International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **13(2)**(2025), 151–165
Available Online: http://ijmaa.in

Solution of Fractional Differential Equation using the Modified Fractional Differential Transform Method

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Abstract

In this paper, the Fractional Differential Transform Method is modified by using the Caputo-Hadamard fractional derivative operator, the modified method results are compared to several existing problems. New examples are also constructed. The results of the existing problems comparison shows that the modified fractional differential transform method is efficient in approximating solution of fractional differential equations.

Keywords: Fractional Differential Equation; Caputo-Hadamard fractional derivative operator; Modified Fractional Differential Transform Method.

1. Introduction

In recent years fractional differential equations are widely studied by Engineers, scientist and social sciences.The increase in computer capability and its efficiency is one of the catalyst for this current development from the field of fractional Calculus. As we model real life problems, often times, the solutions of these models are needed for the better understanding and interpretation of these models.Various solution methods are studied, some of the solution method may be analytic, semi analytic or numerical methods. The most widely commonly used classical solution techniques for differential equations include: the Mellin transform method [1], the fractional Green's Function method [2], Orthogonal polynomial method [3], Laplace transform method [4] and the Furrier transform method [5]. Further more, there are several other solution method for solving fractional differential equations. The most frequently used technique are: Variational Iteration method [7], power series method [7], Adomian decomposition method [8], Fractional differential transform method [9], Collocation shooting method [10]. One of the most famous fractional integral order operator is the one developed by Riemman Liouville, his definition was born out of the generalization

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of the usual Riemman integral

$$\int_a^t g(x)dx.$$

The left and right Riemman Liouville fractional integral operator are defined respectively as:

$$
\begin{aligned}
{}_aI^{\beta}g(t) &= \frac{1}{\Gamma(\beta)}\int_a^t (t-\lambda)^{\beta-1}g(\lambda)d\lambda,\\
I_b^{\beta}g(t) &= \frac{1}{\Gamma(\beta)}\int_t^b (\lambda-t)^{\beta-1}g(\lambda)d\lambda,
\end{aligned}
$$

where, $\beta > 0$ $m-1 < \beta < m$, $\Gamma(\beta)$ represent the gamma function. The left and right fractional derivative are defined

$$
\begin{aligned}
{}_aD^{\beta}g(t) &= \frac{1}{\Gamma(m-\beta)}\left(\frac{d}{dt}\right)^m \int_a^t (t-\lambda)^{m-\beta-1}g(\lambda)d\lambda,\\
D_b^{\beta}g(t) &= \frac{1}{\Gamma(m-\beta)}\left(\frac{d}{dt}\right)^m \int_t^b (\lambda-t)^{m-\beta-1}g(\lambda)d\lambda,
\end{aligned}
$$

The derivative of a constant K is defined as:

$${}_aD^{\beta}K = K\frac{(t-a)^{\beta}}{\Gamma(1-\beta)},$$

and the derivative of a power function is given by

$${}_aD^{\beta}(t-a)^{\gamma} = \frac{\Gamma(\beta+1)(t-a)^{\gamma-\beta}}{\Gamma(\gamma-\beta+1)},$$

for $\gamma > -1$, $\beta \geqslant 0$. In handling the real world problems, the initial conditions interpretations using the Riemman Liouville fractional derivative operator poses a significant challenge, to over come this impediment, Michele Caputo proposed the left and right fractional derivative which the initial conditions are similar to the classical derivatives. The left and right Caputo fractional derivative are defined as follows:

$$
\begin{aligned}
{}_a^CD^{\beta}g(t) &= \frac{1}{\Gamma(m-\beta)}\int_a^t (t-\lambda)^{m-\beta-1}\left(\frac{d}{dt}\right)^m g(\lambda)d\lambda,\\
{}^CD_b^{\beta}g(t) &= \frac{1}{\Gamma(m-\beta)}\int_t^b (\lambda-t)^{m-\beta-1}\left(\frac{d}{dt}\right)^m g(\lambda)d\lambda,
\end{aligned}
$$

where, β represent the order of the derivative $m-1 < \beta < m$. The Riemman Liouville fractional derivative and the Caputo fractional derivatives are related by the following formulae.

$${}_a^CD^{\beta}g(t) = {}_aD^{\beta}\left(g(t) - \sum_{i=0}^{m-1}\frac{g^i(a)}{\Gamma(i-\beta+1)}(t-a)^{i-\beta}\right)$$

$$ {}^{C}D_b^{\beta} g(t) = D_b^{\beta}\left(g(t) - \sum_{i=0}^{m-1} \frac{(-1)^i g^i(b)}{\Gamma(i-\beta+1)}(b-t)^{i-\beta} \right) $$

