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Server Breakdown and Delayed Repair in Three Phases of Service for an M/G/1 Retrial Queueing System

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Abstract: This paper deals with an unreliable server having three phases of heterogeneous service on the basis of M/G/1 queueing system. We suppose that customers arrive and join the system according to a Poisson's process with arrival rate λ. When the server is working with any phase of service, it may breakdown at any instant. After breakdown, when the server is sent for repair then server stops its service and arrival customers are waiting for repair, which we may called as waiting period of the server. This waiting time stands for delay time/delay repair. In this model, first we derive the joint probability distribution for the server. Secondly, we derive the probability generating function of the stationary queue size distribution at a departure epoch as a classical generalisation of Pollaczek - Khinchin formula. Third, we derive Laplace Stieltjes transform of busy period distribution and waiting time distribution. Finally, we obtain some important performance measures and reliability analysis of this model. By using a supplementary variable method, we obtain the transient and the steady state solutions for both queueing and reliability measures.

Keywords: First phase of service, second phase of service, third phase of service, random breakdowns, delayed repair, M/G/1 queue, stationary queue size distribution and reliability index.

1. Introduction

It is difficult to develop a server which is available for permanent basis because it looks to be unrealistic. So, we consider a queueing system where the server may breakdown at any instant during any phase of service, while serving the customers. Such a system is also known as queue with service interruptions or queue with unreliable server. In practical system, we often meet the case where service stations may fail and can be repaired. Most probably the study of service interruptions queueing models was starting at early 1950's. Queueing models wherein the server provide two phases of service to each customer are known as two-phase service queueing model. Such types of queueing situations naturally arise in many real time system namely in *manufacturing system* wherein the machine producing certain items may require two phases of service in succession. For completing the processing of raw materials, the periodic checking (first phase of service) followed by usual processing (second phase of service) of raw material is required. Another example is in a cyber cafe, the customers arrive for utilizing internet surfing but some of them may require scanning of some files also. Madan [1] studied an M/G/1 queue with second optional service in which first essential service time follows a general distribution but second optional service is assumed to be exponentially distributed. Medhi [2] generalized this model by considering that the second optional is also governed by a general distribution. Choudhury [3] investigated this model and generalized for batch arrival case. Choudhury [4] and Paul [5] investigated such a model under Bernoulli feedback mechanism. Krishnakumar and Arivudainambi [6] obtained

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the explicit expression for transient probabilities for this type of finite capacity model M/G/1/1 Bernoulli feedback queue and M/G/1/1 queue with unreliable server [7]. Recently, Wang [8] investigated such a model with the assumption that the server is subject to breakdowns and repairs.

Those queueing systems where the server serves the customers in finite number of phases are said to be multiphase queueing systems. All these phases are essential for each customer to complete the whole service process. Example- In a clinical physical examination wherein the patient goes through a series of physical check-up such as ear, throat ,nose, blood test, eye examination which all are considered as essential phases of service to reach the final decision by any medical doctor to declare a candidate to be fit. Recently, Jain and Agrawal [17,18] examined a batch arrival queueing system with modified Bernoulli vacation under N-policy wherein the customer needs l-stage of service in succession that is, the first stage service (FSS) is followed by the second stage service (SSS), the second stage service followed by third stage service (TSS) and so on up to l-stages of service. Jain and Upadhyaya [15] dealt with unreliable server batch arrival queueing system with essential and multioptional services under N-policy. We investigate M/G/1 unreliable server queue with three phases of service, delayed repair and server breakdown during the service together. The remaining overview of this paper is as follows.

In point 2, we represent the description of mathematical model. Point 3 stands for derivations of the stationary distribution of the queue size for the server state at a random epoch. Point 4 stands for distribution of busy period and waiting time. Point 5 stands for reliability analysis of this model. Finally conclusion is drawn in last one. To derive the probability generating function for queue size distribution at different phases of service, we apply the supplementary variable technique by introducing one or more supplementary variables.

2. The Mathematical Model

Let us suppose an M/G/1 retrial queueing system. In this system arrivals of customers are according to a Poisson process with arrival rate 'λ'. It is a single server model which provides its services in three phases of heterogeneous service in successive service: first phase of service (FPS) denoted by C_1, followed by a second phase of service (SPS) denoted by C_2 which again followed by third phase of service (TPS) denoted by C_3. The system serves according to first come, first serve (FCFS) service discipline. All the arriving customers join the queue and serves the essential preliminary first phase of regular service denoted by C_1. When FPS is completed then the customer can leave the system with probability q $(= 1-p)$ or it may continue for second phase of service denoted by C_2 with probability p $(0 \leq p \leq 1)$. Furthermore, when SPS is completed then the customer can leave the system with probability q $(= 1-p)$ or it may continue for third phase of service denoted by C_3 with probability p $(0 \leq p \leq 1)$. The service time for $\mathrm{i^{th}}$ phases are independent random variables follow general law of distribution with probability distribution function *d.f.* $C_i(x)$, $i = 1, 2, 3$. Laplace–Stieltjes transform (LST) $\beta_i^*(\theta) = E[e^{-\theta C i}]$ and finite moments are $\beta_i^{(k)}$, $k \geq 1$ for $i = 1, 2, 3$. During working with any phase of service, the server can be breakdown at any time. Thus service of the server becomes fail for a short time. This breakdowns are generated by exogenous Poisson process with rates α_1 for FPS, α_2 for SPS and α_3 for TPS respectively. Now when this breakdown occurs, it is sent for repair during which the server stops providing service to the arriving customers till service channel is repaired. It means server waits for repair to start, which is known as waiting period of the server. We define this waiting time as delay time. The delay time D_i of the server of the $\mathrm{i^{th}}$ phase of service follows general law of distribution with probability distribution function *d.f.* $D_i(y)$, $i = 1, 2, 3$. Laplace–Stieltjes transform (LST) $\Psi_i^*(\theta) = E[e^{-\theta D^i}]$ and finite moments are $\Psi_i^{(k)}$, $k \geq 1$ for $i = 1, 2, 3$. The customers which were just being served before server breakdown wait for the service to complete its remaining service. The repair time (denoted by S_1 for FPS, S_2 for SPS and S_3 for TPS) distributions of the server for three phases of service are assumed to be arbitrarily distributed with *d.f.* $G_1(y)$, $G_2(y)$ and $G_3(y)$, Laplace–

