Int. J. Math. And Appl., 8(2)(2020), 1–5
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Fixed Point of Pseudo Contractive Mapping in a Banach Space

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Abstract: Let X be a Banach space, B a closed ball centred at origin in X, $f : B \to X$ a pseudo contractive mapping i.e. $(\alpha-1)\|x-y\| \leq \|(\alpha I-f)(x)-(\alpha I-f)(y)\|$ for all x and y in B and $\alpha > 1$. Here we shown that Mapping f satisfies the property that $f(x) = -f(-x)\ \ \forall\ \ x$ in ∂B called antipodal boundary condition assures existence of fixed point of f in B provided that ball B has a fixed point property with respect to non expansive self mapping. Also included some fixed point theorems which involve the Leray-Schauder condition.

Keywords: Fixed point, Banach space, Non expansive mapping, Pseudo Contractive Mapping, Cauchy Sequence, Lipschitzian Mapping.

1. Introduction

Let X be a real Banach space and D be a subset of X. An operator $f : D \to X$ is said to be k-pseudo contractive $(k > 0)$ if for each x and y in D and $\alpha > k$

$$(\alpha-k)\|x-y\| \leq \|(\alpha I-f)(x)-\|(\alpha I-f)(y)\| \quad for \quad k \leq 1$$

such operator is called strongly pseudo contractive. In addition to generalizing the non-expansive mappings. The pseudo-contractive mappings are characterized by the important fact that a mapping $f : D \to X$ is pseudo-contractive if and only if the mapping $T = I - f$ is accretive on D. It is well known that if D is a bounded open convex subset of a uniformly convex Banach space X and if f is a non-expansive mapping defined from the closure $\overline{D}$ of D into X, then the Leray-Schauder boundary condition which asserts that for $z \in D$, $(L-S)T(x) - z \neq k(x-z)$ for all $x \in \partial D$, $k > 1$ is sufficient of guarantee existence of a fixed point for T. Our main objective here is to study the question mentioned above under two different boundary conditions apparently stronger than $(L-S)$.

2. Preliminaries

This section is devoted to some basic definitions, prepositions and lemmas which are needed for the further study of this Article.

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Definition 2.1. *Let X be a vector space over the field F. Then a semi-norm on X is a function $\|\| : X \to R$ such that*

(a). $\|x\| \geq 0$ *for all* $x \in X$,

(b). $\|\alpha x\| = |\alpha|\|x\|$ *for all* $x \in X$ *and* $\alpha \in F$,

(c). Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$. *A norm on X is a semi-norm which also satisfies:*

(d). $\|x\| = 0 \Rightarrow x = 0$. *A vector space X together with a norm $\|\|$ is called a normed linear space, a normed vector space, or simply a normed space*

Definition 2.2. *A normed space X is called a Banach space if it is complete, i.e., if every Cauchy sequence is convergent. That is, $\{f_n\}$, $n \in N$ is Cauchy in X $\exists$ $f \in X$ such that $f_n \to f$.*

Definition 2.3. *Let X be a Complete Metric Space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $k \in (0, 1)$ such that $d(T(x), T(y)) \leq kd(xy)$ for all x and y in X.*

Definition 2.4. *Let E be a Banach space, X a subset of E, and f a mapping of X into E. Then f is said non expansive if for all $x, y \in X$, $\|f(x) - f(x)\| \leq \|x - y\|$.*

Definition 2.5. *Let (M, ρ) be a metric space and let $T : M \to M$ be a mapping. We say that T satisfies a Lipschitz condition with constant $k \geq 0$ if for all $x, y \in M$, $\rho(Tx, Ty) \leq k\rho(x, y)$ then T is called Lipschitzian mapping and k is called Lipshhitzian constant.*

Proposition 2.6. *Let X be a Banach space, D be a open subset of X and T be a strongly pseudo contractive mapping on $\overline{D}$ in to X. If there exist $z \in D$ such that $T(x) - z \neq \lambda(x - z)$ for $x \in D$, $\lambda > 1$ then T has a fixed point in $\overline{D}$.*