The fractional power $\left(\frac{d}{dt}\right)^{\beta}$ of differentiation $\frac{d}{dt}$ is the Riemman Liouville fractional derivative and is invariant with respect to translation on the whole axis [1]. Hardamard gave a modification to the fractional power as follows

$$ \left(t\frac{d}{dt} \right)^{\beta} $$

Hardamard definition was based on the n$^{\text{th}}$ integral generalization.

$$ {}_aI^m(t) = \int_a^t \frac{dx_1}{x_1} \int_a^{t_1} \frac{dx_2}{x_2} \cdots \int_a^{t_{m-1}} g(t_m)\frac{dx_m}{x_m} = \frac{1}{(m-1)!}\int_a^x \left(log\frac{t}{x} \right)^{m-1} g(x)\frac{dx}{x} $$

The left and right Hardamard fractional integrals of order $\beta \in \mathcal{C}$, $\mathcal{R}(\beta) > 0$ are defined respectively as follow:

$$ {}_aJ^{\beta} g(t) = \frac{1}{\Gamma(\beta)}\int_a^t \left(log\frac{t}{x} \right)^{\beta-1} g(x)\frac{dx}{x}, $$
$$ J_b^{\beta} g(t) = \frac{1}{\Gamma(\beta)}\int_t^b \left(log\frac{x}{t} \right)^{\beta-1} g(x)\frac{dx}{x}. $$

The Hardamard fractional derivatives are defined as follow:

$$ {}_aD^{\beta} g(t) = \delta^m \left({}_aJ^{m-\beta} g \right)(t) = \left(t\frac{d}{dt} \right)^m \frac{1}{\Gamma(m-\beta)} \int_a^t \left(log\frac{t}{x} \right)^{m-\beta-1} g(x)\frac{dx}{x}, $$

and

$$ D_b^{\beta} g(t) = \left(-\delta \right)^m \left(J_b^{m-\beta} g \right)(t) = \left(-t\frac{d}{dt} \right)^m \frac{1}{\Gamma(m-\beta)} \int_t^b \left(log\frac{x}{t} \right)^{m-\beta-1} g(x)\frac{dx}{x}, $$

where, $\delta = t\frac{d}{dt}$, $\delta^0 = g(t)$, $\beta \in \mathbb{C}$, $\mathbb{R}(\beta) \geqslant 0$.

The Caputo-type Hardamard fractional derivative C-HFrD: Let $\mathbb{R}(\beta) \geqslant 0$ and $m = \lfloor \mathbb{R}(\beta) \rfloor + 1$, if $g(t) \in AC_{\delta}^m[a,b]$, where $0 < a < b < \infty$ and $AC_{\delta}^m[a,b] = \left\{ f : [a,b] \to \mathbf{C}, \delta^{m-1} f(t) \in AC[a,b], \delta = t\frac{d}{dt} \right\}$. The left and right Caputo- type Hardamard fractional derivative (C-HFrD) is defined respectively as follow:

$$ {}_a^CD^{\beta} g(t) = {}_aD^{\beta}\left[g(t) - \sum_{i=0}^{m-1} \frac{\delta^i g(a)}{i!}\left(log\frac{x}{a} \right)^i \right](t) \tag{1} $$

and

$$ {}^{C}D_b^{\beta} g(t) = {}_aD^{\beta}\left[g(t) - \sum_{i=0}^{m-1} \frac{(-1)^i\delta^i g(b)}{i!}\left(log\frac{b}{x} \right)^i \right](t) \tag{2} $$

The Caputo-type Hardamard fractional derivatives have some group properties: Let $\mathbb{R}(\beta) \geqslant 0$, $\lfloor \mathbb{R}(\beta) \rfloor + 1$ and $\mathbb{R}(\gamma) > 0$, then

$$
\begin{aligned}
{}_a^C D^{\beta} \left(log \frac{t}{a} \right)^{\gamma-1} &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)} \left(log \frac{t}{a} \right)^{\gamma-\beta-1}, \\
{}^C D_b^{\beta} \left(log \frac{b}{t} \right)^{\gamma-1} &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)} \left(log \frac{b}{t} \right)^{\gamma-\beta-1},
\end{aligned}
$$

where, $\mathbb{R}(\beta) > m$.

$$
\begin{aligned}
{}_a^C D^{\beta} \left(log \frac{t}{a} \right)^{k} &= 0 \\
{}^C D_b^{\beta} \left(log \frac{b}{t} \right)^{k} &= 0
\end{aligned}
$$

for $K = 0, 1, 2, \ldots, m-1$.

$$
{}^C D_{a^+}^{\gamma} \, {}^C D_{a^+}^{\beta} g(t) = {}^C D_{a^+}^{\gamma+\beta} g(t)
$$

for detail of the proofs see [1] and [2].

Various solutions technics are studied for example [3] implemented a well known transform technique, the Differential Transform method, and applied it to the area of fractional Calculus, particularly differential equations. Latter, [4] utilize the method to solve system of differential equations. Arising from the forgoing we modified the fractional differential transform method proposed by [3] by using the Caputo-type Hadamard fractional derivative operator.

