

Split Total Domination Number of Some Special Graphs

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Abstract: A dominating set for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D . The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G . In this paper a new parameter, Split Total Dominating Set S and the Split Total Domination Number $\gamma_{st}(G)$ has been introduced. A dominating set is called split total dominating set if $\langle V - D \rangle$ is disconnected and every vertex $v \in V$ is adjacent to an element of D . The split total domination number is given by $\gamma_{st}(G)$. In this paper the split total domination number for some standard graphs like star, path, cycle, complete, ladder, wheel, bistar, tadpole, comb, barbell, butterfly and fan graphs are found. Also the complement of graphs are obtained.

Keywords: Split Dominating Set, Total Dominating Set, Split Total Dominating Set.

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1. Introduction

Claude Berge was the first to introduce domination in graph theory and referred the domination number as the co-efficient of external stability. Oystein Ore introduced the terms dominating set and domination number in his book on graph theory which was published in 1962, and also introduced the concept of minimal and minimum dominating set in graphs. In 1977, Cockayne and Hedetniemi introduced the notation $\gamma(G)$ to denote the domination number. The study of split domination number was introduced by V. R. Kulli and B. Janagiram [5]. The study of total domination in graphs was first introduced by E. J. Cockayne, T. M. Dawes, S. T. Hedetniemi [2]. In this paper, a new parameter called the split total dominating sets and split total domination number $\gamma_{st}(G)$ has been introduced.

1.1. Graph theory terminology and concepts

In this paper, Finite, undirected, non-trivial, connected and simple graphs are used. Let G be a graph with vertex set V and edge set E . A graph is simple if it has neither self loop nor parallel edges. An undirected graph is a graph without any directions. The degree of a vertex v is denoted by $\deg(v)$. The maximum and minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$ or Δ and δ respectively. A vertex v of G is said to be a pendent vertex if and only if it has degree one and a vertex adjacent to pendent vertex is called support s . Let $S \subset V$ be any subset of vertices of a given graph G , then the induced subgraph $\langle S \rangle$ is a graph whose vertex set is S and whose edge set consists of all the edges in E that have both endpoints in S .

A set D is said to be a dominating set if every vertex $v \in V - D$ is adjacent to some vertex in D . The minimum cardinality of the dominating set is the domination number denoted as $\gamma(G)$. A dominating set D is said to be a split dominating set

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if $\langle V - D \rangle$ is disconnected. The split domination number is denoted by $\gamma_s(G)$. A set D of vertices in a graph G is called the total dominating set if every vertex $v \in V$ is adjacent to an element of D . The total domination number is denoted by $\gamma_t(G)$. In this paper the Split Total Domination number for some standard graphs like star graph, bistar, ladder graph, comb graph, barbell graph, butterfly graph, fan graph, tadpole graph, path, cycle, wheel graph and complete graph has been obtained. Also, the complement of graphs has been obtained.

2. Results

Theorem 2.1. *Let G be a star graph with n vertices then $\gamma_{st}(G) = 2$ for $n \geq 4$.*

Proof. Let G be a star graph with n vertices such that the vertex set $V = \{u, v_1, \dots, v_{n-1}\}$ where u is the centre vertex. We prove this theorem by induction hypothesis.

For $n = 4$, let G be a star graph with the vertex set $V = \{u, v_1, v_2, v_3\}$ where u is the centre vertex. If we take the dominating set D is $\{u\}$ then the induced graph $\langle V - D \rangle$ becomes totally disconnected graph. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 1$. But D does not satisfy the condition for total dominating set. Hence the neighbour vertex of u is added to D to satisfy the condition for total dominating set. Therefore, the set becomes $D = \{u, v_i\}$ where $1 \leq i \leq n - 1$ which implies $\gamma_t(G) = 2$. The split total domination set is not satisfied for $n = 4$ as the domination number differs. Hence by adding a vertex v_i , $1 \leq i \leq n - 1$, the split dominating set becomes $D = \{u, v_i\}$, $1 \leq i \leq n - 1$. Clearly, the split domination number becomes, $\gamma_s(G) = 2$. Therefore, the split total domination number for a star graph with $n = 4$ vertices is $\gamma_{st}(G) = 2$.

