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A Study on W_8-curvature Tensor in Kenmotsu Manifolds

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Abstract: In this paper we study the curvature properties of Kenmotsu manifolds satisfying the conditions ξ-W_8-flat, ϕ-W_8-semisymmetric, $R(\xi,X)\cdot W_8=0$, $W_8\cdot R=0$, $W_8\cdot W_8=0$, W_8-Ricci pseudosymmetric, $W_8\cdot Q=0$.

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1. Introduction

K. Kenmotsu [14] introduced a new class of almost contact Riemann manifold which is known as Kenmotsu Manifold. Kenmotsu investigated fundamental properties on local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with one dimensional base and Kahler fiber. Kenmotsu studied that if a Kenmotsu manifold satisfies the condition $R(X,Y)Z=0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor and $R(X,Y)Z$ is the derivative of tensor algebra at each point of the tangent space. It is well known that odd dimensional spheres admit Sasakian structures whereas odd dimensional hyperbolic spaces can not admit Sasakian structure, so odd dimensional hyperbolic spaces admits Kenmotsu structure. Kenmotsu manifolds are normal almost contact Riemannian manifolds. Several properties of Kenmotsu Manifold have been studied by many authors like ([1–3, 7, 9, 11, 13, 15–17, 19, 20]). Motivated by all these work in this paper we study W_8-curvature tensor in Kenmotsu manifold.

The present paper is organized as follows: After a brief review of Kenmotsu manifold we study ξ-W_8-flat, ϕ-W_8-semisymmetric, $R(\xi,X)\cdot W_8=0$, $W_8\cdot R=0$, $W_8\cdot W_8=0$, W_8-Ricci pseudosymmetric, $W_8\cdot Q=0$.

2. Preliminaries

In this section, we briefly recall some general definitions of Kenmotsu manifolds:

An n-dimensional differential manifold M is said to be an almost contact metric manifold [2] if it admits an almost contact metric structure (ϕ,ξ,η,g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η and a Riemannian metric g compatible with (ϕ,ξ,η) satisfying

$$\phi^2=-I+\eta\otimes\xi,\quad \eta(\xi)=1,\quad \eta\circ\phi=0,\quad \phi\xi=0. \tag{1}$$

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$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X). \tag{2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [14] if it satisfies

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{3}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{4}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold [14] the following relations hold:

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{5}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{6}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{7}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{8}$$

$$Q\xi = -(n-1)\xi, \tag{9}$$

for any vector fields X, Y, Z on M, where R, S and Q denotes the curvature tensor, Ricci tensor and Ricci operator $g(QX,Y) = S(X,Y)$ on M.

3. ξ-W_8-flat Kenmotsu Manifold

In this section, we study ξ-W_8-flat in Kenmotsu manifold:

Definition 3.1. *A Kenmotsu manifold is said to be ξ-W_8-flat if*

$$W_8(X,Y)\xi = 0 \tag{10}$$

for any vector fields X, Y on M. W_8-curvature tensor [22] is defined as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X], \tag{11}$$

where R and S are the curvature tensor and Ricci tensor of the manifold respectively.

By using (11) in (10), we get

$$R(X,Y)\xi + \frac{1}{n-1}[S(X,Y)\xi - S(Y,\xi)X] = 0. \tag{12}$$

By virtue of (6), (8) in (12) and on simplification, we obtain

$$\eta(X)Y - \eta(Y)X + \frac{1}{n-1}[S(X,Y)\xi + (n-1)\eta(Y)X] = 0. \tag{13}$$

By taking innerproduct with ξ in (13) and on simplification, we have

$$S(X,Y) = -(n-1)\eta(Y)\eta(X). \tag{14}$$

Hence, we state the following theorem:

Theorem 3.2. *If a Kenmotsu manifold satisfying ξ-W_8-flat condition then the manifold is a special type of η-Einstein manifolds.*

4. ϕ-W_8-semisymmetric Condition in Kenmotsu Manifold

In this section, we study ϕ-W_8-semisymmetric condition in Kenmotsu manifold:

Definition 4.1. *A Kenmotsu manifold is said to be ϕ-W_8-semisymmetric if*

$$W_8(X,Y)\cdot\phi = 0, \tag{15}$$

for any vector fields X, Y on M.