The paper is organized as follows: In section I we give the introduction and some definitions and basic properties of Hadamard fractional derivative operator. Section II is dedicated for the derivation of the modified Fractional Differential Transform Method (MFrDTM). In section III we applied the derived method to solve existing problems and new constructed examples. The summary and conclusions are given in section IV.

2. The Modified Fractional Differential Transform Method

The generalisation of the concept of differentiation to fractional orders are approached in different ways. The Hadamard fractional derivatives to order β of a function $g(t)$ with respect to t with constant of differentiation t_0 is defined for a general $\beta \in \mathbb{C}$ for the non-negative real part is as follows:

$$
{}_{t_0}D^{\beta} g(t) = \left(t \frac{d}{dt} \right)^{m} \frac{1}{\Gamma(m-\beta)} \left[\int_{t_0}^{t} \left(log \frac{t}{x} \right)^{m-\beta-1} g(x) \frac{dx}{x} \right] \tag{3}
$$

$$
{}_{t_0}D^{\beta} g(t) = \frac{1}{\Gamma(m-\beta)} \left(t \frac{d}{dt} \right)^{m} \left[\int_{t_0}^{t} \frac{1}{x} \frac{g(x)}{log(t-x)}^{1+\beta-m} dx \right] \tag{4}
$$

for $m-1 \leqslant \beta < m$, $m \in \mathbb{Z}^+$, $t > t_0$. We expand the analytical and continuous function $g(t)$ in terms of a fractional power series as follows:

$$g(t) = \sum_{i=0}^{\infty} F(i)(t-t_0)\frac{i}{\beta} \tag{5}$$

where β is the fractional order and $F(i)$ is the fractional differential transform of $g(t)$. In most cases, the practical applications of fractional differential equations in different branches of science and engineering using some operators appear to difficult, due to the problem of the initial values. The fractional initial conditions are frequently not available, and it may not be clear what there physical meanings are. For ease of understanding, the behaviour of the model problem and efficient application, the definition in equation (**??**) shall be modified to handle the integral ordered initial conditions in the sense of Caputo-type Hadamard as follows.

$$D_{t_0}^{\beta}\left[g(t) - \sum_{i}^{m-1}\frac{1}{i!}(t-t_0)^i g^{(i)}(0)\right] = \frac{1}{\Gamma(m-\beta)}\left(t\frac{d}{dt}\right)^m \cdot \left\{\int_0^t \left[g(t) - \sum_{i=0}^{m-1}\frac{1}{i!}\frac{(x-t_0)^i g^{(i)}(0)}{log\left(\frac{t}{x}\right)^{1+\beta-m}}\right]dx\right\} \tag{6}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follow.

$$G(i) = \begin{cases} if \quad \frac{i}{\beta} \in \mathbb{Z}^+ \frac{1}{(\frac{i}{\beta})!}\left[\frac{d^{\frac{i}{\beta}} g(t)}{dt^{\frac{i}{\beta}}}\right]_{t=t_0} & for \quad i = 0, 1, \ldots, (m\beta - 1) \\ if \quad \frac{i}{\beta} \in \mathbb{Z}^+ \quad for \quad 0 & \end{cases} \tag{7}$$

where m is the order of the fraction differential equations (FrDE). Applying equation (3) and (5), the theorems of fractional Differential Transform (Frc-DT) are stated below.

Theorem 2.1. *If*

$$g(t) = f(t) \pm q(t) \quad then, \tag{8}$$

$$G(i) = F(i) \pm Q(i) \tag{9}$$

Proof.

$$G(i) = \sum_{i=0}^{\infty} F(i)\left(t-t_0\right)^{\frac{i}{\beta}} \pm \sum_{i=0}^{\infty} Q(i)\left(t-t_0\right)^{\frac{i}{\beta}},$$

$$G(i) = \sum_{i=0}^{\infty}\left[F(i) \pm Q(i)\right]\left(t-t_0\right)^{\frac{i}{\beta}}. \tag{10}$$

Using the definition of the transform in (5), the result follows

$$G(i) = F(i) \pm Q(i)$$

□

Theorem 2.2. *If*

$$g(t) = f(t)q(t) \quad then, \tag{11}$$

$$G(i) = \sum_{r=0}^{i} F(r)Q(i-r) \tag{12}$$

Proof.