Stieltjes transform (LST) $G_1^*(\theta) = E[e^{-\theta S_1}]$, $G_2^*(\theta) = E[e^{-\theta S_2}]$ and $G_3^*(\theta) = E[e^{-\theta S_3}]$ and also finite k$^{\text{th}}$ moments $g_1^{(k)}$, $g_2^{(k)}$ and $g_3^{(k)}$ respectively. Immediately when the server is fixed i.e., repaired, the server is again ready to start its remaining service to customers in all the three phases of service and in this case the service times are cumulative, which may be referred to as generalized service times. In addition, we assume that input process, server's life time, server's repair time, service time and vacation time random variables are mutually independent of each other. The notation "PH" is used for phase type distribution. This distribution is characterized by a Markov chain with states $1, 2, \ldots, k$ (the so called phases) and a transition probability matrix P which is transient. The random variable "X" has a phase-type distribution if X is the total time elapsing from start in the Markov chain till departure from the Markov chain. Let us define H_i as generalised service time for i$^{\text{th}}$ phase of service, distribution function $H_i(x)$ and Laplace–Stieltjes transform $H_i^*(\theta) = E[e^{-\theta H}i]$ respectively then by [28] we have

$$H_i^*(\theta) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\theta x} e^{-\alpha_i x} \left[\frac{(\alpha_i x)^n}{n!}\right] [\psi_i^*(\theta) G_i^*(\theta)]^n dC_i(x) = \beta_i^*(\theta + \alpha_i(1 - \psi_i^*(\theta) G_i^*(\theta))) \quad for \quad i = 1, 2, 3.$$

The first two moments are found to be,

$$h_i^{(1)} = (-1)^1 \left[\frac{dH_i^*(\theta)}{d\theta}\right]_{\theta=0} = \beta_i^{(1)}\{1 + \alpha_i(\psi_i^{(1)} + g_i^{(1)}) \quad and$$
$$h_i^{(2)} = (-1)^2 \left[\frac{d^2 H_i^*(\theta)}{d\theta^2}\right]_{\theta=0} = \beta_i^{(2)}\{1 + \alpha_i(\psi_i^{(1)} + g_i^{(1)})\}^2 + \alpha_i \beta_i^{(1)}(\psi_i^{(2)} + g_i^{(2)} + 2\psi_i^{(1)} g_i^{(1)})$$

where $h_i^{(k)}$ is the k$^{\text{th}}$ moment of the i$^{\text{th}}$ phase of the generalised service time distribution.

3. Stationary Queue Size Distribution

We derive the system state equation for its stationary queue size distribution by treating the elapsed service time, the elapsed delay time of the server and the elapsed repair time of the server for three phases of the service as supplementary variables. The Supplementary variable technique is a method which is used for solution of Non–Markovian queueing problems. We can convert by this technique a Non–Markovian queueing model in Markovian queueing model by introducing one or more supplementary variables. These supplementary variables are introduced corresponding to either elapsed time or remaining time of the random variables. Then we solve the equations and derive the probability generating function (PGF)s of the stationary queue size distribution. It is assumed that system is in steady state condition. D. R. Cox [24] was first to study Non–Markovian stochastic process by the inclusion of supplementary variables.

$M_Q(t)$: the queue size (including the one being served, if any) at time t.

$C_1^0(t)$: the elapsed FPS time at time t.

$C_2^0(t)$: the elapsed SPS time at time t.

$C_3^0(t)$: the elapsed TPS time at time t.

$D_1^0(t)$: the elapsed delay time of the ser ver in FPS at time t.

$D_2^0(t)$: the elapsed delay time of the server in SPS at time t.

$D_3^0(t)$: the elapsed delay time of the server in TPS at time t.

$S_1^0(t)$: the elapsed repair time for FPS during which breakdown occurs time at time t.

$S_2^0(t)$: the elapsed repair time for SPS during which breakdown occurs time at time t.

$S_3^0(t)$: the elapsed repair time for TPS during which breakdown occurs time at time t.