For $n = 5$, let G be a star graph with the vertex set $V = \{u, v_1, v_2, v_3, v_4\}$ where u is the centre vertex. If we take the dominating set D is $\{u\}$ then the induced graph $\langle V - D \rangle$ becomes totally disconnected graph. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 1$. But D does not satisfy the condition for total dominating set. Hence the neighbour vertex of u is added to D to satisfy the condition for total dominating set. Therefore, the set becomes $D = \{u, v_i\}$ where $1 \leq i \leq n - 1$ which implies $\gamma_t(G) = 2$. The split total domination set is not satisfied for $n = 5$ as the domination number differs. Hence by adding a vertex v_i , $1 \leq i \leq n - 1$, the split dominating set becomes $D = \{u, v_i\}$, $1 \leq i \leq n - 1$. Clearly, the split domination number becomes, $\gamma_s(G) = 2$. Therefore, the split total domination number for a star graph with $n = 5$ vertices is $\gamma_{st}(G) = 2$.

Proceeding further, the split total dominating set for a star graph with n vertices is $D = \{u, v_i\}$ and the split total domination number $\gamma_{st}(G) = 2, 1 \leq i \leq n - 1, n \geq 4$. \square

Remark 2.2. *The split total domination set does not exist when G is a star graph with $n = 2, 3$.*

Theorem 2.3. *Let G be a wheel graph with n vertices then $\gamma_{st}(G) = 3$ for $n \geq 5$.*

Proof. Let G be a wheel graph with the vertex set becomes $V = \{v_1, \dots, v_n\}$ where v_n is the universal vertex. By induction hypothesis.

For $n = 5$, let G be a wheel graph with the vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ where v_5 is the universal vertex. If the dominating set $D = \{v_2, v_5\}$ then the condition for split dominating set is not satisfied as the induced graph $\langle V - D \rangle$ is connected. Thus adding a vertex to D it becomes $D = \{v_2, v_4, v_5\}$, then the induced graph $\langle V - D \rangle$ is disconnected satisfying the condition for split dominating set. Hence $\gamma_s(G) = 3$. Also D satisfies the condition for total dominating set, clearly $\gamma_t(G) = 3$. Therefore, the split total dominating set $D = \{v_2, v_4, v_5\}$ and the split total domination number $\gamma_{st}(G) = 3$.

For $n = 6$, let G be a wheel graph with the vertex set $V = \{v_1, v_2, \dots, v_6\}$ where v_6 is the universal vertex. If the dominating set $D = \{v_2, v_6\}$ then the condition for split dominating set is not satisfied as the induced graph $\langle V - D \rangle$ is connected. Thus

adding a vertex to D it becomes $D = \{v_2, v_5, v_6\}$, then the induced graph $\langle V - D \rangle$ is disconnected satisfying the condition for split dominating set. Hence $\gamma_s(G) = 3$. Also D satisfies the condition for total dominating set, clearly $\gamma_t(G) = 3$. Therefore, the split total dominating set $D = \{v_2, v_5, v_6\}$ and the split total domination number $\gamma_{st}(G) = 3$.

Proceeding further, the split total dominating set for a wheel graph with n vertices is $D = \{v_n, v_i, v_j\}$ and the split total domination number $\gamma_{st}(G) = 3, n \geq 5, i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n, i \neq j$. \square

Theorem 2.4. *Let G be a fan graph, $F_n = P_n + K_1$ with n vertices then $\gamma_{st}(G) = 2$ for $n \geq 3$.*

Proof. If G is a fan graph $F_n = P_n + K_1$ where P_n is a path and K_1 is the vertex adjacent to all the other vertices. We prove this theorem by induction hypothesis.

For $n = 3$, $F_3 = P_3 + K_1$ is a graph with vertex set $V = \{v_1, v_2, v_3, k_1\}$. If we consider the dominating set $D = \{k_1, v_2\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 2$. Here D satisfies the condition for total dominating set. Hence, the total domination number $\gamma_t(G) = 2$. Since D satisfies both the condition for split total domination set and the split total domination number is $\gamma_{st}(G) = 2$.