$$\begin{aligned}
G(t) &= \sum_{i=0}^{\infty} F(i)\left(t-t_0\right)^{\frac{i}{\beta}} \times \sum_{i=0}^{\infty} Q(i)\left(t-t_0\right)^{\frac{i}{\beta}}, \\
&= \left[F(0) + F(1)(t-t_0)^{\frac{1}{\beta}}F(2)(t-t_0)^{\frac{2}{\beta}} + \cdots + F(m)(t-t_0)^{\frac{m}{\beta}}\right] \\
&\times \left[Q(0) + Q(1)(t-t_0)^{\frac{1}{\beta}}Q(2)(t-t_0)^{\frac{2}{\beta}} + \cdots + Q(m)(t-t_0)^{\frac{n}{\beta}}\right] \\
&= [F(0)Q(0)] + \left[F(0)Q(1) + F(1)Q(0)\right](t-t_0)^{\frac{1}{\beta}} + \left[F(0)Q(2) + F(1)Q(1) + F(2)Q(0)\right](t-t_0)^{\frac{2}{\beta}} + \cdots \\
&+ \left[F(0)Q(m) + F(1)Q(m-1) + \cdots + F(m-1)Q(1) + F(m)Q(0)\right](t-t_0)^{\frac{m}{\beta}}
\end{aligned}$$

we have in general as follows

$$g(t) = \sum_{i=0}^{\infty}\sum_{r=0}^{i} F(r)Q(i-r)(t-t_0)^{\frac{i}{\beta}}$$

Following from the definition of Differential transform, the result follows

$$G(i) = \sum_{r=0}^{i} F(r)Q(i-r)$$

□

Theorem 2.3. *If*

$$\begin{aligned}
g(t) &= f_1(t)f_2(t)\cdots f_{m-1}(t)f_m(t), \quad then \\
G(i) &= \sum_{i_{m-1=0}}^{i}\sum_{i_{m-2}=0}^{i_{m-1}}\cdots\sum_{i_2}^{i_3}\sum_{i_1}^{i_2} F_1(i_1)F_2(i_2-i_1)\cdots F_{m-1}(i_{m_1}-i_{m-2})F_m(i-i_{m-1})
\end{aligned}$$

Proof. We can write the form of $g(t)$ by using power series expansion of $f_1(t), f_2(t), \cdots, f_m(t)$

$$g(t) = \sum_{i=0}^{\infty} F_1(i)(t-t_0)^{\frac{i}{\beta}} F_2(i)(t-t_0)^{\frac{i}{\beta}} \cdots \sum_{i=0}^{\infty} F_{m-1}(i)(t-t_0)^{\frac{i}{\beta}} \times \sum_{m}^{\infty} F_m(i)(t-t_0)^{\frac{i}{\beta}}$$

$$\Big[F_1(0) + F_1(1)(t-t_0)^{\frac{1}{\beta}} F_1(2)(t-t_0)^{\frac{2}{\beta}} + \cdots\Big]\Big[F_2(0) + F_2(1)(t-t_0)^{\frac{1}{\beta}} F_2(2)(t-t_0)^{\frac{2}{\beta}} + \cdots\Big] \cdots$$

$$\Big[F_{m-1}(0) + F_{m-1}(1)(t-t_0)^{\frac{1}{\beta}} F_{m-1}(2)(t-t_0)^{\frac{2}{\beta}} + \cdots\Big]\Big[F_m(0) + F_m(1)(t-t_0)^{\frac{1}{\beta}} F_m(2)(t-t_0)^{\frac{2}{\beta}} + \cdots\Big].$$

$$= \Big[F_1(0)F_2(0)\cdots F_{m-1}(0)F_m(0)\Big] + \Big[F_1(0)F_2(0)\cdots F_{m-1}(0)F_m(0) + F_1(0)F_2(1)\cdots F_{m-1}(0)F_m(0) + \cdots$$

$$+ F_1(0)F_2(0)\cdots F_{m-1}(1)F_m(1) + F_1(0)F_2(0)\cdots F_{m-1}(0)F_m(1)\Big](t-t_0)^{\frac{1}{\beta}}\Big[F_1(1)F_2(1)F_3(0)\cdots F_m(0)$$

$$+ F_1(1)F_2(0)F_3(01\cdots F_m(0) + \cdots + F_1(1)F_2(0)F_3(0)\cdots F_m(1) + F_1(0)F_2(1)F_3(1)\cdots F_m(0)$$

$$+ \cdots + F_1(0)F_2(1)F_3(0)\cdots F_m(1) + \cdots + F_1(0)F_2(0)\cdots F_{m-1}(1)F_m(1)F_1(1)F_2(0)\cdots$$

$$F_{m-1}(0)F_m(0) + \cdots + F_1(0)F_2(0)\cdots F_{m-1}(0)F_m(2)\Big](t-t_0)^{\frac{2}{\beta}} + \cdots$$