Proposition 2.7. *Let X be a uniformly convex Banach space and B a closed ball in X centred at the origin. Let $f : B \to X$ be a continuous pseudo contractive mapping satisfying $f(x) = -f(-x)$. Then T has a fixed point in B.*

Lemma 2.8. *Let X be a Banach space for each non-empty bounded closed convex subset has a fixed point property for non-expansive self-mappings and let $f : D \subset X \to X$ a mapping such that $I - T$ is m-accretive. Then T has a fixed point in D if and only if there exists x_0 in D and a bounded open neighbourhood U of x_0 such that $f(x) - x_0 \neq \lambda(x - x_0)$ for $x \in \partial D \cap D$ and $\lambda > I$.*

Lemma 2.9. *Let X be a Banach space and D a subset of X with $0 \in D$. Suppose $T : D \to X$ is a mapping such that $I - T$ is m-accretive. Then there exists a mapping $\Phi : (1, \infty) \to D$ define by $\Phi(\lambda) = x_\lambda$ where $T(x_\lambda) = \lambda x_\lambda$.*

3. Main Results

Our main results are as follows.

Theorem 3.1. *Let X be a Banach space and B a closed ball centred at the origin in X, and suppose B has the fixed point property with respect to non-expansive self-mappings. Suppose $f : B \to X$ is a continuous pseudo contractive mappings, which satisfies $f(x) = -f(-x)$ for all x in ∂B Then f has a fixed point in B.*

Proof. Since mapping f is pseudo contractive therefore for each x in ∂B and for each $\lambda > 1$.

$$2(\lambda - 1)\|x\| = (\lambda - 1)\|2x\|$$

$$
\begin{aligned}
&= (\lambda - 1)\|x - (-x)\| \\
&\leq \|(\lambda I - f)(x) - (\lambda I - f)(-x)\| \\
&= \|\lambda x - fx - \lambda(-x) + f(-x)\| \\
&= \|\lambda x - fx + \lambda x - fx\| \quad \because \quad f(-X) = -f(x) \\
&= \|2(\lambda x - fx)\|
\end{aligned}
$$

Thus

$$(\lambda - 1)\|x\| \leq \|(\lambda x - fx)\| \tag{1}$$

Let us define $f_r(x) = rf(x) + (1-r)y$ for $r \in (0,1)$ and $y \in B$ and if $f_r(x) = ax$ for $x \in \partial B$ and $a > 1$ then by (1) we have

$$ax = rf(x) + (1-r)y \Longrightarrow rf(x) = ax - (1-r)y \Longrightarrow f(x) = r^{-1}(ax) - r^{-1}(1-r)y$$

Putting this value in (1) we get

$$(\lambda - 1)\|x\| \leq \|\lambda x = r^{-1}(ax) - r^{-1}(1-r)y\| \tag{2}$$

$$= \|(\lambda - r^{-1}a)x + r^{-1}(1-r)y\| \tag{3}$$

Choose $\lambda = r^{-1}a$ then from (3) we get

$$
\begin{aligned}
&(\lambda - 1)\|x\| \leq \|(\lambda - \lambda)x + r^{-1}(1-r)y\| \\
&\Longrightarrow \|x\| \leq \frac{(I-r)}{r(\lambda - 1)}\|y\| \\
&\Longrightarrow \|x\| \leq \frac{(1-r)}{(a-r)}\|y\| \\
&\Longrightarrow \|x\| \leq \|y\| \quad \because \quad (1-r) \leq (a-r)
\end{aligned}
$$

This contradicts $x \in \partial B$. Therefore f satisfies condition $(L-S)$ on ∂B. Since f_r is strongly pseudo contractive therefore f_r have fixed point B i.e., $f(x) = x$. Now $f_r(x) = x \Longrightarrow rf(x) + (1-r)y = x \Longrightarrow (I - rf)(x) = (1-r)y \Longrightarrow (I - rf)$ maps B in to $(1-r)B$. Since f is pseudo contractive therefore

$$
\begin{aligned}
&(r^{-1} - 1)\|u - v\| \leq \|(r^{-1}I - f)u - (r^{-1}I - f)\| \\
&\Longrightarrow (1-r)\|u - v\| \leq \|(I - rf)u - (I - rf)v\|
\end{aligned}
$$