Further, let us introduce the following random variable:

$$Y(t) = \begin{cases} 0, & if\ the\ system\ is\ idle\ at\ time\ t, \\ 1, & if\ the\ server\ is\ busy\ with\ FPS\ at\ time\ t, \\ 2, & if\ the\ server\ is\ busy\ with\ SPS\ at\ time\ t, \\ 3, & if\ the\ server\ is\ busy\ with\ TPS\ at\ time\ t, \\ 4, & if\ the\ server\ is\ waiting\ for\ repair\ during\ FPS\ at\ time\ t, \\ 5, & if\ the\ server\ is\ waiting\ for\ repair\ during\ SPS\ at\ time\ t, \\ 6, & if\ the\ server\ is\ waiting\ for\ repair\ during\ TPS\ at\ time\ t, \\ 7, & if\ the\ server\ is\ under\ repair\ during\ FPS\ at\ time\ t, \\ 8, & if\ the\ server\ is\ under\ repair\ during\ SPS\ at\ time\ t, \\ 9, & if\ the\ server\ is\ under\ repair\ during\ TPS\ at\ time\ t. \end{cases}$$

In M/G/1 queue, we assume that arrival of customers take place according to Poisson's process with rate λ. Let $P_{i,n}(x)$ represents the steady state probability that there are $n \geq 0$ customers in the queue excluding one in i$^{\text{th}}$ $(= 1, 2, 3)$ phase of service and the elapsed service time of this customer is x. Let $Q_{i,n}(x)$ represents probability when the server is in waiting period for repair with $n \geq 0$ customers in the queue excluding one in i$^{\text{th}}$ $(= 1, 2, 3)$ phase of service and the elapsed service time of this customer is x. Let $S_{i,n}(x)$ represents the steady state probability that there are $n \geq 0$ customers in the queue excluding one customer who is repeating the i$^{\text{th}}$ $(= 1, 2, 3)$ phase of service and the elapsed service time of this customer is x.

Now, the supplementary variable $C_i^0(t), D_i^0(t)$ and $S_i^0(t)$ for $i = 1, 2, 3$ are introduced in order to obtain a bivariate Markov process $\{M_Q(t), X(t)\}$, where $X(t) = 0$ if $Y(t) = 0$, $X(t) = C_1^0(t)$ if $Y(t) = 1$, $X(t) = C_2^0(t)$ if $Y(t) = 2$, $X(t) = C_3^0(t)$ if $Y(t) = 3$, $X(t) = D_1^0(t)$ if $Y(t) = 4$, $X(t) = D_2^0(t)$ if $Y(t) = 5$, $X(t) = D_3^0(t)$ if $Y(t) = 6$, $X(t) = S_1^0(t)$ if $Y(t) = 7$, $X(t) = S_2^0(t)$ if $Y(t) = 8$ and $X(t) = S_3^0(t)$ if $Y(t) = 9$.

Further we define the following probabilities:

$$U_{0,0}(t) = \lim_{t\to\infty} P_r\{M_Q(t) = 0, X(t) = 0\},$$

and for $i = 1, 2, 3$ and $n \geq 0$,

$$P_{i,n}(x;t)dx = \lim_{t\to\infty} P_r\{M_Q(t) = n, X(t) = C_i^0(t); \quad x < C_i^0(t) \leq x + dx\}; \quad x > 0, \quad n \geq 0$$
$$Q_{i,n}(x,y;t)dy = \lim_{t\to\infty} P_r\{M_Q(t) = n, X(t) = D_i^0(t); \quad y < D_i^0(t) \leq y + dy \,|C_i^0(t) = x\}; \quad (x,y) > 0, \quad n \geq 0$$
$$S_{i,n}(x,y;t)dy = \lim_{t\to\infty} P_r\{M_Q(t) = n, X(t) = S_i^0(t); \quad y < S_i^0(t) \leq y + dy \,|C_i^0(t) = x\}; \quad (x,y) > 0, \quad n \geq 0$$

Further, it is assumed that $D_i(0) = 0, D_i(\infty) = 1, C_i(0) = 0, C_i(\infty) = 1$, $G_i(0) = 0, G_i(\infty) = 1$ for $i = 1, 2, 3$. For $i = 1, 2, 3$; $C_i(x)$ is continuous at $x = 0$. $D_i(y)$ and $G_i(y)$ are continuous at $y = 0$ for $i = 1, 2, 3$ respectively, so that

$$\phi_i(y)dy = \frac{dD_i(y)}{1 - D_i(y)},$$
$$\mu_i(x)dx = \frac{dC_i(x)}{1 - C_i(x)} \quad and$$
$$\xi_i(y)dy = \frac{dG_i(y)}{1 - G_i(y)}$$

are the first order differential (Hazard rate) functions of C_i, D_i and S_i respectively for $i = 1, 2, 3$.

3.1. The Steady State Equations

The Kolmogorov forward equation to govern the system under steady state conditions [24] can be written as follows:

$$\frac{d}{dx}P_{1,n}(x)+[\lambda+\alpha_1+\mu_1(x)]P_{1,n}(x)=\lambda(1-\delta_{n,0})P_{1,n-1}(x)+\int_0^\infty \xi_1(y)S_{1,n}(x,y)dy;\quad n\geq 0, \tag{1}$$

$$\frac{d}{dx}P_{2,n}(x)+[\lambda+\alpha_2+\mu_2(x)]P_{2,n}(x)=\lambda(1-\delta_{n,0})P_{2,n-1}(x)+\int_0^\infty \xi_2(y)S_{2,n}(x,y)dy;\quad n\geq 0, \tag{2}$$

$$\frac{d}{dx}P_{3,n}(x)+[\lambda+\alpha_3+\mu_3 x)]P_{3,n}(x)=\lambda(1-\delta_{n,0})P_{3,n-1}(x)+\int_0^\infty \xi_3(y)S_{3,n}(x,y)dy;\quad n\geq 0, \tag{3}$$