For $n = 4$, $F_4 = P_4 + K_1$ is a graph with vertex set $V = \{v_1, v_2, v_3, v_4, k_1\}$. If we consider the dominating set $D = \{k_1, v_3\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 2$. Here D satisfies the condition for total dominating set. Hence, the total domination number $\gamma_t(G) = 2$. Since D satisfies both the condition for split total domination set and the split total domination number is $\gamma_{st}(G) = 2$.

Proceeding further for n vertices, the split total dominating set $D = V - \{k_1, v_i\}$ and the split total domination number $\gamma_{st}(G) = 2, i = 2, 3, \dots, n - 1, n \geq 3$. \square

Remark 2.5. *The split total domination set does not exist for a complete graph. Hence by adding an edge $(C_n \cup \{e\})$ it satisfies the condition for the split total domination set.*

Theorem 2.6. *Let G be a $C_n \cup \{e\}$ graph with $n + 1$ vertices then $\gamma_{st}(G) = 2$ for $n \geq 3$.*

Proof. Let G be a $C_n \cup \{e\}$ graph with $n + 1$ vertices. Let u be the vertex adjacent to v_n . We prove this theorem by induction hypothesis.

For $n = 3$, let G be a $C_3 \cup \{e\}$ graph with the vertex set $V = \{u, v_1, v_2, v_3\}$ where u is the adjacent vertex of v_3 . If we take the dominating set $D = \{u\}$ then the induced graph $\langle V - D \rangle$ becomes a disconnected graph. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 1$. But D does not satisfy the condition for total dominating set. Hence a vertex is added to D to satisfy the condition for total dominating set. Therefore, the set becomes $D = \{u, v_i\}$ where $1 \leq i \leq n - 1$ which implies $\gamma_t(G) = 2$. The split total domination set is not satisfied as the domination number differs. Hence by adding a vertex D , the split dominating set becomes $D = \{u, v_i\}, 1 \leq i \leq n - 1$. Clearly, the split domination number becomes, $\gamma_s(G) = 2$. Therefore, the split total domination number for a $C_3 \cup \{e\}$ graph is $\gamma_{st}(G) = 2$.

For $n = 4$, let G be a $C_4 \cup \{e\}$ graph with the vertex set $V = \{u, v_1, \dots, v_4\}$ where u is the adjacent vertex of v_4 . If we take the dominating set $D = \{u\}$ then the induced graph $\langle V - D \rangle$ becomes a disconnected graph. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 1$. But D does not satisfy the condition for total dominating set. Hence a vertex is added to D to satisfy the condition for total dominating set. Therefore, the set becomes $D = \{u, v_i\}$ where $1 \leq i \leq n - 1$ which implies $\gamma_t(G) = 2$. The split total domination set is not satisfied for $n = 4$ as the domination number differs. Hence by adding a vertex D , the split dominating set becomes $D = \{u, v_i\}, 1 \leq i \leq n - 1$. Clearly, the split domination number becomes, $\gamma_s(G) = 2$. Therefore, the split total domination number for a $C_4 \cup \{e\}$ graph $\gamma_{st}(G) = 2$.

Proceeding further, the split total domination set for $C_n \cup \{e\}$ graph with $n + 1$ vertices is $D = V - \{u, v_i\}$ and the split total domination number $\gamma_{st}(G) = 2, 1 \leq i \leq n - 1, n \geq 3$. \square

Theorem 2.7. *Let G be a Bistar graph with n vertices then $\gamma_{st}(G) = 2$ for $n \geq 4$.*

Proof. Let G be a bistar graph with n vertices. We prove this theorem by induction hypothesis.

For $n = 4$, If we consider the dominating set D is $\{v_2\}$ then the induced graph $\langle V-D \rangle$ becomes a disconnected graph. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 1$. But D does not satisfy the condition for total dominating set. Hence by adding a vertex adjacent to $\{v_2\}$, the set becomes $D = \{v_2, v_3\}$ then the condition for total dominating set is satisfied which implies $\gamma_t(G) = 2$. The split total domination set is not satisfied as the domination number differs. Hence by adding a vertex to D , the split dominating set and the split total dominating set conditions are satisfied. Thus the split total domination set for a bistar with $n = 4$ is $D = \{\frac{v_n}{2}, \frac{v_{(n+1)}}{2}\}$. Therefore, $\gamma_{st}(G) = 2$.

