta + m, \gamma - \beta) B\left(\beta' + n, \gamma' - \beta'\right)}{m! n! B(\beta, \gamma - \beta) B\left(\beta', \gamma' - \beta'\right)} \\ & z^m t^n \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-(m+n+a)x} dx \end{aligned} \tag{14}$$

Interchanging the order of integration and summation on the right - hand side of (14), we get:

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta+m,\gamma-\beta) B(\beta'+n,\gamma'-\beta')}{m!n! B(\beta,\gamma-\beta) B(\beta',\gamma'-\beta')}$$
$$e^{-(m+n+a)x} z^m t^n dx \tag{15}$$

or,

$$\phi_(\alpha,\alpha',\beta,\beta';\gamma,\gamma')(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta+m,\gamma-\beta) B(\beta'+n,\gamma'-\beta')}{m!n! B(\beta,\gamma-\beta) B(\beta',\gamma'-\beta')}$$
$$z^m t^n \left(e^{-x}\right)^m \left(e^{-x}\right)^n dx \tag{16}$$

which is the desired result, that is (13), which is wanted to prove. □

Corollary 3.2. *When $\beta = \gamma$ and $\beta' = \gamma'$ in* (13), *we get:*

$$\phi_{\alpha,\alpha'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n}{m!n!} z^m t^n \left(e^{-x}\right)^m \left(e^{-x}\right)^n dx \tag{17}$$

(i) $\mathbb{R}(a) > 0, \mathbb{R}(s) > 0$ *provided* $|z| \leq 1, |t| \leq 1$

(ii) $\mathbb{R}(s) > 1$ *provided* $z = 1, t = 1$

Theorem 3.3. *For*

(i) $\mathbb{R}(s) > 0$, $\mathbb{R}(a) > 0$ *provided* $|z| \leq 1$ *and* $|t| \leq 1$;

(ii) $\mathbb{R}(s) > 1$, *provided* $z = 1$ *and* $t = 1$ *then,*

the following integral representation for $\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ *holds true:*

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma\alpha} \frac{1}{\Gamma\alpha'} \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\alpha'-1} e^{-(x+y)} \phi_{\beta,\beta';\gamma,\gamma'}(zx,ty,s,a) dx dy \tag{18}$$

Proof. Using (7) for $(\alpha)_m$ and $(\alpha')_n$ in right -hand side of (5), we get:

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{B(\beta+m,\gamma-\beta) B(\beta'+n,\gamma'-\beta')}{m!n! B(\beta,\gamma-\beta) B(\beta',\gamma'-\beta')} \frac{z^m t^n}{(m+n+a)^s}$$
$$\frac{1}{\Gamma\alpha} \int_0^\infty x^{\alpha+m-1} e^{-x} dx \frac{1}{\Gamma\alpha'} \int_0^\infty y^{\alpha'+n-1} e^{-y} dy \tag{19}$$

On interchanging the signs of integration and summation on the right - hand side of the last equation we get:

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma\alpha} \frac{1}{\Gamma\alpha'} \int_0^\infty \int_0^\infty x^{\alpha+m-1} e^{-x} \sum_{m=0}^{\infty} \frac{B(\beta+m,\gamma-\beta)}{B(\beta,\gamma-\beta)} y^{\alpha'+n-1} e^{-y}$$
$$\sum_{n=0}^{\infty} \frac{B(\beta'+n,\gamma'-\beta')}{B(\beta',\gamma'-\beta')} \frac{1}{m!n!} \frac{z^m t^n}{(m+n+a)^s} dx dy \tag{20}$$

or,

$$= \frac{1}{\Gamma\alpha}\frac{1}{\Gamma\alpha'}\int_0^\infty\int_0^\infty x^{\alpha-1}e^{-x}\sum_{m=0}^{\infty}\frac{B(\beta+m,\gamma-\beta)}{B(\beta,\gamma-\beta)}y^{\alpha'-1}e^{-y}\sum_{n=0}^{\infty}\frac{B\left(\beta'+n,\gamma'-\beta'\right)}{B\left(\beta',\gamma'-\beta'\right)}\frac{1}{m!n!}\frac{(zx)^m(ty)^n}{(m+n+a)^s}dxdy \tag{21}$$