In general form we get,

$$g(t) = \sum_{i_{m-1}=0}^{i} \sum_{i_{m-2}=0}^{i_{m-1}} \cdots \sum_{i_2}^{i_3} \sum_{i_1}^{i_2} F_1(i_1)F_2(i_2-i_1)\cdots F_{m-1}(i_{m_1}-i_{m-2})F_m(i-i_{m-1})(t-t_0)^{\frac{i}{\beta}},$$

applying the definition of differential transform in equation (3) the results follows,

$$g(t) = \sum_{i_{m-1}=0}^{i} \sum_{i_{m-2}=0}^{i_{m-1}} \cdots \sum_{i_2}^{i_3} \sum_{i_1}^{i_2} F_1(i_1)F_2(i_2-i_1)\cdots F_{m-1}(i_{m_1}-i_{m-2})F_m(i-i_{m-1}).$$

□

Theorem 2.4. *If $g(t) = (t-t_0)^h$, then $G(i) = \delta(i-\beta h)$, where*

$$\delta(i) = \begin{cases} 1 \;\; if \;\; i = 0 \\ 0 \;\; if \;\; i \notin 0 \end{cases}$$

Proof. The expansion in terms of dirac-delta function is written for $g(t)$ as

$$g(t) = \sum_{i=0}^{\infty} \delta(i-\beta h)(t-t_0)^{\frac{i}{\beta}}$$

by the definition of transform we obtained expression $G(i) = \delta(i-\beta h)$. □

Theorem 2.5. *If $g(t) = D_{t_0}^{\alpha}[f(t)]$, then*

$$F(i) = \frac{\Gamma(\alpha + 1 + \frac{i}{\beta})}{\Gamma(1 - m + \frac{i}{\beta})} G(i + \beta\alpha)$$

Proof. By making use of equation (6), the Caputo-type Hadamard sense fractional differentiation of a function f(t) can be written as

$$D_{t_0}^{\alpha}\Big[f(t)\Big] = \frac{1}{\Gamma(m-\alpha)}\left(t\frac{d}{dt}\right)^m \left\{ \left[f(t) - \frac{\sum_{i=0}^{m-1}\frac{1}{i!}(x-t_0)^i f^{(i)}(0)}{log\left(\frac{t}{x}\right)^{1+\alpha-m}} \right] dx \right\} \tag{13}$$

Using equation (5) and (7), we have

$$D_{t_0}^{\alpha}\Big[f(t)\Big] = \frac{1}{\Gamma(m-\alpha)}\left(t\frac{d}{dt}\right)^m \left[\int_{t_0}^{t} \frac{\sum_{i=0}^{m-1} G(i)(x-t_0)^{\frac{i}{\beta}}}{log\left(\frac{t}{x}\right)^{1+\alpha-m}} dx \right]$$

$$D_{t_0}^{\alpha}\Big[f(t)\Big] = \frac{1}{\Gamma(m-\alpha)} \sum_{i=\alpha\beta}^{\infty} G(i) \left(t\frac{d}{dt}\right)^m \left[\int_{t_0}^{t} \frac{(x-t_0)^{\frac{i}{\beta}}}{log\left(\frac{t}{x}\right)^{1+\alpha-m}} dx \right] \tag{14}$$

we defined the Hadamard integral J_H^{β} of function f by

$$J_H^{\beta} g(t) = \frac{1}{\beta}\int_{t_0}^{t} \left(log\frac{t}{x}\right)^{\beta-1} g(x)\frac{dx}{x}, \quad \beta > 0$$

from equation (14), the integrand is

$$\frac{(x-t_0)^{\frac{i}{\beta}}}{log\left(\frac{t}{x}\right)^{1+\alpha-m}} = \frac{(x-t_0)^{\omega}}{\left[log(\frac{t}{x})\right]^{1+\alpha-m}}, \quad (\omega = \frac{i}{\beta})$$

hence, from Hadamard fractional calculus theory, we have

$$\left(t\frac{d}{dt}\right)^m \left[\int_{t_0}^{t} \frac{(x-t_0)^{\omega}}{\left[\log(\frac{t}{x})\right]^{1+\alpha-m}} dx \right] = \Gamma(m-\alpha)\frac{\Gamma(\omega+1)}{\Gamma(\omega-(m-\alpha)+1)}(t-t_0)^{\omega-(m-\alpha)} \tag{15}$$

substituting (15) into (13) we have

$$= \frac{1}{\Gamma(m-\alpha)} \sum_{i=\alpha\beta}^{\infty} G(i) \frac{\Gamma(m-\alpha)\Gamma(\frac{i}{\beta}+1)}{\Gamma(\frac{i}{\beta}-m+\alpha+1)}(t-t_0)^{\frac{i}{\beta}-m+\alpha} \tag{16}$$

simplifying (16) we have

$$\sum_{i=\alpha\beta}^{\infty} \frac{\Gamma(\frac{i}{\beta}+1)}{\Gamma(\frac{i}{\beta}-m+\alpha+1)} G(i)(t-t_0)^{\frac{i}{\beta}-m+\alpha} \tag{17}$$

starting the index of the series from $i-0$, we obtained the equation,

$$f(t)=\sum_{i=0}^{\infty}\frac{\Gamma(\alpha+\frac{i}{\beta}+1)}{\Gamma(\frac{i}{\beta}-m+1)}G(i+\beta\alpha)(t-t_0)^{\frac{i}{\beta}}, \tag{18}$$

from the definition of transform in equation (5) we have the expression

$$F(i)=\frac{\Gamma(\alpha+1+\frac{i}{\beta})}{\Gamma(1-m+\frac{i}{\beta})}G(i+\beta\alpha)$$