Now, (15) turns into

$$(W_8(X,Y)\cdot\phi)Z = W_8(X,Y)\phi Z - \phi W_8(X,Y)Z = 0. \tag{16}$$

Making use of (11) in (16), we get

$$R(X,Y)\phi Z - \phi R(X,Y)Z + \frac{1}{(n-1)}[S(Y,Z)\phi X - S(Y,\phi Z)X] = 0. \tag{17}$$

Putting $X = \xi$ in (17) and by virtue of (7), (8) and on simplification, we obtain

$$-g(Y,\phi Z)\xi - \eta(Z)\phi Y - \frac{1}{(n-1)}S(Y,\phi Z)\xi = 0. \tag{18}$$

Replace Z by ϕZ in (18) and on simplification, we get

$$S(Y,Z)\xi = -(n-1)g(Y,Z)\xi. \tag{19}$$

By taking innerproduct with ξ in (19), we have

$$S(Y,Z) = -(n-1)g(Y,Z). \tag{20}$$

Hence, we state the following theorem:

Theorem 4.2. *If a Kenmotsu manifold satisfying ϕ-W_8-semisymmetric condition then the manifold is an Einstein manifolds.*

5. Kenmotsu Manifold Satisfying $R(\xi,X)\cdot W_8 = 0$ Condition

In this section, we study Kenmotsu manifold satisfying $R(\xi,X)\cdot W_8 = 0$. Then, we have

$$R(\xi,X)W_8(U,V)W - W_8(R(\xi,X)U,V)W - W_8(U,R(\xi,X)V)W - W_8(U,V)R(\xi,X)W = 0. \tag{21}$$

By using (7) in (21), we obtain

$$\begin{aligned}&\eta(W_8(U,V)W)X - g(X,W_8(U,V)W)\xi - \eta(U)W_8(X,V)W + g(X,U)W_8(\xi,V)W\\&-\eta(V)W_8(U,X)W + g(X,V)W_8(U,\xi)W - \eta(W)W_8(U,V)X + g(X,W)W_8(U,V)\xi = 0.\end{aligned} \tag{22}$$

By taking innerproduct with ξ in (22) and by virtue of (11) and on simplification, we get

$$-g(X, R(U, V)W) - g(X, U)g(V, W) + g(X, W)\eta(U)\eta(V) + g(X, V)g(U, W)$$
$$-g(U, X)\eta(V)\eta(W) - \frac{1}{(n-1)}[S(U, X)\eta(W)\eta(V) - S(X, W)\eta(U)\eta(V)] = 0. \quad (23)$$

Let $\{e_i : i = 1, 2, \ldots, n\}$ be an orthonormal frame field at any point of the manifold. Then contracting $X = U = e_i$ in (23), we have

$$S(V, W) = (1-n)g(V, W) - \left[\frac{r}{(n-1)} + n\right]\eta(V)\eta(W). \quad (24)$$

Hence, we state the following theorem:

Theorem 5.1. *If a Kenmotsu manifold satisfying $R(\xi, X) \cdot W_8 = 0$ then the manifold is an η-Einstein manifolds.*

6. Kenmotsu Manifold Satisfying $W_8 \cdot R = 0$ Condition

In this section, we study Kenmotsu manifold satisfying $W_8 \cdot R = 0$. Then, we get

$$W_8(\xi, U)R(X, Y)Z - R(W_8(\xi, U)X, Y)Z - R(X, W_8(\xi, U)Y)Z - R(X, Y)W_8(\xi, U)Z = 0. \quad (25)$$

Putting $Z = \xi$ in (25), we have

$$W_8(\xi, U)R(X, Y)\xi - R(W_8(\xi, U)X, Y)\xi - R(X, W_8(\xi, U)Y)\xi - R(X, Y)W_8(\xi, U)\xi = 0. \quad (26)$$

By using (6) in (26) and on simplification, we obtain

$$-\eta(W_8(\xi, U)X)Y + \eta(W_8(\xi, U)Y)X - R(X, Y)W_8(\xi, U)\xi = 0. \quad (27)$$

By using (7), (8), (11) in (27) and on simplification, we get

$$g(U, X)Y - g(U, Y)X - R(X, Y)U + \eta(U)\eta(X)Y - \eta(U)\eta(Y)X + \frac{1}{(n-1)}[S(U, X)Y - S(U, Y)X] = 0. \quad (28)$$

Putting $Y = \xi$ in (28) and by virtue of (7), (8), we obtain

$$S(X, U)\xi = -(n-1)\eta(U)\eta(X)\xi. \quad (29)$$

By taking innerproduct with ξ in (29), we have

$$S(X, U) = -(n-1)\eta(U)\eta(X). \quad (30)$$

Hence, we state the following:

Theorem 6.1. *If a Kenmotsu manifold satisfying $W_8 \cdot R = 0$ then the manifold is a special type of η-Einstein manifolds.*

7. Kenmotsu Manifold Satisfying $W_8 \cdot W_8 = 0$ Condition

In this section, we study Kenmotsu manifold satisfying $W_8 \cdot W_8 = 0$. Then, we get

$$W_8(\xi, U)W_8(X, Y)Z - W_8(W_8(\xi, U)X, Y)Z - W_8(X, W_8(\xi, U)Y)Z - W_8(X, Y)W_8(\xi, U)Z = 0. \tag{31}$$