or,

$$= \frac{1}{\Gamma\alpha}\frac{1}{\Gamma\alpha'}\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\alpha'-1}e^{-(x+y)}\sum_{m=0}^{\infty}\frac{B(\beta+m,\gamma-\beta)}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{B\left(\beta'+n,\gamma'-\beta'\right)}{B\left(\beta',\gamma'-\beta'\right)}\frac{1}{m!n!}\frac{(zx)^m(ty)^n}{(m+n+a)^s}dxdy \tag{22}$$

or,

$$= \frac{1}{\Gamma\alpha}\frac{1}{\Gamma\alpha'}\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\alpha'-1}e^{-(x+y)}\sum_{m=0,n=0}^{\infty}\frac{B(\beta+m,\gamma-\beta)B\left(\beta'+n,\gamma'-\beta'\right)}{B(\beta,\gamma-\beta)B\left(\beta',\gamma'-\beta'\right)}\frac{1}{m!n!}\frac{(zx)^m(ty)^n}{(m+n+a)^s}dxdy \tag{23}$$

On comparison with (5):

$$\phi_{(\alpha,\alpha',\beta,\beta';\gamma,\gamma')}(z,t,s,a) = \frac{1}{\Gamma\alpha}\frac{1}{\Gamma\alpha'}\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\alpha'-1}e^{-(x+y)}\phi_{\beta,\beta';\gamma,\gamma'}(zx,ty,s,a)dxdy \tag{24}$$

And hence we get the desired result i.e., (18), which is wanted to prove. □

3.2 Summation Formula

Theorem 3.4. *For*

(i) $R(s) > 0, \mathbb{R}(a) > 0$ *provided* $|\mathbf{z}| \leq 1$ *and* $|t| \leq 1$;

(ii) $R(s) > 1$, *provided* $z = 1$ *and* $t = 1$ *and*

(iii) $(|x| < |a|; s \neq 1)$ *then,*

the following summation formula for $\phi_{\alpha,\alpha',\beta,\beta';;}, \gamma'(z,t,s,a)$ *holds true:*

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a-x) = \sum_{p=0}^{\infty}\frac{(s)_p}{p!}\left\{\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s+p,a)\right\}X^p \tag{25}$$

Proof. Applying (5) on left-hand side of (25) we get:

$$\text{Left - hand side of (25)} = \sum_{m,n=0}^{\infty}\frac{(\alpha)_m\left(\alpha'\right)_n B(\beta+m,\gamma-\beta)B\left(\beta'+n,\gamma'-\beta'\right)}{m!n!B(\beta,\gamma-\beta)B\left(\beta',\gamma'-\beta'\right)}\frac{z^m t^n}{(m+n+a-x)^s} \tag{26}$$

Or,

$$= \sum_{m,n=0}^{\infty}\frac{(\alpha)_m\left(\alpha'\right)_n B(\beta+m,\gamma-\beta)B\left(\beta'+n,\gamma'-\beta'\right)}{m!n!}\frac{z^m t^n}{(m+n+a)^s}\left(1-\frac{x}{m+n+a}\right)^{-s} \tag{27}$$

Using (8) in above (27) we get:

$$= \sum_{p=0}^{\infty} \frac{(s)_p}{p!} \left\{ \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta+m, \gamma-\beta) B(\beta'+n, \gamma'-\beta')}{m!n!B(\beta, \gamma-\beta)} \frac{z^m t^n}{(m(\beta', \gamma'-\beta')} \right\} X^p \tag{28}$$