$$\frac{d}{dy}Q_{1,n}(x,y)+[\lambda+\phi_1(y)]Q_{1,n}(x,y)=\lambda(1-\delta_{n,0})Q_{1,n-1}(x,y);\quad n\geq 0, \tag{4}$$

$$\frac{d}{dy}Q_{2,n}(x,y)+[\lambda+\phi_2(y)]Q_{2,n}(x,y)=\lambda(1-\delta_{n,0})Q_{2,n-1}(x,y);\quad n\geq 0, \tag{5}$$

$$\frac{d}{dy}Q_{3,n}(x,y)+[\lambda+\phi_3(y)]Q_{3,n}(x,y)=\lambda(1-\delta_{n,0})Q_{3,n-1}(x,y);\quad n\geq 0, \tag{6}$$

$$\frac{d}{dy}S_{1,n}(x,y)+[\lambda+\xi_1(y)]S_{1,n}(x,y)=\lambda(1-\delta_{n,0})S_{1,n-1}(x,y);\quad n\geq 0, \tag{7}$$

$$\frac{d}{dy}S_{2,n}(x,y)+[\lambda+\xi_2(y)]S_{2,n}(x,y)=\lambda(1-\delta_{n,0})S_{2,n-1}(x,y);\quad n\geq 0, \tag{8}$$

$$\frac{d}{dy}S_{3,n}(x,y)+[\lambda+\xi_3(y)]S_{3,n}(x,y)=\lambda(1-\delta_{n,0})S_{3,n-1}(x,y);\quad n\geq 0, \tag{9}$$

$$\lambda U_0=\int_0^\infty \phi(y)Q_0(y)dy+q\int_0^\infty \mu_3(x)P_{3,0}(x)dx; \tag{10}$$

Where $\delta_{n,m}$ denotes Kronecker's delta function. These set of equations are to be solved under the following boundary condition at $x=0$;

$$P_{1,n}(0)=\lambda\delta_{n,0}U_{0,0}+\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx+\int_0^\infty \mu_3(x)P_{3,n+1}(x)dx+q\int_0^\infty \mu_1(x)P_{1,n+1}(x)dx;\quad n\geq 0, \tag{11}$$

$$P_{2,n}(0)=p\int_0^\infty \mu_1(x)P_{1,0}(x)dx;\quad n\geq 0 \tag{12}$$

$$P_{3,n}(0)=p\int_0^\infty \mu_2(x)P_{2,0}(x)dx;\quad n\geq 0 \tag{13}$$

And at $y=0$ for $i=1,2,3$ and fixed values of x:

$$Q_{i,n}(x,0)=\alpha_i P_{i,n}(x);\quad x>0,\quad n\geq 0. \tag{14}$$

$$S_{i,n}(x,0)=\int_0^\infty \phi_i(y)Q_{i,n}(x,y)dy;\quad x>0,\quad n\geq 0 \tag{15}$$

With normalizing condition

$$U_{0,0}+\sum_{i=1}^{3}\sum_{n=0}^{\infty}\left[\int_0^\infty P_{i,n}(x)dx+\int_0^\infty\int_0^\infty Q_{i,n}(x,y)dxdy+\int_0^\infty\int_0^\infty S_{i,n}(x,y)dxdy\right]=1. \tag{16}$$

3.2. The Model Solution

To solve the system of Equations (1)-(15), let us introduce the following PGFs for $i=1,2,3$ and $|z|<1$:

$$S_i(x,y;z)=\sum_{n=0}^{\infty} z^n S_{i,n}(x;y);\ S_i(x,0;z)=\sum_{n=0}^{\infty} z^n S_{i,n}(x;0),$$

$$Q_i(x,y;z)=\sum_{n=0}^{\infty} z^n Q_{i,n}(x,y);Q_i(x,0;z)=\sum_{n=0}^{\infty} z^n Q_{i,n}(x,0),$$

$$P_i(x,z)=\sum_{n=0}^{\infty} z^n P_{i,n}(x);\qquad P_i(0,z)=\sum_{n=0}^{\infty} z^n P_{i,n}(0)$$

Let $\omega(z) = \lambda(1-z)$, then proceeding in usual manner with Equations (4) - (9), we get a set of differential equation of Lagrangian type whose solutions are given by:

$$Q_i(x,y;z) = Q_i(x,0;z)[1-D_i(y)]\exp\{-\omega(z)y\}; \quad (x,y)>0, \quad for \quad i=1,2,3 \tag{17}$$

$$and \quad S_i(x,y;z) = S_i(x,0;z)[1-G_i(y)]\exp\{-\omega(z)y\}; \quad (x,y)>0 \quad for \quad i=1,2,3 \tag{18}$$

Where $Q_i(x,0;z)$ and $S_i(x,0;z)$ for $i=1,2,3$ can be obtained from Equations (14), (15) and (17) which after simplification yields

$$Q_i(x,0;z) = \alpha_i P_i(x;z) \quad for \quad i=1,2,3 \tag{19}$$

$$S_i(x,0;z) = Q_i(x,0;z)\Psi_i^*(\omega(z)) \quad for \quad i=1,2,3 \tag{20}$$

Now solving the differential equation (1), (2) and (3) we get

$$P_i(x;z) = P_i(0;z)[1-C_i(x)]\exp\{-A_i(z)x\}; \quad x>0 \tag{21}$$

Where $A_i(z) = \omega(z)+\alpha_i(1-G_i^*(\omega(z))\Psi_i^*(\omega(z)))$ for $i=1,2,3$. Multiplying equation (11) by z^n and then taking summation over all possible values of $n \geq 0$, and utilizing (21) we get on simplification

$$\begin{aligned} zP_1(0;z) = {} & qP_1(0,z)\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\} + P_2(0,z)\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\} \\ & + P_3(0,z)\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\} - \omega(z)U_{0,0}. \end{aligned} \tag{22}$$