By using (11) in (31), we obtain

$$\begin{aligned}
&\eta(W_8(X, Y)Z)U - g(U, W_8(X, Y)Z)\xi - \eta(X)W_8(U, Y)Z - \eta(Y)W_8(X, U)Z \\
&-\eta(Z)W_8(X, Y)U + 2\eta(U)W_8(X, Y)Z + g(U, X)W_8(\xi, Y)Z + g(U, Y)W_8(X, \xi)Z + g(U, Z)W_8(X, Y)\xi \\
&+\frac{1}{n-1}[-S(U, W_8(X, Y)Z)\xi + S(U, X)W_8(\xi, Y)Z + S(U, Y)W_8(X, \xi)Z + S(U, Z)W_8(X, Y)\xi] = 0.
\end{aligned} \tag{32}$$

Putting $Z = \xi$ in (32) and by virtue of (6), (8), (11) and on simplification, we have

$$\begin{aligned}
&\frac{2}{n-1}[\eta(U)S(X, Y)\xi - \eta(X)S(U, Y)\xi - \eta(Y)S(U, X)\xi] - R(X, Y)U + g(U, X)Y \\
&+\eta(U)\eta(X)Y - g(U, X)\eta(Y)\xi - g(U, Y)\eta(X)\xi + \frac{1}{n-1}[S(U, X)Y + S(U, Y)X] = 0.
\end{aligned} \tag{33}$$

Putting $Y = \xi$ in (33) and by virtue of (7), (8) and on simplification, we obtain

$$S(X, U)\xi = -(n-1)g(X, U)\xi. \tag{34}$$

By taking innerproduct with ξ in (34), we have

$$S(X, U) = -(n-1)g(X, U). \tag{35}$$

This leads the following:

Theorem 7.1. *If a Kenmotsu manifold satisfying $W_8 \cdot W_8 = 0$ then the manifold is an Einstein manifolds.*

8. Kenmotsu Manifold Satisfying W_8-Ricci Pseudosymmetric Condition

In this section, we study W_8-Ricci pseudosymmetric Kenmotsu manifold.

The concept of Ricci pseudosymmetric manifold was introduced by Deszcz ([4, 5]). A geometrical interpretation of Ricci pseudosymmetric manifolds in the Riemannian case is given in [12]. A Riemannian manifold (M, g) is called Ricci pseudosymmetric ([4, 5, 10, 19]) if the tensor $R \cdot S$ and the Tachibana tensor $Q(g, S)$ are linearly dependent, where

$$(R(X, Y) \cdot S)(Z, U) = -S(R(X, Y)Z, U) - S(Z, R(X, Y)U), \tag{36}$$

$$Q(g, S)(Z, U; X, Y) = -S((X \wedge_g Y)Z, U) - S(Z, (X \wedge_g Y)U) \tag{37}$$

and

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y \tag{38}$$

for all vector fields X, Y, Z, U on M. R denotes the curvature tensor of M.

A Kenmotsu manifold is said to be W_8-Ricci pseudosymmetric if its curvature tensor satisfies

$$(W_8(X,Y)\cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y), \tag{39}$$

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \ at \ x\}$, where L_S is some function on U_S. From (39), we get

$$S(W_8(X,Y)Z,U) + S(Z,W_8(X,Y)U) = L_S[g(Y,Z)S(X,U) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) - g(X,U)S(Y,Z)]. \tag{40}$$

Putting $Z = \xi$ in (40) and by using (8), (11) and on simplification, we obtain

$$\begin{aligned} 2[S(Y,U)\eta(X) \quad & -S(X,Y)\eta(U)] + (n-1)[g(Y,U)\eta(X) - g(X,U)\eta(Y)] \\ & = L_S[\eta(Y)S(X,U) - \eta(X)S(Y,U) - (n-1)g(Y,U)\eta(X) + (n-1)g(X,U)\eta(Y)]. \end{aligned} \tag{41}$$

Putting $Y = \xi$ in (41) and by virtue of (8) and on simplification, we get

$$S(X,U) = -\frac{(n-1)(L_S+1)}{L_S}g(X,U) + \frac{(n-1)}{L_S}\eta(X)\eta(U). \tag{42}$$

Hence, we state the following:

Theorem 8.1. *If a Kenmotsu manifold satisfying W_8-Ricci pseudosymmetric, then the manifold is an η-Einstein manifolds.*