On comparison with (5) we get:

$$= \sum_{p=0}^{\infty} \frac{(s)_p}{p!} \left\{ \phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s+p,a) \right\} X^p \tag{29}$$

$$= \text{Right } - \text{hand side of (25)}$$

Hence theorem (3) is proved. □

3.3 Differential Formula

Theorem 3.5. *The following differential formula for $\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ hold true:*

$$\frac{\delta^2}{\delta z \delta t} \left(\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) \right) = \frac{\alpha\beta}{\gamma} \frac{\alpha'\beta'}{\gamma'} \phi_{\alpha+1,\alpha'+1,\beta+1,\beta'+1;\gamma+1,\gamma'+1}(z,t,s,a+2) \tag{30}$$

Proof. Partial derivative of (5) with respect to t yields:

$$\frac{\delta}{\delta t} \left(\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) \right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta+m, \gamma-\beta) B(\beta'+n, \gamma'-\beta')}{m!n!n(\beta, \gamma-\beta) \quad B(\beta', \gamma'-\beta')} \frac{t^{n-1}}{(m+n+a)^s} \tag{31}$$

And again, partial derivative of above (31) with respect to z yields:

$$\frac{\delta^2}{\delta z \delta t} \left(\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) \right) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta+m, \gamma-\beta) B(\beta'+n, \gamma'-\beta')}{m!n!B(\beta, \gamma-\beta) B(\beta', \gamma'-\beta')} \frac{mnz^{m-1}t^{n-1}}{(m+n+a)^s} \tag{32}$$

or,

$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n B(\beta+m, \gamma-\beta) B(\beta'+n, \gamma'-\beta')}{(m-1)!(n-1)! \quad B(\beta, \gamma-\beta) B(\beta', \gamma'-\beta')} \frac{z^{m-1}t^{n-1}}{(m+n+a)^s} \tag{33}$$

Replacing m by $(m+1)$ and n by $(n+1)$ on the right - hand side (33) we get:

$$\frac{\delta^2}{\delta z \delta t} \left(\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) \right)$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+1} (\alpha')_{n+1} B(\beta+m+1, \gamma-\beta) B(\beta'+n+1, \gamma'-\beta')}{m!n!B(\beta, \gamma-\beta) \quad B(\beta', \gamma'-\beta')} \frac{z^m t^n}{(m+n+a+2)^s} \tag{34}$$

Using the identity (9) and (10) on (34):

$$\frac{\delta^2}{\delta z \delta t} (\phi_(\alpha, \alpha', \beta, \beta'; \gamma, \gamma')(z,t,s,a))$$
$$= \sum_{m,n=0}^{\infty} \frac{\alpha\alpha\alpha'(\beta(\alpha+1)_m(\alpha'+1))_n B(\beta+1+m, \gamma-\beta) B(\beta'+1+n, \gamma'-\beta')}{\gamma\gamma' m!n!B(\beta+1, \gamma-\beta) B(\beta'+1, \gamma'-\beta')} X \frac{z^m t^n}{(m+n+a+2)^s}$$

On comparison with (5) we get:

$$\frac{\delta^2}{\delta z \delta t}\left(\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a)\right) = \frac{\alpha\beta}{\gamma}\frac{\alpha'\beta'}{\gamma'}\phi_{\alpha+1,\alpha'+1,\beta+1,\beta'+1;\gamma+1,\gamma'+1}(z,t,s,a+2) \tag{35}$$

Hence Theorem (4) is proved. □

3.4 Representation in terms Generalized Hypergeometric Function

Theorem 3.6. *For $a \neq \{-1,-2,\dots\}$ and $z \neq 0$, the following explicit series representation hold true:*

$$\begin{aligned}\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m=0}^{\infty} & \frac{(\alpha)_m B(\beta+m,\gamma-\beta)}{m! B(\beta,\gamma-\beta)}\frac{z^m}{(m+a)^s} \\ & XF\left\{(-m),(1-\gamma-m),(\beta');(1-\alpha-m),(1-\beta-m),(\gamma');\frac{t}{z'}\right\}\end{aligned} \tag{36}$$

where $F(-)$ is the Generalized Hypergeometric Function ${}_pF_q$ defined in (11).