$\Longrightarrow (1-r)(I - rf)^{-1}$ non expansive which has a fixed point in $(1-r)B \Longrightarrow \ \exists \ z \in B$ such that

$$
\begin{aligned}
&(1-r)(I - rf)^{-1}(1-r)z = (1-r)z \\
&\Longrightarrow (I - rf)^{-1}(1-r)z = z \\
&\Longrightarrow (1-r)z = (I - rf)z \\
&\Longrightarrow z - rz = Iz - rfz \\
&\Longrightarrow f(z) = z
\end{aligned}
$$

$\Longrightarrow$ f has a fixed point in B. $\square$

Theorem 3.2. *Let X be a Banach space and $B_z(r)$ a closed ball which satisfies fixed point Property with respect to non expansive mapping. Suppose $f : B \to X$ is a pseudo contractive mapping and suppose that $\exists \quad j \in J(x-z)$ for which $((x - fx), j) \geq 0$ for $x \in \partial B$ then f has a fixed point in B.*

Proof. Let us suppose that f does not satisfy Leray-Schauder boundary condition define the mapping $f : B \to X$ such that $f_r(x) = rf(x) + (1-r)y$ for some $r \in (0,1)$ and $y \in B$, let $x \in B$ and $f_r(x) = ax$ then

$$\begin{aligned} a(x,j) &= (f_r(x), j) \\ &= (rfx + (1-r)y, j) \\ &= (rfx, j) + ((1-r)y, j) \\ &\leq r(x,j) + (1-r)(x,j) \quad \because \quad \|x\| \leq \|y\| \\ &= (r + 1 - r)(x,j) \\ &= (x,j) \end{aligned}$$

which implies that $a \leq 1$ which is not true because $a > 1$. Thus f satisfies Leray-Schauder boundary condition, consequently by Theorem 3.1, f has a fixed point. □

Theorem 3.3. *Let X be a reflexive Banach space and D an open convex subset of X with $0 \in D$ Let f be a generalized contractive mapping of $\overline{D}$ into X such that for each x in the boundary of D, $f(x) \neq \lambda x$ if $\lambda > 1$. Then tf has a fixed point in $\overline{D}$.*

Proof. Let $f(x) = \lambda x$ for $\lambda > 1$ and consider the set $M = \{x \in D : f(x) = \lambda x \text{ for some } \lambda > 1\}$. Now $\|\lambda x - f(0)\| \leq a(0)\|x\| \implies \|x\| \leq \frac{\|f(0)\|}{1-a(0)}$, where $a(0) < 1 \implies M$ is a bounded set for $t \in (0,1)$ the mapping tf satisfies Leray-Schauder boundary condition on ∂D therefore tf has a fixed point in D i.e., $tf(x) = x$ for some $x \in D$. Choose now $f(x_n) = \lambda_n x_n$ with $\lambda_n \to 1^+$ and $x_n \in M$ then $\|x_n - f(x_n)\| \to 0$ as $n \to \infty$. Now consider the set $T = \left\{x \in \overline{D} : \limsup_{n\to\infty} \|x - x_n\| = L\right\}$, where L is given by $L = \limsup_{n\to\infty} \|x_n\|$. It is obvious that T is non empty, bounded, closed and convex subset of $\overline{D}$. Now

$$\begin{aligned} \|f(x) - x_n\| &= \|f(x) - f(x_n) + f(x_n) - x_n\| \\ &= \|f(x) - f(x_n)\| + \|f(x_n) - x_n\| \\ &= \|x - x_n\| \quad \because \quad \|x_n - f(x_n)\| \to 0 \quad \text{as } n \to \infty \end{aligned}$$

Which shows that set T is invariant under f. Hence f has a fixed point in $\overline{D}$. □

4. Conclusion

Finding fixed points of nonlinear mappings especially, nonexpansive mappings has received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing. It is well known that pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to construct the methods for computing the fixed points of pseudocontractive mappings.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous referees for their valuable comments and useful suggestions for improving the article.

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