For $n = 5$, If we consider the dominating set D is $\{v_3\}$ then the induced graph $\langle V-D \rangle$ becomes a disconnected graph. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 1$. But D does not satisfy the condition for total dominating set. Hence by adding a vertex adjacent to $\{v_3\}$, the set becomes $D = \{v_3, v_4\}$ then the condition for total dominating set is satisfied which implies $\gamma_t(G) = 2$. The split total domination set is not satisfied as the domination number differs. Hence by adding a vertex to D , the split dominating set and the split total dominating set conditions are satisfied. Thus the split total domination set for a bistar with $n = 4$ is $D = \{\frac{v_{(n+1)}}{2}, \frac{v_{(n+3)}}{2}\}$. Therefore, $\gamma_{st}(G) = 2$.

Proceeding further for n vertices, the split total domination set,

$$D = \begin{cases} \left\{ \frac{v_n}{2}, \frac{v_{(n+1)}}{2} \right\}, & \text{if } n \text{ is even.} \\ \left\{ \frac{v_{(n+1)}}{2}, \frac{v_{(n+3)}}{2} \right\}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus the split total domination number $\gamma_{st}(G) = 2$ for $n \geq 4$. □

Result 2.8. *Let G be a path P_n with $n = 3, 4, \dots, 20$ vertices namely $v_3, v_4, v_5, \dots, v_{20}$, then*

$$\gamma_{st}(G) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for all } n \text{ except } n = 10, 14, 16, 18. \\ \lceil \frac{n}{2} \rceil + 1, & \text{for } n = 10, 14, 16, 18. \end{cases}$$

Result 2.9. *Let G be a cycle graph with $n = 3, 4, \dots, 20$ vertices namely $v_3, v_4, v_5, \dots, v_{20}$, then*

$$\gamma_{st}(G) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{for all } n \text{ except } n = 6, 8, 10, 14, 18. \\ \lceil \frac{n}{2} \rceil + 1, & \text{for } n = 6, 8, 10, 14, 18. \end{cases}$$

Theorem 2.10. *Let G be a ladder graph with $2n$ vertices, then*

$$\gamma_{st}(G) = \begin{cases} n, & \text{if } n \text{ is even, for } n \geq 2. \\ n - 1, & \text{if } n \text{ is odd, for } n \geq 2. \end{cases}$$

Proof. In this theorem there are two cases.

Case 1: When n is even.

Subcase 1: When $n = 2$, $L_2 = 4$ vertices with the vertex set $V = \{v_1, v_2, v_3, v_4\}$. Take the dominating set $D = \{v_2, v_4\}$ then the induced graph is disconnected. But D does not satisfy the condition for total dominating set and D satisfies split dominating set condition. Hence split total domination set does not exist for ladder graph when $n = 2$.

Subcase 2: When $n = 4$, $L_4 = 8$ vertices with the vertex set $V = \{v_1, v_2, \dots, v_8\}$. Taking the dominating set $D = \{v_2\}$, the induced graph $\langle V - D \rangle$ is not disconnected. Hence if $D = \{v_2, v_7\}$ then the $\langle V - D \rangle$ graph is disconnected. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 2$. But D does not satisfy the condition for total dominating

set. Hence adding two vertices the total dominating set becomes $D = \{v_2, v_3, v_6, v_7\}$. Clearly, $\gamma_t(G) = 4$. Now adding two vertices to D , it satisfies the condition for split dominating set and split total dominating set. Clearly, $\gamma_{st}(G) = 4$. Therefore, split total dominating set $D = \{v_2, v_3, v_6, v_7\}$ and the split total domination number $\gamma_{st}(G) = 4$.