□

Theorem 2.6. *The most general form of fractional derivatives are produce if*

$$g(t)=\frac{d^{\alpha_1}}{dt^{\alpha_1}}\left[f_1(t)\right]\frac{d^{\alpha_2}}{dt^{\alpha_2}}\left[f_2(t)\right]\cdots\frac{d^{\alpha_{m-1}}}{dt^{\alpha_{m-1}}}\left[f_{m-1}(t)\right]\frac{d^{\alpha_m}}{dt^{\alpha_m}}\left[f_m(t)\right]$$

then,

$$G(i)=\sum_{i_{m-1}=0}^{i}\sum_{i_{m-2}=0}^{i_{m-1}}\cdots\sum_{i_2=0}^{i_3}\sum_{i_1-0}^{i_2}\frac{\Gamma[\alpha_1+1+\frac{i}{\beta}]}{\Gamma[1+\frac{i}{\beta}-m]}\frac{\Gamma[\alpha_2+1+\frac{(i_2-i_1)}{\beta}]}{\Gamma[1+\frac{(i_2-i_1)}{\beta}-m]}\cdots\frac{\Gamma[\alpha_{m-1}+1+\frac{(i_{m-1}-i_{m-2})}{\beta}]}{\Gamma[1+\frac{(i_{m-1}-i_{m-2})}{\beta}-m]}$$
$$\frac{\Gamma[\alpha_m+1+\frac{(i-i_{m-1})}{\beta}]}{\Gamma[1+\frac{(i-i_{m-1})}{\beta}-m]}F_1(i_1+\beta\alpha_1)F_2((i_2-i_1)+\beta\alpha_2)\cdots F_{m-1}(i_{m-1}-i_{m-2})+\beta\alpha_{m-1})$$
$$.F_m(i_m-i_{m-1})+\beta\alpha_m)$$

where $\beta\alpha_j\in\mathbb{Z}^+$ *for* $j=1,2,\cdots,m$.

Proof. We let the differential transform (DTr) of $\frac{d^{\alpha j}}{dt^{\alpha_j}}\left[f_j(t)\right]$ be $\beta_j(i)$ at $t-t_0$ for $j-1,2,\cdots,m$, then by making use of Theorem (2.3) we have the modified fractional differential transform (MFrDTr) of g(t) as follows

$$G(i)=\sum_{i_{m-1}=0}^{i}\sum_{i_{m-2}=0}^{i_{m-1}}\cdots\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}B_1(i_1)B_2(i_2-i_1)\cdots B_{m-1}(i_{m-1}-i_{m-2})B_m(i-i_{m-1}) \tag{19}$$

and applying Theorem **??**, we can deduced that

$$B_1(i_1)=\frac{\Gamma[\alpha_1+\frac{i_1}{\beta}+1]}{\Gamma[\frac{i_1}{\beta}+1-m]}F_1(i_1+\beta\alpha)$$
$$B_2(i_2-i_1)=\frac{\Gamma[\alpha_2+\frac{i_2-i_1}{\beta}+1]}{\Gamma[\frac{i_2-i_1}{\beta}+1-m]}F_2(i_2-i_1+\beta\alpha)$$
$$B_{m-1}(i_{m-1}-i_{m-2})=\frac{\Gamma[\alpha_{m-1}+\frac{i_{m-1}-i_{m-2}}{\beta}+1]}{\Gamma[\frac{i_{m-1}-i_{m-1}}{\beta}+1-m]}F_{m-1}(i_{m-1}-i_{m-2}+\beta\alpha_{m-1})$$

$$B_m(i-i_{m-1}) = \frac{\Gamma[\alpha_m + \frac{i-i_{m-1}}{\beta}+1]}{\Gamma[\frac{i-i_{m-1}}{\beta}+1-m]} F_{m-1}(i-i_{m-1}+\beta\alpha_m)$$

substituting the $B(j)$ for $j = 1, 2, \cdots, m$ into equation (19) we have

$$G(i) = \sum_{i_{m-1}=0}^{i} \sum_{i_{m-2}=0}^{i_{m-1}} \cdots \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{i_2} B_1(i_1)B_2(i_2-i_1)\cdots B_{m-1}(i_{m-1}-i_{m-2})B_m(i-i_{m-1}) \tag{20}$$

where $\beta\alpha_k \in \mathbb{Z}^+$ for $k = 1, 2, \cdots, m$ □

From our proved theorem one can see that Differential transform (DTrM) is sitting right inside Fractional Differential Transform Method (MFrDTrM) for the special case $\beta = 1$.