Similarly from Equations (12) and (13) we get

$$\begin{aligned} P_2(0,z) &= pP_1(0,z)\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\} \\ P_3(0,z) &= pP_2(0,z)\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\} \end{aligned} \tag{23}$$

Putting (23) in equation (22) gives

$$P_1(0;z) = \frac{\omega(z)U_{0,0}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z} \tag{24}$$

Now utilizing (21) and (19) in (17) we get for $i=1,2,3$.

$$Q_i(x,y:z) = \alpha_i P_i(0;z)[1-C_i(x)]\exp\{-(\omega(z)+\alpha_i(1-G_i^*(\omega(z))\Psi_i^*(\omega(z))))x\} \times [1-D(y)]\exp\{-\omega(z)y\} \tag{25}$$

Similarly using (19) and (21) in (18) we get for i = *1,2,3.*

$$S_i(x,y:z) = \alpha_i\Psi_i^*(\omega(z))P_i(0;z)[1-C_i(x)]\exp\{-(\omega(z)+\alpha_i(1-G_i^*(\omega(z))\Psi_i^*(\omega(z))))x\} \times [1-D(y)]\exp\{-\omega(z)y\} \tag{26}$$

Let $z \to 1$ in Eqquation (24), we obtain by the L'Hospital's rule

$$P_1(0,1) = \frac{\omega U_{0,0}}{(1-\rho_H)}; \tag{27}$$

Where $\rho_H = \rho_1[1+\alpha_1(\Psi_1^{(1)}+g_1^{(1)})] + p\rho_2[1+\alpha_2(\Psi_2^{(1)}+g_2^{(1)})] + p\rho_3[1+\alpha_3(\Psi_3^{(1)}+g_3^{(1)})]$ is the utilizing factor of the system, $\rho_i = \omega\beta_i^{(1)}$ for $i = 1,2,3$. This gives for $i = 1,2,3$.

$$P_1(x,1) = \frac{\lambda U_{0,0}[1-C_1(x)]}{(1-\rho_H)} \tag{28}$$

$$P_2(x,1) = \frac{p\lambda U_{0,0}[1-C_2(x)]}{(1-\rho_H)} \tag{29}$$

$$P_3(x,1) = \frac{p\lambda U_{0,0}[1-C_3(x)]}{(1-\rho_H)} \tag{30}$$

$$Q_1(x,y,1) = \frac{\alpha_1\lambda U_{0,0}[1-C_1(x)][1-D_1(y)]}{(1-\rho_H)}$$

$$Q_2(x,y,1) = \frac{p\alpha_2\lambda U_{0,0}[1-C_2(x)][1-D_2(y)]}{(1-\rho_H)}$$

$$Q_3(x,y,1) = \frac{p\alpha_3\lambda U_{0,0}[1-C_3(x)][1-D_3(y)]}{(1-\rho_H)}$$

$$S_1(x,y,1) = \frac{\alpha_1\lambda U_{0,0}[1-C_1(x)][1-G_1(y)]}{(1-\rho_H)}$$

$$S_2(x,y,1) = \frac{p\alpha_2\lambda U_{0,0}[1-C_2(x)][1-G_2(y)]}{(1-\rho_H)}$$

$$S_3(x,y,1) = \frac{p\alpha_3\lambda U_{0,0}[1-C_3(x)][1-G_3(y)]}{(1-\rho_H)}$$

Now utilizing the normalizing condition (16), we get

$$U_{0,0} = (1-\rho_H); \tag{31}$$

Note that equation (31) represents steady-state probability that the server is idle but available in the system, Also, from equation (31). We have $\rho_H < 1$, which is the necessary and sufficient condition under which steady-state solution exists. Thus we summarize our results in the following Theorem 3.1.

Theorem 3.1. *Under the stability condition $\rho_H < 1$, the joint distribution of the state of the server and the queue size has the following partial PGFs.*

$$P_1(x,z) = \frac{(1-\rho_H)\omega(z)[1-C_1(x)]\exp.\{-\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z} \tag{32}$$

$$P_2(x,z) = \frac{p(1-\rho_H)\omega(z)[1-C_2(x)]\exp.\{-\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\times\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\} \tag{33}$$

$$P_3(x,z) = \frac{(1-\rho_H)\omega(z)[1-C_3(x)]\exp.\{-\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\} \tag{34}$$

$$Q_1(x,y,z) = \frac{\alpha_1(1-\rho_H)\omega(z)[1-C_1(x)]\exp.\{-\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\times[1-D_1(y)]\exp\{-\omega(z)y\} \tag{35}$$

$$Q_2(x,y,z) = \frac{p\alpha_2(1-\rho_H)\omega(z)[1-C_2(x)]\exp.\{-\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\times[1-D_2(y)]\exp\{-\omega(z)y\}\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\} \tag{36}$$

$$Q_3(x,y,z) = \frac{p\alpha_3(1-\rho_H)\omega(z)[1-C_3(x)]\exp.\{-\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\times[1-D_3(y)]\exp\{-\omega(z)y\}\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\} \tag{37}$$