Subcase 3: When $n = 6$, $L_6 = 12$ vertices with the vertex set $V = \{v_1, v_2, \dots, v_{12}\}$. Taking the dominating set $D = \{v_i\}$, $i = \{1, 2, \dots, 12\}$, the induced graph $\langle V - D \rangle$ is connected. Hence adding a vertex to D , it becomes $D = \{v_2, v_{11}\}$ then the $\langle V - D \rangle$ graph is disconnected. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 2$. But D does not satisfy the condition for total dominating set. Hence adding two vertices in such a way that it satisfies the total dominating set then the set becomes $D = \{v_2, v_4, v_6, v_7, v_9, v_{11}\}$. Clearly, $\gamma_t(G) = 6$. But the condition for split total dominating set is not satisfied as the domination number varies. Now adding two vertices to D , it satisfies the condition for split dominating set and split total dominating set. Therefore, D becomes $D = \{v_2, v_4, v_6, v_7, v_9, v_{11}\}$. Clearly, $\gamma_s(G) = 6$. Now the condition for split total dominating set is satisfied. Therefore, split total dominating set $D = \{v_2, v_3, v_6, v_7, v_9, v_{11}\}$ and the split total domination number $\gamma_{st}(G) = 6$.

Proceeding further for $n - \text{even}$ vertices the split total dominating set $D = \{v_2, v_4, \dots, v_n, v_{n+1}, v_{n+3}, \dots, v_{2n-1}\}$ and the split total domination number $\gamma_{st}(G) = n$ for $n \geq 6$.

Case 2: When n is odd.

Subcase 1: When $n = 3$, $L_3 = 6$ vertices with the vertex set $V = \{v_1, v_2, \dots, v_6\}$. Take the dominating set $D = \{v_2, v_5\}$ then the induced graph is disconnected. Thus, $\gamma_s(G) = 2$ and also D satisfies the condition for total dominating set and so $\gamma_t(G) = 2$. Since the domination numbers are same the split total domination number for ladder graph when $n = 3$ is $\gamma_{st}(G) = 2$.

Subcase 2: When $n = 5$, $L_5 = 10$ vertices with the vertex set $V = \{v_1, v_2, \dots, v_{10}\}$. Taking the dominating set $D = \{v_2\}$, the induced graph $\langle V - D \rangle$ is not disconnected. Hence if $D = \{v_2, v_9\}$ then the $\langle V - D \rangle$ graph is disconnected. Therefore, D satisfies the condition for split dominating set. Clearly, $\gamma_s(G) = 2$. But D does not satisfy the condition for total dominating set. Hence adding two vertices the total dominating set becomes $D = \{v_2, v_4, v_7, v_9\}$. Clearly, $\gamma_t(G) = 4$. But the split total domination set is not satisfied as the domination number varies. Now adding two vertices to D , it satisfies the condition for split dominating set and split total dominating set. Therefore, D becomes $D = \{v_2, v_4, v_7, v_9\}$. Clearly, $\gamma_s(G) = 4$. Therefore, split total dominating set $D = \{v_2, v_3, v_6, v_7\}$ and the split total domination number $\gamma_{st}(G) = 4$ for $n = 5$.

Proceeding further, for $n - \text{odd}$ vertices the split total dominating set $D = \{v_2, v_4, \dots, v_{n-1}, v_{n+2}, v_{n+4}, \dots, v_{2n-1}\}$ for $n \geq 5$. □

Result 2.11. Let G be a Tadpole graph $T_{m,n}$ where $m = 3$ is a cycle C_3 and $n = 1, 2, \dots, 20$ is a path P_n with vertices namely $c_1, c_2, c_3, v_1, v_2, \dots, v_{20}$, then

$$\gamma_{st}(G) = \begin{cases} \lceil \frac{m+n}{2} \rceil, & \text{for all } n \text{ except } n = 2, 6, 10, 14, 18. \\ \lceil \frac{m+n}{2} \rceil - 1, & \text{for } n = 2, 6, 10, 14, 18. \end{cases}$$

Theorem 2.12. Let G be a butterfly graph with 5 vertices and 6 edges, then $\gamma_{st}(G) = 2$.

Proof. Let G be a butterfly graph with 5 vertices and 6 edges and let $D = \{v_3\}$ be the dominating set. Since every vertex of V is adjacent to an element of D , $\gamma(G) = 1$. Since the condition for split total dominating set is not satisfied, one more vertex is added to D to satisfy the condition for split total dominating set. Hence the split total dominating set becomes $D = \{v_2, v_3\}$. Therefore, the split total domination number $\gamma_{st}(G) = 2$. □

Theorem 2.13. Let G be a comb graph $P_n \odot K_1$ with $2n$ vertices, then $\gamma_{st}(G) = n$ for $n \geq 2$.