3. Numerical Examples

Under this section, we construct Examples to demonstrate the applicability and the efficiency of the modified method.

Example 3.1. *Consider the fractional differential equation [3,5,6]*

$$A_1\frac{d^2x(t)}{dt^2} + A_2\frac{d^{\frac{3}{2}}x(t)}{dt^{\frac{3}{2}}} + A_3X(t) = g(t)$$

where $g(t) = A_3(t+1)$ and the boundary conditions $x(0) - 1x^{'}(0) = 1$. We take the $\beta = 2$ we transform the boundary values using equation (7) as follows:

$$X(0) = x(0) = 0$$
$$X(2) = \frac{1}{1}! = 1$$
$$X(i) = 0, \ \ for \ \ i \in (0,2)$$

using Theorem 2.4 and 2.5, we find the recurrence relation,

$$A_1\frac{\Gamma(3+\frac{i}{2})}{\Gamma(1+\frac{i}{2})}X(i+4) + A_2\frac{\Gamma(\frac{5}{2}+\frac{i}{2})}{\Gamma(1+\frac{i}{2}-m)}X(i+3) + A_3X(i) = A_3\delta X(i) + A_3\delta(i-2)$$

making $X(i+4)$ the subject of formula we have

$$X(i+4) = \frac{\Gamma(1+\frac{i}{2})}{A_1\Gamma(3+\frac{i}{2})}\left[A_3\delta X(i) + A_3\delta(i-2) - A_2\frac{\Gamma(\frac{5}{2}+\frac{i}{2})}{\Gamma(1+\frac{i}{2}-m)}X(i+3)\right] \tag{21}$$

using the transformed boundary conditions and equation (23) and evaluating up to certain number of terms and *applying the inverse transform we found the $x(t)$ to be*

$$x(t) = \sum_{i=0}^{\infty} X(i)(t - t_0)^{\frac{i}{\beta}} = \sum_{i=0}^{\infty} X(i))t^{\frac{i}{\beta}}$$

$$x(t) = X(0)t^0 + X(2)t^{\frac{1}{2}} = t + 1$$

The result is exactly same with the work of [3,5] and [6].

Example 3.2. *Consider the non-homogeneous fractional differential equation*

$$B_1 \frac{d^2y}{dx^2} + B_2 \frac{d^\alpha y}{dx^\alpha} + B_3 y = \cos(x) \tag{22}$$

where $x > 0$, $0 < \alpha \leq 2$ with the initial conditions $u(0) = 0$ and $u^{'}(0) = 0$. Using equation (22), Theorem 2.4 and Theorem 2.5 we have the transform results

$$Y(i+4) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \delta(i - 4m) - \frac{\Gamma(3/2 + i/2)}{\Gamma(i/2 - 1)} X(i+1) - X(i) \tag{23}$$

We transform the initial conditions by using Theorem 2.4, $Y(0) = y(0), Y(2) = 1/1!y^{'}(0) = 0$. By applying equation (23) and the transform initial conditions we find $Y(i)$ and inverse transform is used to find the $y(x)$. The graph of the $y(x)$ is shown in Figure 1

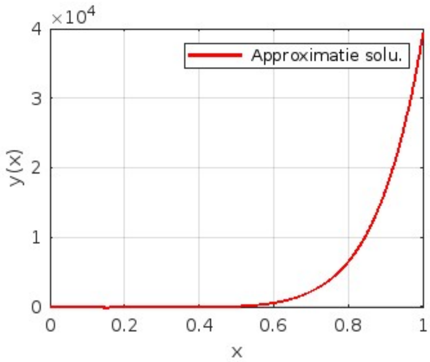

Figure 1: The graph of the approximate solution for Example 3.2

Example 3.3 ([3,7])**.** *The fractional ricatti equation that is frequently encountered in optimal control problem is consider,*

$$\frac{d^\alpha z}{dz^\alpha} - 2z - z^2 + 1, \qquad 0 < \alpha < 1, \tag{24}$$

$$z(0) = 0, \tag{25}$$

with the initial conditions $x(0) = 0$. Using Theorem 2.2, 2.4 and 2.5, equation (24) is transform,

$$Z(i + \alpha\beta) = \frac{\Gamma(\frac{i}{\alpha})}{\Gamma(\alpha + \frac{i}{\beta})} \left[2Z(i) - \sum_{i_1}^{i} Z(i)Z(i - i_1) + \delta(i) \right] \tag{26}$$