$$S_1(x,y,z) = \frac{\alpha_1\Psi_1^*(\omega(z))(1-\rho_H)\omega(z)[1-C_1(x)]\exp.\{-\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\times[1-G_1(y)]\exp\{-\omega(z)y\} \tag{38}$$

$$S_2(x,y,z) = \frac{p\alpha_2\Psi_2^*(\omega(z))(1-\rho_H)\omega(z)[1-C_2(x)]\exp.\{-\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$\times[1-G_2(y)]\exp\{-\omega(z)y\}\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\} \tag{39}$$

$$S_3(x,y,z) = \frac{p\alpha_3\Psi_3^*(\omega(z))(1-\rho_H)\omega(z)[1-C_3(x)]\exp.\{-\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\}x\}}{[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}(1+p\beta_3^*\{\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))\})]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}-z}$$
$$[1-D_3(y)]\exp\{-\omega(z)y\}\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))\}\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\} \tag{40}$$

Where $\lambda(z) = \lambda(1-z)$.

Remark 3.2. *It is important to note here that such types of joint distributions are important to obtain the distribution of each state of the server in more comprehensive manner, which helps us to obtain marginal distributions of the server's states as well as stationary queue size distribution at a departure epoch.*

Theorem 3.3. *Under the stability condition* $\rho_H < 1$ *the marginal PGFs of the server's state queue size distributions are given by*

$$P_1(z) = \frac{(1-\rho_H)\omega(z)(1-H_1^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))]} \tag{41}$$
$$P_2(z) = \frac{p(1-\rho_H)\omega(z)(1-H_2^*(\omega(z)))H_1^*(\omega(z))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))]} \tag{42}$$
$$P_3(z) = \frac{p(1-\rho_H)\omega(z)(1-H_3^*(\omega(z)))H_1^*(\omega(z))H_2^*(\omega(z))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))]} \tag{43}$$
$$Q_1(z) = \frac{\alpha_1(1-\rho_H)(1-H_1^*(\omega(z)))(1-\Psi_1^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))]} \tag{44}$$
$$Q_2(z) = \frac{p\alpha_2(1-\rho_H)(1-H_2^*(\omega(z)))H_1^*(\omega(z))(1-\Psi_2^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))]} \tag{45}$$
$$Q_3(z) = \frac{p\alpha_3(1-\rho_H)(1-H_3^*(\omega(z)))H_1^*(\omega(z))H_2^*(\omega(z))(1-\Psi_3^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))]} \tag{46}$$
$$S_1(z) = \frac{\alpha_1\Psi_1(\omega(z))(1-\rho_H)(1-H_1^*(\omega(z)))(1-G_1^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))]} \tag{47}$$
$$S_2(z) = \frac{p\alpha_2\Psi_2(\omega(z))(1-\rho_H)(1-H_2^*(\omega(z)))H_1^*(\omega(z))(1-G_2^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))]} \tag{48}$$
$$S_3(z) = \frac{p\alpha_3\Psi_3(\omega(z))(1-\rho_H)(1-H_3^*(\omega(z)))H_1^*(\omega(z))H_2^*(\omega(z))(1-G_3^*(\omega(z)))}{[\{q+pH_2^*(\omega(z))\}H_1^*(\omega(z))-z]\,[\omega(z)+\alpha_3(1-G_3^*(\omega(z))\Psi_3^*(\omega(z)))]} \tag{49}$$

where $H_i^*(\omega(z)) = \beta_i^*\{\omega(z)+\alpha_i(1-G_i^*(\omega(z))\Psi_i^*(\omega(z)))\}$ *for* $i = 1,2,3$.

Proof. Integrating equations (32), (33) and (34) with respect to x and y respectively and then using the well known result of renewal theory.

$$\int_0^\infty e^{-\theta x}(1-C_i(x))dx = \frac{[1-\beta_i^*(\theta)]}{\theta} \quad for \quad i = 1,2,3.$$

We get formulae equations (41), (42) and (43). Similarly, integrating equation (25) and (26) with respect to y, get for $i = 1,2,3$.

$$Q_i(x,z) = \int_0^\infty Q_i(x,y;z)dy = \alpha_i[\omega(z)]^{-1}[1-\Psi_i^*(\omega(z))]P_i(0;z)[1-C_i(x)]\exp\{-(\omega(z)+\alpha_i(1-G_i^*(\omega(z))\Psi_i^*(\omega(z))))x\}. \tag{50}$$

and

$$S_i(x,z) = \int_0^\infty S_i(x,y;z)dy$$
$$= \alpha_i[\omega(z)]^{-1}\Psi_i^*(\omega(z))[1-G_i^*(\omega(z))]P_i(0;z)[1-C_i(x)]\exp\{-(\omega(z)+\alpha_i(1-G_i^*(\omega(z))\Psi_i^*(\omega(z))))x\}. \tag{51}$$

□

Further integrating equations (50) and (51) with respect to x, we claimed in formulae (44) - (49). Next the system state probabilities are given in Corollary 3.1.