Proof. Using (12) on right -hand side of (4) we get:

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(\alpha)_{m-n}(\alpha')_n(\beta)_{m-n}(\beta')_n}{(m-n)!n!(\gamma)_{m-n}(\gamma')_n}\frac{z^{m-n}t^n}{(m+a)^s} \tag{37}$$

Since:

$$(m-n)! = \frac{(-1)^n}{(-m)_n}m!, \qquad 0 \le n \le m \tag{A}$$

$$(\alpha)_{m-n} = \frac{(-1)^n(\alpha)_m}{(1-\alpha-m)_n}, \quad 0 \le n \le m \tag{B}$$

$$(\beta)_{m-n} = \frac{(-1)^n(\beta)_m}{(1-\beta-m)_n}, \quad 0 \le n \le m \tag{C}$$

$$(\gamma)_{m-n} = \frac{(-1)^n(\gamma)_m}{(1-\gamma-m)_n}, \quad 0 \le n \le m \tag{D}$$

Therefore in light of (A), (B), (C) and (D), (38) reduces To

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(-1)^n(\alpha)_m(-m)_n(\alpha')_n(-1)^n(\beta)_m(1-\gamma-m)_n(\beta')_n}{(1-\alpha-m)_n(-1)^n m!n!(1-\beta-m)_n(-1)^n(\gamma)_m(\gamma')_n}\frac{z^{m-n}t^n}{(m+a)^s} \tag{38}$$

or,

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(\alpha)_m(\alpha')_n(-m)_n(1-\gamma-m)_n(\beta)_m(\beta')_n}{m!n!(1-\alpha-m)_n(1-\beta-m)_n(\gamma)_m(\gamma')_n}\frac{z^{m-nt^n}}{(m+a)^s} \tag{39}$$

or,

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(\alpha)_m(\alpha')_n B(\beta+m,\gamma-\beta)B(\beta'+n,\gamma'-\beta')B(1-\gamma-m+n,\gamma-\beta)B(-m+n,1-\alpha)}{m!n!B(\beta,\gamma-\beta)B(\beta',\gamma'-\beta')B(1-\gamma-m,\gamma-\beta)B(-m,1-\alpha)}\frac{z^m(t/z)^n}{(m+a)^s} \tag{40}$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m B(\beta+m,\gamma-\beta)}{m! B(\beta,\gamma-\beta)} \frac{z^m}{(m+a)^s} XF\left\{(-m),(1-\gamma-m),(\beta');(1-\alpha-m),(1-\beta-m),(\gamma');\frac{t}{z'}\right\} \tag{41}$$

where $F(-)$ is the generalized Hypergeometric Function ${}_pF_q$ defined in (11).

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \text{Right} - \text{hand side of (37)}$$

Hence Theorem (5) is proved. □

Corollary 3.7. *If we set $\beta = \gamma$ in (37)*

$$\phi_{\alpha,\alpha',\gamma,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} \frac{z^m}{(m+a)^s} XF\left\{(-m),(\beta');(1-\alpha-m),(\gamma');\frac{t}{z'}\right\} \tag{42}$$

Corollary 3.8. *If we set $\beta = \gamma$, $\beta' = \gamma'$ in (37)*

$$\phi_{\alpha,\alpha',\gamma,\gamma';\gamma,\gamma'}(z,t,s,a) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} \frac{z^m}{(m+a)^s} \quad X \quad F\left\{(-m);(1-\alpha-m);\frac{t}{z}\right\} \tag{43}$$

4. Special Cases

Case 1:

If we put $\alpha = \alpha' = 1$ in (5) we obtained

$$\phi_{1,1,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{B(\beta+m,\gamma-\beta) B(\beta'+n,\gamma'-\beta')}{B(\beta,\gamma-\beta)(\beta',\gamma'-\beta')} \frac{z^m t^n}{(m+n+a)^s} \tag{44}$$

$(\beta, \beta' \in \mathbb{C}; s, z, t \in \mathbb{C})$ and $a, \gamma, \gamma' \neq \{0,-1,-2,\ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}(s+\gamma+\gamma'-\beta-\beta') > 0$ when $|z| = 1$ and $|t| = 1$.