Proof. To prove this theorem by induction on the number of vertices. Let G be a comb graph with $2n$ vertices. Let $V = U \cup W$ where $U = \{u_1, u_2, \dots, u_n\}$ is the set of vertices in P_n and $W = \{w_1, w_2, \dots, w_n\}$ be the set of vertices attached to P_n .

For $n = 2$, $P_2 \odot K_1 = 4$ vertices with the vertex set $V = U \cup W = \{u_1, u_2, w_1, w_2\}$. Taking $D = \{u_1, u_2\}$ to be the dominating set D , then D is adjacent to all the other vertices. Thus the domination number $\gamma(G) = 2$. Also, the condition for split total dominating set is satisfied. Hence the split total dominating set becomes $D = \{u_1, u_2\}$. Therefore, the split total domination number $\gamma_{st}(G) = 2$.

For $n = 3$, $P_3 \odot K_1 = 6$ vertices with the vertex set $V = U \cup W = \{u_1, u_2, u_3, w_1, w_2, w_3\}$. Taking $D = \{u_1, u_2, u_3\}$ to be the dominating set D , then D is adjacent to all the other vertices. Thus the domination number $\gamma(G) = 3$. Also, the condition for split total dominating set is satisfied. Hence the split total dominating set becomes $D = \{u_1, u_2, u_3\}$. Therefore, the split total domination number $\gamma_{st}(G) = 3$.

Proceeding further, the split total dominating set for a comb graph with $2n$ vertices is $D = V - \{W\}$, $n \geq 2$. Thus, the split total domination number $\gamma_{st}(G) = n$, $n \geq 2$. \square

Theorem 2.14. *Let G be a barbell graph with $2n$ vertices then $\gamma_{st}(G) = 2$ for $n \geq 3$.*

Proof. To prove this theorem by induction on the number of vertices. Let G be a barbell graph with the vertex set $V = \{v_1, v_2, v_3, \dots, v_n\}$.

For $n = 3$, the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Taking $\{v_3, v_4\}$ to be the dominating set D , then D is adjacent to all the other vertices. Thus the domination number $\gamma(G) = 2$. Also, the condition for split total dominating set is satisfied. Hence the split total dominating set becomes $D = \{v_3, v_4\}$. Therefore, the split total domination number $\gamma_{st}(G) = 2$.

For $n = 4$, the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. Taking $\{v_4, v_5\}$ to be the dominating set D , then D is adjacent to all the other vertices. Thus the domination number $\gamma(G) = 2$. Also, the condition for split total dominating set is satisfied. Hence the split total dominating set becomes $D = \{v_4, v_5\}$. Therefore, the split total domination number $\gamma_{st}(G) = 2$.

Proceeding further for n vertices the split total dominating set, $D = \{\frac{v_n}{2}, \frac{v_n}{2} + 1\}$, $n \geq 3$. Thus the split total domination number $\gamma_{st}(G) = 2$. \square

3. Complement of Graphs

Theorem 3.1. *Let G be a path and \bar{G} be the complement of the path G then $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 5$.*

Proof. Let G be a graph with n vertices and \bar{G} be the complement of the graph G . By induction hypothesis.

For $n = 5$, the vertex set is $V = \{v_1, v_2, v_3, v_4, v_5\}$. Taking the dominating set $D = \{v_1\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{v_5\}$ to D , it becomes $D = \{v_1, v_5\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 2$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma_t(\bar{G}) = 2$. Thus the split total dominating set becomes $D = \{v_1, v_5\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 2$.

For $n = 6$, the vertex set is $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Taking the dominating set $D = \{v_1\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{v_6\}$ to D , it becomes $D = \{v_1, v_6\}$ then the induced graph $\langle V - D \rangle$ is connected. Hence adding another vertex $\{v_5\}$ to D , it becomes $D = \{v_1, v_5, v_6\}$ the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 3$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma_t(\bar{G}) = 3$. Thus the split total domination set becomes $D = \{v_1, v_5, v_6\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 3$.

Proceeding further for n vertices the split total dominating set becomes $D = V - \{v_2, v_3, v_4\}$ and the split total domination number $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 5$. \square

Remark 3.2. The split total domination number does not exist for the complement of a graph with $n = 3$ and $n = 4$.