Corollary 3.4. *If the system is in steady-state conditions, then*

(1). The probability that the system is idle is

$$P_I = 1 - \rho_1[1 + \alpha_1(\Psi_1^{(1)} + g_1^{(1)})] - p\rho_2[1 + \alpha_2(\Psi_2^{(1)} + g_2^{(1)})] - p\rho_3[1 + \alpha_3(\Psi_3^{(1)} + g_3^{(1)})]$$

(2). the probability that the server is busy with FPS is $P_{C_1} = \rho_1$*;*

(3). the probability that the server is busy with SPS is $P_{C_2} = p\rho_2$*;*

(4). the probability that the server is busy with TPS is $P_{C_3} = p\rho_3$*;*

(5). the probability that the server is waiting for repair during FPS is, $P_{w_1} = \rho_1\alpha_1\psi_1^{(1)}$*;*

(6). the probability that the server is waiting for repair during SPS is, $P_{w_2} = p\rho_2\alpha_2\psi_2^{(1)}$*;*

(7). the probability that the server is waiting for repair during TPS is, $P_{w_3} = p\rho_3\alpha_3\psi_3^{(1)}$*;*

(8). the probability that the server is under repair during FPS is, $P_{S_1} = \rho_1\alpha_1 g_1^{(1)}$*;*

(9). the probability that the server is under repair during SPS is, $P_{S_2} = p\alpha_2\rho_2 g_2^{(1)}$*;*

(10). the probability that the server is under repair during TPS is, $P_{S_3} = p\alpha_3\rho_3 g_3^{(1)}$*.*

Proof. Here we have

$$P_{w_i} = \lim_{z\to 1} Q_i(z), P_{C_i} = \lim_{z\to 1} P_i(z), P_{S_i} = \lim_{z\to 1} S_i(z), \quad for \;\; i = 1, 2, 3 \;\; and \;\; P_I = 1 - \sum_{i=1}^{3}\{P_{C_i} + P_{S_i} + P_{w_i}\}$$

The stated formulae follow by direct calculation. □

Finally, the derivation of the stationary queue size distribution at a departure epoch of this model is given in the proof of Theorem 3.3.

Theorem 3.5. *Under the steady-state condition, the PGF of the stationary queue size at a departure epoch of this model is given by*

$$\pi(z) = \frac{(1-\rho_H)(1-z)\{q + pH_2^*(\omega(z))\}H_1^*(\omega(z))}{[\{q + pH_2^*(\omega(z))\}H_1^*(\omega(z)) - z]} \tag{52}$$

Proof. Following the argument of PASTA (see Wolf [25]). We state that a departing customer will see 'j' customer in the queue just after a departure if and only if there were 'j' customer in the queue TPS or a vacation just before the departure. Now denoting $\{\pi_j : j \geq 0\}$ as the probability that there are j units in the queue at a departure epoch, then for $j \geq 0$ we may write.

$$\pi_j = K_0 q\int_0^\infty \mu_1(x)P_{1,j}(x)dx + K_0 q\int_0^\infty \mu_2(x)P_{2,j}(x) + K_0\int_0^\infty \mu_3(x)P_{3,j}(x)dx \tag{53}$$

Where K_0 is the normalizing constant. Now multiplying both sides of Equation (53) by z^j and then taking summation over $j \geq 0$ and utilizing equations (23) and (24), we get on simplification.

$$\pi(z) = \frac{K_0 U_{0,0}\omega(z)\{q + pH_2^*(\omega(z))\}H_1^*(\omega(z))}{[\{q + pH_2^*(\omega(z))\}H_1^*(\omega(z)) - z]} \tag{54}$$

Utilizing normalizing condition $\pi(1) = 1$, we get $K_0 = \frac{(1-\rho_H)}{\lambda U_{0,0}}$. Hence formula (52) follows by inserting (55) in (54). □

Next the mean queue size of this model is given in Corollary 3.2

Corollary 3.6. *Under the stability conditions, the mean number of customers in the system (i.e. mean queue length)* $E[M_Q(t)]$ *is given by*

$$\begin{aligned}E[M_Q(t)] = \rho_H &+ \frac{p\rho_1\rho_2\rho_3[1+\alpha_1(\Psi_1^{(1)}+g_1^{(1)})][1+\alpha_2(\Psi_2^{(1)}+g_2^{(1)})][1+\alpha_3(\Psi_3^{(1)}+g_3^{(1)})]}{(1-\rho_H)} \\ &+ \frac{\lambda[\alpha_1\rho_1\{\Psi_1^{(2)}+g_1^{(2)}+2g_1^{(1)}\Psi_1^{(1)}\}+p\alpha_2\rho_2\{\Psi_2^{(2)}+g_2^{(2)}+2g_2^{(1)}\Psi_2^{(1)}\}+p\alpha_3\rho_3\{\Psi_3^{(2)}+g_3^{(2)}+2g_3^{(1)}\Psi_3^{(1)}\}]}{2(1-\rho_H)} \\ &+ \frac{\lambda[\rho_1[1+\alpha_1(\Psi_1^{(1)}+g_1^{(1)})]^2\beta_s^{(1)}+p\rho_2[1+\alpha_2(\Psi_2^{(1)}+g_2^{(1)})]^2\beta_s^{(2)}+p\rho_3[1+\alpha_3(\Psi_3^{(1)}+g_3^{(1)})]^2\beta_s^{(3)}]}{2(1-\rho_H)} \end{aligned} \tag{55}$$

Where $\beta_S^{(i)} = \frac{\beta_i^{(2)}}{2\beta_i^{(1)}}$ *is the residual service time of ith phase of service for* $i = 1, 2, 3$.