The initial condition in equation 25 is also transform using Theorem 2.4,

$$Z(i) = 0, \ \ for \ \ i = 0, 1, \cdots, \alpha\beta - 1 \tag{27}$$

We take the value of $\alpha = 2$ and $\beta - 1/2$ the transform equation (26) and (27) are evaluated and $Z(i)$ is also evaluated to certain number of i. Finally by using the inverse transform definition given equation (5), we find $z(x)$

$$z(x) = \frac{2}{\sqrt{\pi} x^{1/2}} + 2x + \frac{16(\pi - 1)}{3\pi^{3/2}} x^{3/2} + \frac{\pi - 4}{\pi} x^3 - \frac{32(3\pi^2 + 44\pi - 32)}{45\pi^{5/2}} x^{5/2} + \left(\frac{128}{9\pi^2} - \frac{71}{9\pi} - \frac{37}{4} \right) x^3 + \cdots$$

Using MATLAB software we graph the solution as shown below

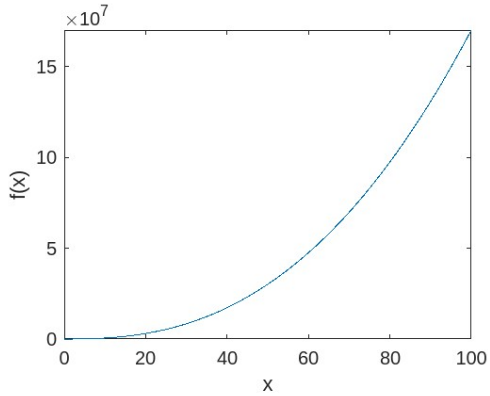

Figure 2: The graph of the approximate solution for Example 3.3

Example 3.4.

$$D^{\gamma} x = x^2 + 1, \ \ m - 1 < \gamma < m, \ \ 0 < t < 1 \tag{28}$$

$$x^k(0) = 0, \ \ k = 0 \cdots, m - 1 \tag{29}$$

We using Theorem 2.2 and 2.5 the non-linear fractional differential equation is transform.

$$X(k + \beta\gamma) = \frac{\Gamma(1 + -m + k/\beta)}{\Gamma(1 + \gamma + \frac{k}{\beta})} \left[\sum_{k_1=0}^{\infty} X(k - k_1) - \delta(k) \right] \tag{30}$$

by using Theorem 2.4, we transform the initial conditions as follows,

$$X(k) = 0, \quad for \quad k = 0, 1, \cdots, \gamma\beta - 1 \tag{31}$$

We obtained $X(k)$ for different values of γ by using equation (30) and equation (31). We find the inverse transform $x(t)$ by using (5), $x(t)$ is evaluated and the numerical results are shown in Table 1. This result is compared with other existing results given by the method of Adomian decomposition and Fractional differential transform.

Table 1: A table shown the approximation for Example 3.4

t	$\alpha = 1$			$\alpha = 1.5$			$\alpha = 2.5$		
	$u_h(ADM)$	$u_h(FDTM)$	$u_h(MFrTrM)$	$u_h(ADM)$	$u_h(FDTM)$	$u_h(MFrTrM)$	$u_h(ADM)$	$u_h(FDTM)$	$u_h(MFrTrM)$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.100335	0.100335	0.100335	0.023790	0.023790	0.023790	0.000952	0.000952	0.000946
0.2	0.202710	0.202710	0.202710	0.067330	0.067330	0.067330	0.005382	0.005383	0.005391
0.3	0.309336	0.309336	0.309336	0.123896	0.123896	0.123897	0.014833	0.014833	0.014833
0.4	0.422793	0.422793	0.422793	0.191362	0.191362	0.191362	0.030449	0.030450	0.030492
0.5	0.546302	0.546302	0.546302	0.268856	0.268856	0.268856	0.053196	0.053197	0.053197
0.6	0.684131	0.684137	0.684137	0.356238	0.356238	0.356229	0.083924	0.083924	0.083914
0.7	0.842245	0.842288	0.842288	0.453950	0.453950	0.453950	0.123412	0.123412	0.123412
0.8	0.029370	0.029639	0.029639	0.563007	0.563007	0.563006	0.172391	0.172391	0.172373
0.9	1.258800	1.260158	1.260158	0.685056	0.685056	0.685056	0.231574	0.231574	0.231563
1.0	1.551370	1.557406	1.557406	0.822511	0.822509	0.822507	0.301676	0.301676	0.301697

4. Summary and Conclusion

A fractional differential transform method for solving fractional differential equation is modified by using the Caputo type Hadamard fractional derivative operator. The modified fractional differential transform method (MFrTrM) is applied to existing problems in literature and new constructed examples. The results show a good agreement with results of some selected methods.

Acknowledgement

This work is supported by the department of mathematics, Federal University of Agriculture Makurdi, Benue state, Nigeria.

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