Theorem 3.3. Let G be a tadpole graph with $m + n$ vertices where $m = 3$ and $n = \{1, 2, \dots, n\}$ then \bar{G} be the complement of the graph then $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 2$.

Proof. Let G be a tadpole graph with $m + n$ vertices where m is a cycle with the vertices $\{c_1, c_2, c_3\}$ and $n = \{1, 2, \dots, n\}$ is a path and the degree of $c_3 = 3$ and the degree of $c_1 = \text{degree of } c_2 = 2$. Let \bar{G} be the complement of the graph then by induction hypothesis.

For $n = 2$, consider the complement of $T_{3,2}$ graph with the vertex set is $V = \{c_1, c_2, c_3, v_1, v_2\}$. Taking the dominating set $D = \{c_1\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{v_2\}$ to D , it becomes $D = \{c_1, v_2\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 2$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma_t(\bar{G}) = 2$. Thus the split total dominating set becomes $D = \{c_1, v_2\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 2$.

For $n = 3$, consider the complement of $T_{3,3}$ graph with the vertex set is $V = \{c_1, c_2, c_3, v_1, v_2, v_3\}$. Taking the dominating set $D = \{c_1\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{v_3\}$ to D , it becomes $D = \{c_1, v_3\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 2$. But D does not satisfy the condition for total dominating set so adding a vertex to D , the total domination set becomes $D = \{c_1, v_2, v_3\}$. Therefore, D satisfies the condition for total dominating set and so the total domination number is $\gamma_t(\bar{G}) = 3$. But the split total domination set is not satisfied as the domination number varies. Now adding a vertex to D , it satisfies the condition for split dominating set and split total dominating set. Therefore, D becomes $D = \{c_1, v_2, v_3\}$. Clearly, $\gamma_s(\bar{G}) = 3$. Therefore, split total dominating set $D = \{c_1, v_2, v_3\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 3$ for $n = 3$.

Proceeding further, the split total dominating set is $D = V - \{c_2, c_3, v_1\}$ and the split total domination number $\gamma_{st}(\bar{G}) = n$ for $n \geq 2$. \square

Theorem 3.4. Let G be a bistar graph with n vertices and \bar{G} be the complement of the graph then $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 5$.

Proof. Let G be a bistar graph with n vertices and \bar{G} be the complement of the graph then by induction hypothesis.

For $n = 5$, the vertex set is divided into two parts V_1 and V_2 where $V_1 = \{v, v_5\}$ and $V_2 = \{u, u_1, u_2\}$ where u and v forms a path between the two set of vertices V_1 and V_2 . Taking the dominating set $D = \{u_1\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{v_5\}$ to D , it becomes $D = \{u_1, v_5\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 2$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma_t(\bar{G}) = 2$. Thus the split total dominating set becomes $D = \{u_1, v_5\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 2$.

For $n = 6$, the vertex set is divided into two parts V_1 and V_2 where $V_1 = \{v, v_5, v_6\}$ and $V_2 = \{u, u_1, u_2\}$ where u and v forms a path between the two set of vertices V_1 and V_2 . Taking the dominating set $D = \{u_1\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{v_5\}$ to D , it becomes $D = \{u_1, v_5\}$ then the induced graph $\langle V - D \rangle$ is connected. Again adding an vertex $\{v_6\}$ to D it becomes $D = \{u_1, v_5, v_6\}$. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 3$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma_t(\bar{G}) = 3$. Thus the split total dominating set becomes $D = \{u_1, v_5, v_6\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 3$.

Proceeding further for n vertices, the split total dominating set becomes $D = \{(V_1 - v) \cup (V_2 - (u, u_i))\}, 1 \leq i \leq n$ and the split total domination number $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 5$. \square

Theorem 3.5. *Let G be a ladder graph with $2n$ vertices and \bar{G} is the complement of the graph then $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 4$.*

Proof. Let G be ladder graph with $2n$ vertices and \bar{G} is the complement of G with the vertex set $V = \{v_1, v_2, \dots, v_n\}$. By induction,

For $n = 4, L_4 = 8$ vertices, the vertex set $V = \{v_1, v_2, \dots, v_8\}$. If the dominating set is $D = \{v_6, v_7\}$ then the induced graph is connected. Thus, adding vertices to D to satisfy the condition for split dominating set the set becomes $D = \{v_4, v_5, v_6, v_7, v_8\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Clearly, $\gamma_s(\bar{G}) = 5$. Also D satisfies the condition for the total dominating set, hence $\gamma_t(\bar{G}) = 5$. Therefore, the split total dominating set for a ladder graph with $2n$ vertices for $n = 4$ is $D = \{v_4, v_5, v_6, v_7, v_8\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 5$.