Proof. The result follows directly by differentiating Equation (52) with respect to z and then taking limit $z \to 1$ by using the L-hospital rule. □

4. Busy Period Distribution and Waiting Time Distribution

In this section we provide main results for busy period distribution and waiting time distribution of this model. Since the derivation of busy period distribution is standard and it follows from existing literature of classical $M/G/1$ queue hence present the result without derivation in Theorem 4.1. Now we define 'T_B' as length of time interval that makes the server busy and this continues to the instant when the system becomes empty.

Theorem 4.1. *Let* $T_B^*(\theta) = E[e^{-\theta T_B}]$ *be the LST of* T_B. *Then Taka'cs functional equation under the steady state condition is given by*

$$T_B^*(\theta) = H^*(\theta + \lambda(1 - T_B^*(\theta)))$$

Where $H^*(\theta) = [q + p\beta_2^*\{\omega(z) + \alpha_2(1 - G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}]\beta_1^*\{\omega(z) + \alpha_1(1 - G_1^*(\omega(z))\Psi_1^*(\omega(z)))$.

The mean busy period is found to be

$$E(T_B) = \frac{\beta_1^{(1)}[1+\alpha_1(\Psi_1^{(1)}+g_1^{(1)})]}{(1-\rho_H)} + \frac{p\beta_2^{(1)}[1+\alpha_2(\Psi_2^{(1)}+g_2^{(1)})]}{(1-\rho_H)} + \frac{p\beta_3^{(1)}[1+\alpha_3(\Psi_3^{(1)}+g_3^{(1)})]}{(1-\rho_H)}$$

Similarly, the waiting time distribution of a test customer for our model has the following LST.

Theorem 4.2. *Let* $W_Q^*(\theta)$ *be the LST of the waiting time distribution of a test customer for this model under steady state condition, then*

$$W_Q^*(\theta) = \frac{\theta(1-\rho_H)[q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\}]\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))}{\theta-\lambda[1-(q+p\beta_2^*\{\omega(z)+\alpha_2(1-G_2^*(\omega(z))\Psi_2^*(\omega(z)))\})\beta_1^*\{\omega(z)+\alpha_1(1-G_1^*(\omega(z))\Psi_1^*(\omega(z)))]} \tag{56}$$

$$\text{and} \quad E[W_Q] = \frac{E[M_Q(t)]}{\lambda} \tag{57}$$

Proof. The results follows directly from formula (52) by utilizing relationship (see Takagi [26]);

$$W_Q^*(\lambda - \lambda z) = \pi(z). \tag{58}$$

Now setting $\lambda - \lambda z = \theta$ in eq. (58) and utilizing Eqution (52), we get (56). Similarly formula (57) follows directly by routine differentiation in (56) with respect to θ and then taking limit $\theta \to 0$ by using the L'Hospital's rule. □

5. Reliability Analysis

Our final goal is to derive some reliability indices of this model. Now we will discuss two reliability indices of the system viz. – the system availability and failure frequency under the steady state conditions. Suppose that the system is initially empty. Let $A_E(t)$ be the point wise availability of the server at time $'t'$that is, the probability that the server is either serving a customer or the server is available if the server is free and up during an idle period, such that the steady state availability of the server will be

$$A_E = \lim_{t\to\infty} A_E(t).$$

Theorem 5.1. *The steady state availability of the server is given by.*

$$A_E = 1 - \rho_1\alpha_1(\Psi_1^{(1)} + g_1^{(1)}) - p\rho_2\alpha_2(\Psi_2^{(1)} + g_2^{(1)}) - p\rho_3\alpha_3(\Psi_3^{(1)} + g_3^{(1)}) \tag{59}$$

Proof. The result follows directly from Theorem (2) by considering the following equation.

$$A = U_{0,0} + \sum_{i=1}^{3}\int_0^\infty P_i(x,1)dx = U_{0,0} + \lim_{z\to 1}[P_1(z) + P_2(z) + P_3(z)].$$

By using (31), (41) (42) and (43) we get (59). □

Theorem 5.2. *The steady state failure frequency of the server is given by.*

$$M_f = \alpha_1\rho_1 + p\alpha_2\rho_2 + p\alpha_3\rho_3 \tag{60}$$

Proof. The result follows directly from equation (28) and (29) by utilizing the argument of Li et. Al. [16]

$$M_f = \alpha_1\int_0^\infty P_1(x,1)dx + \alpha_2\int_0^\infty P_2(x,1)dx + \alpha_3\int_0^\infty P_3(x,1)dx.$$

Now since $\int_0^\infty[1 - C_{i\_}(x)]dx = \int_0^\infty xdC_i(x) = \beta_i^{(1)}$; for $i = 1,2,3$; therefore from Equation (28), (29) and (30) we have (60). □

6. Concluding Remarks

In this paper we are studying M/G/1 queueing system where arrival of customers are serving specific characteristic according to which - each customer requires three successive phases of service whereas the server is unreliable and it can may breakdown during any phase of serving service. After breakdown when the server is sent for repair then customers have to wait for the repairing of server. This waiting time refers to delay time / delay repair. The obtained results are the following - the probability generating function of the joint distributions of the server state and queue size, the queue size distribution at the departure epoch, waiting time distribution, busy period distribution, the system availability, the failure frequency and the Laplace transform of the system reliability function. Supplementary variable method has been applied to obtain results. We can also analysis this using embedded Markov chain technique and Markov renewal process. This study can be complemented in various ways by introducing concepts of new vacation policies like modified vacation policy, work vacation policy etc. Further present model can be generalized for the arrival process to the case of a compound Poisson process.

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