Case 2:

If we put $\beta = \gamma, \beta' = \gamma'$ in (5) we get the Generalized Hurwitz - Lerch Zeta Function of [14]:

$$\phi_{\alpha,\alpha'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n}{m! n!} \frac{z^m t^n}{(m+n+a)^s} \tag{45}$$

$(\alpha, \alpha' \in \mathbb{C}; s, z, t \in \mathbb{C})$ and $a \neq \{0,-1,-2,\ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}(s-\alpha-\alpha') > 0$ when $|z| = 1$ and $|t| = 1$.

Case 3:

If we put $\alpha = \gamma = 1, \alpha' = \gamma' = 1$ in (5) we get:

$$\phi_{1,1,\beta,\beta';1,1}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{B(\beta+m, 1-\beta)B\left(\beta'+n, 1-\beta'\right)}{B(\beta, 1-\beta) \quad B\left(\beta', 1-\beta'\right)} \frac{z^m t^n}{(m+n+a)^s} \tag{46}$$

$(\beta, \beta' \in \mathbb{C}; s, z, t \in \mathbb{C})$ and $a \neq \{0, -1, -2, \ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}\left(s + \beta - \beta'\right) > 0$ when $|z| = 1$ and $|t| = 1$.

Case 4:

If we put $\alpha = \alpha' = 1, \beta = \gamma, \beta' = \gamma'$ in (5) we get:

$$\phi_{\alpha,\alpha'}(z,t,s,a) = \sum_{m,n=0}^{\infty} \frac{z^m t^n}{(m+n+a)^s} \tag{47}$$

$(\alpha, \alpha' \in \mathbb{C}; s, z, t \in \mathbb{C})$ and $a \neq \{0, -1, -2, \ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}\left(s - \alpha - \alpha'\right) > 0$ when $|z| = 1$ and $|t| = 1$.

Case 5:

If $\beta \to \infty$ in (5) we get:

$$\begin{aligned}
\phi_{\alpha,\alpha',\beta';\gamma,\gamma'}(z,t,s,a) &= \lim_{\beta \to \infty} \left\{ \phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'} \left(\frac{z}{\beta}, t, s, a \right) \right\} \\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m \left(\alpha'\right)_n \left(\beta'\right)_n}{m!n!(\gamma)_m \left(\gamma'\right)_n} \frac{z^m t^n}{(m+n+a)^s} \\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m \left(\alpha'\right)_n}{m!n!(\gamma)_m} \frac{B\left(\beta'+n, \gamma'-\beta'\right)}{B\left(\beta', \gamma'-\beta'\right)} \frac{z^m t^n}{(m+n+a)^s} \\
&= \sum_{m,n=0}^{\infty} \frac{\left(\alpha'\right)_n}{m!n!} \frac{B(\alpha+m, \gamma-\alpha)}{B(\alpha, \gamma-\alpha)} \frac{B\left(\beta'+n, \gamma'-\beta'\right)}{B\left(\beta', \gamma'-\beta'\right)} \frac{z^{m^n}}{(m+n+a)^s}
\end{aligned} \tag{48}$$

$(\alpha, \alpha', \beta' \in \mathbb{C}; s, z, t \in \mathbb{C})$ and $a, \gamma, \gamma' \neq \{0, -1, -2, \ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}\left(s + \gamma + \gamma' - \alpha - \alpha' - \beta'\right) > 0$ when $|z| = 1$ and $|t| = 1$.