For $n = 5, L_5 = 10$ vertices, the vertex set $V = \{v_1, v_2, \dots, v_{10}\}$. If the dominating set is $D = \{v_6, v_7\}$ then the induced graph is connected. Thus, adding vertices to D to satisfy the condition for split dominating set the set becomes $D = \{v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Clearly, $\gamma_s(\bar{G}) = 7$. Also D satisfies the condition for the total dominating set, hence $\gamma_t(\bar{G}) = 7$. Therefore, the split total dominating set for a ladder graph with $2n$ vertices for $n = 5$ is $D = \{v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 7$.

Proceeding further, the split total dominating set $D = V - \{v_1, v_2, v_3\}$ and the split total domination number $\gamma_{st}(\bar{G}) = n - 3$ for $n \geq 4$. \square

Theorem 3.6. *Let G be a comb graph with $2n$ vertices and \bar{G} be the complement of the comb graph then $\gamma_{st}(\bar{G}) = 2n - 3$ for $n \geq 3$.*

Proof. Let G be a graph with $2n$ vertices and \bar{G} be the complement of the graph G . Let $V = U \cup W$, where $U = \{u_1, u_2, \dots, u_n\}$ is the set of vertices in P_n and $W = \{w_1, w_2, \dots, w_n\}$ is the set of vertices attached to P_n . By induction hypothesis.

For $n = 3$, the vertex set is $V = \{u_1, u_2, u_3, w_1, w_2, w_3\}$. Taking the dominating set $D = \{w_3, w_2\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{w_1\}$ to D , it becomes $D = \{w_1, w_2, w_3\}$ then the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma(\bar{G}) = 3$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma(\bar{G}) = 3$. Thus the split total domination set becomes $D = \{w_1, w_2, w_3\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 3$.

For $n = 4$, the vertex set is $V = \{u_1, u_2, u_3, u_4, w_1, w_2, w_3, w_4\}$. Taking the dominating set $D = \{w_4, w_3\}$ the induced graph $\langle V - D \rangle$ is not disconnected. Hence adding a vertex $\{w_2\}$ to D , it becomes $D = \{w_2, w_3, w_4\}$ then the induced graph $\langle V - D \rangle$ is connected. Hence adding two vertices $\{w_1, u_4\}$ to D , it becomes $D = \{u_4, w_1, w_2, w_3, w_4\}$ the induced graph $\langle V - D \rangle$ is disconnected. Therefore, D satisfies the condition for split dominating set and $\gamma_s(\bar{G}) = 5$. Also D satisfies the condition for total dominating set so that the total domination number is $\gamma_t(\bar{G}) = 5$. Thus the split total domination set becomes $D = \{u_4, w_1, w_2, w_3, w_4\}$ and the split total domination number $\gamma_{st}(\bar{G}) = 5$.

Proceeding further for $2n$ vertices the split total domination set becomes $D = V - \{u_1, u_2, u_3\}$ where $V = U \cup W$, $U = \{u_1, u_2, \dots, u_n\}$ is the set of vertices in P_n and $W = \{w_1, w_2, \dots, w_n\}$ is the set of vertices attached to P_n , and the split total domination number $\gamma_{st}(\bar{G}) = 2n - 3$ for $n \geq 3$. \square

Result 3.7. *Let G be a barbell graph with $2n$ vertices and \bar{G} be the complement of the barbell graph then $\gamma_{st}(\bar{G}) = n$ for $n \geq 3$.*

4. Conclusion

In this paper the Split Total Domination Number of some simple, undirected, non-trivial, connected and finite graphs like star graph, path, wheel graph, fan graph, complete graph, cycle, comb graph, barbell graph, butterfly graph, tadpole graph, bistar graph, ladder graph are obtained. And also the complement of graphs has been obtained.

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