9. Kenmotsu Manifold Satisfying $W_8 \cdot Q = 0$

In this section, we study Kenmotsu manifold satisfying $W_8 \cdot Q = 0$. Then, we have

$$W_8(X,Y)QZ - Q(W_8(X,Y)Z) = 0. \tag{43}$$

Putting $Y = \xi$ in (43), we obtain

$$W_8(X,\xi)QZ - Q(W_8(X,\xi)Z) = 0. \tag{44}$$

By virtue of (11) in (44), we get

$$R(X,\xi)QZ + \frac{1}{(n-1)}[S(X,\xi)QZ - S(\xi,QZ)X] - Q[R(X,\xi)Z + \frac{1}{(n-1)}\{S(X,\xi)Z - S(\xi,Z)X\}] = 0. \tag{45}$$

By using (7), (8) in (45) and on simplification, we obtain

$$S(X,Z)\xi - \eta(X)QZ - Q[g(X,Z)\xi - \eta(X)Z] = 0. \tag{46}$$

Taking inner product with ξ in (46) and on simplification, we have

$$S(X,Z) = -(n-1)g(Z,X). \tag{47}$$

Hence, we state the following:

Theorem 9.1. *A Kenmotsu manifold satisfying $W_8 \cdot Q = 0$, then the manifold is an Einstein manifold.*

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References

[1] C. S. Bagewadi and Venkatesha, *Some curvature tensors on a Kenmotsu manifolds*, Tensor N.S., 68(2007), 140-147.

[2] D. E. Blair, *Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics*, 509, Springer-Verlag, berlin-New-York, (1976).

[3] S. K. Chaubey and R. H. Ojha, *On the m-projective curvature tensor of a Kenmotsu manifold*, Differential Geometry-Dynamical Systems, 12(2010), 52-60.

[4] R. Deszcz, *On Ricci-pseudosymmetric warped products*, Demonstratio Math., 22(1989), 1053-1065.

[5] R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A, 44(1992), 1-34.

[6] U. C. De and A. A. Shaikh, *Differential Geometry of Manifolds*, Narosa Publishing House Pvt. Ltd., New Delhi, (2007).

[7] U. C. De, *On ϕ-symmetric Kenmotsu manifolds*, International electronic Jour. of geometry, 1(2008), 33-38.

[8] Gurupadavva Ingalahalli and C. S. Bagewadi, *A Study on ϕ-Symmetric τ-curvature tensor in $N(k)$-contact metric manifold*, Carpathian Math. Publ., 6(2014), 203-211.

[9] Gurupadavva Ingalahalli and C. S. Bagewadi, *A study on Conservative C-Bochner curvature tensor in K-contact and Kenmotsu manifolds admitting semi-symmetric metric connection*, ISRN Geometry, (2012), 1-14.

[10] S. K. Hui and R. S. Lemence, *Ricci pseudosymmetric generalized quasi-Einstein manifolds*, SUT J. Math., 51(2015), 195-213.

[11] S. K. Hui and D. Chakraborty, *Infinitesimal CL-transformations on Kenmotsu manifolds*, Bangmod Int. J. Math and Comp. Sci., 3(2017), 1-9.

[12] B. Jahanara, S. Haesen, Z. Sentürk and L. Verstraelen, *On the parallel transport of the Ricci curvatures*, J. Geom. Phys., 57(2007), 1771-1777.

[13] K. K. Baishya and P. R. Chowdhury, *Kenmotsu Manifold with some Curvature Conditions*, Annales Univ. Sci. Budapest., 59(2016), 55-65.

[14] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., 24(1972), 93-103.

[15] H. G. Nagaraja, D. L. Kiran Kumar and V. S. Prasad, *Ricci solitons on Kenmotsu manifolds under D-homothetic deformation*, Khayyam J. Math., 4(2018), 102-109.

[16] C. Özgür, *On weakly symmetric Kenmotsu manifolds*, Differential Geometry-Dynamical Systems, 8(2006), 204-209.

[17] D. G. Prakasha and B. S. Hadimani, *On the conharmonic curvature tensor of Kenmotsu manifolds with generalized Tanaka-Webster connection*, Misolc. Math. Notes, 19(2018), 491-503.

[18] G. P. Pokhariyal, *Relativistic significance of curvature tensors*, International Journal of Mathematics and Mathematical Sciences, 5(1982), 133-139.

[19] A. A. Shaikh and S. K. Hui, *On Locally ϕ-Symmetric β-Kenmotsu Manifolds*, Extracta Mathematicae, 24(2009), 301-316.

[20] B. B. Sinha and A. K. Srivastava, *Curvature on Kenmotsu manifold*, Indian J. Pure and Appl. Math., 22(1991), 23-28.

[21] M. M. Tripathi and Punam Gupta, *τ-curvature tensor on a semi-Riemannian manifold*, J. Adv. Math. Stud., 4(2011), 117-129.

[22] M. M. Tripathi and Punam Gupta, $(N(k), \xi)$*-semi-Riemannian manifolds: Semisymmetries*, arXiv:1202.6138v [math.DG] 28(Feb 2012).