Case 6:

If $\beta \to \infty, \beta' \to \infty$ in (5) we get:

$$\begin{aligned}
\phi_{\alpha,\alpha';\gamma,\gamma'}(z,t,s,a) &= \lim_{\beta,\beta' \to \infty} \left\{ \phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'} \left(\frac{z}{\beta'}, \frac{t}{\beta'}, s, a \right) \right\} \\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m \left(\alpha'\right)_n}{m!n!(\gamma)_m \left(\gamma'\right)_n} \frac{z^m t^n}{(m+n+a)^s}
\end{aligned}$$

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{B(\alpha+m,\gamma-\alpha)}{B(\alpha,\gamma-\alpha)} \frac{B(\alpha'+n,\gamma'-\alpha')}{B(\alpha',\gamma'-\alpha')} \frac{z^{m^n}}{(m+n+a)^n} \tag{49}$$

$(\alpha,\alpha' \in \mathbb{C}; s,z,t \in \mathbb{C})$ and $a,\gamma,\gamma' \neq \{0,-1,-2,\ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}(s+\gamma+\gamma'-\alpha-\alpha') > 0$ when $|z| = 1$ and $|t| = 1$.

Case 7:

If $\alpha \to \infty, \alpha' \to \infty$ in (5) we get:

$$\begin{aligned}
\phi_{\beta,\beta';\gamma,\gamma'}(z,t,s,a) &= \lim_{\alpha,\alpha' \to \infty} \left\{ \phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}\left(\frac{z}{\alpha},\frac{t}{\alpha'},s,a\right)\right\} \\
&= \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_n}{m!n!(\gamma)_m (\gamma')_n} \frac{z^{m_1} t^n}{(m+n+a)^s} \\
&= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{B(\beta+m,\gamma-\beta)}{B(\beta,\gamma-\beta)} \frac{B(\beta'+n,\gamma'-\beta')}{B(\beta',\gamma'-\beta')} \frac{z^m t^n}{(m+n+a)^s}
\end{aligned} \tag{50}$$

$(\beta,\beta' \in \mathbb{C}; s,z,t \in \mathbb{C})$ and $a,\gamma,\gamma' \neq \{0,-1,-2,\ldots\}$ when $|z| < 1$ and $|t| < 1$ and $\mathbb{R}(s+\gamma+\gamma'-\beta-\beta') > 0$ when $|z| = 1$ and $|t| = 1$.

5. Conclusion

We have introduced Hurwitz-Lerch Zeta Function of two variables in terms of Beta Function and thereafter we have obtained two different types of Integral representations for this function. It was interesting to see that corollary of one of our theorems (Integral representation) gives the following result:

$$\phi_{\alpha,\alpha'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax} \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n}{m!n!} z^m t^n \left(e^{-x}\right)^m \left(e^{-x}\right)^n dx$$

(i) $\mathbb{R}(a) > 0, \mathbb{R}(s) > 0$ provided $|z| \leq 1$, $|t| \leq 1$

(ii) $\mathbb{R}(s) > 1$ provided $z = 1$, $t = 1$

In the course of our above study, we have also obtained one Summation Formula and one Differential Formula for $\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a)$:
Summation Formula:

$$\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s,a-x) = \sum_{p=0}^{\infty} \frac{(s)_p}{p!} \left\{\phi_{\alpha,\alpha',\beta,\beta';\gamma,\gamma'}(z,t,s+p,a)\right\} X^p$$

(i) $\mathbb{R}(a) > 0, \mathbb{R}(s) > 0$ provided $|z| \leq 1$, $|t| \leq 1$

(ii) $\mathbb{R}(s) > 1$ provided $z = 1, t = 1$

(iii) $(|x| < |a|; s \neq 1)$

References

[1] H. M. Srivastava, *A New Family of the λ Generalized Hurwitz-Lerch Zeta Function with applications*, Applied Mathematics & Information Sciences 8(4)(2014), 1485-1500.

[2] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Springe, (2001).

[3] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, (2012).

[4] H. M. Srivastava, R. K. Saxena, T. K. Pogany and R. Saxena, *Integral and computational representations of the extended Hurwitz-Lerch Zeta function*, Integral Transforms and Special Functions, 22(7)(2011), 487-506.

[5] Kottakkaran Sooppy Nisar, *Further Extension of the Generalized Hurwitz-Lerch Zeta Function of Two Variables*, Mathematics, 7(1)(2019).

[6] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, (1960).