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Certain Curvature Conditions on $N(k)$-Contact Metric Manifold

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Abstract

We consider a concircularly semi-symmetric $N(k)$-contact metric manifold and we classify $N(k)$-contact metric manifolds satisfying the curvature conditions $\widetilde{Z}.S = 0$, $Q.\widetilde{Z} = 0$, $\widetilde{Z}.P = 0$.

Keywords: $N(k)$-contact metric manifold; Einstein manifold; concircular curvature tensor; projective curvature tensor.

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1. Introduction

Let M be a $(2n+1)$-dimensional Riemannian manifold. The transformation for every geodesic circle of M into a geodesic circle is called a concircular transformation [7,11]. In [7] W. Kuhnel found that a concircular transformation is always a conformal transformation. In 1940, K. Yano [11] introduced the concircular curvature tensor $\widetilde{Z}$ defined by [10,11]:

$$\widetilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{(2n)(2n+1)}[g(Y,W)X - g(X,W)Y], \tag{1}$$

for $X, Y, W \in T(M)$ and r is the scalar curvature. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature.

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [8]:

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{2}$$

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for all $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. By virtue of (2), M is projectively flat if and only if it is of constant curvature. Hence, the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

In [2], D. E. Blair et al. studied the $N(k)$-contact metric manifold satisfying the curvature conditions $\widetilde{Z}.\widetilde{Z} = 0$, $\widetilde{Z}.R = 0$ and $R.\widetilde{Z} = 0$, where $\widetilde{Z}$ is the concircular curvature tensor. U. C. De et al. [6] studied $N(k)$-contact metric manifold satisfying $\widetilde{Z}.S = 0$, where S is the Ricci tensor. Recently, motivated by the above studies, in this paper we consider a concircularly semi-symmetric $N(k)$-contact metric manifold and different curvature conditions on $N(k)$-contact manifolds $\widetilde{Z}.S = 0$, $Q.\widetilde{Z} = 0$, $\widetilde{Z}.P = 0$.

2. Preliminaries

A $(2n + 1)$-dimensional differentiable manifold M is said to admit an almost contact structure if it admits a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying [1,3]:

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{3}$$

An almost contact structure is said to be normal if the induced almost contact structure J on the product manifold $M \times \mathbb{R}$ defined by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta \otimes \frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and λ is a smooth function on $M \times \mathbb{R}$. The condition of almost contact metric structure being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ. Let g be the compatible Riemannian metric with almost contact structure (φ, ξ, η), that is

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{4}$$

for all vector fields $X, Y \in T(M)$. A manifold M together with this almost contact metric structure is said to be an almost contact metric manifold denoted by $M(\varphi, \xi, \eta, g)$. An almost contact metric structure reduces to a contact metric structure if $g(X, \varphi Y) = d\eta(X, Y)$. Moreover, if ∇ denotes the Riemannian connection of g, then the following relation holds:

$$\nabla_X\xi = -\varphi X - \varphi hX. \tag{5}$$

A contact metric manifold is said to be η-Einstein if

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{6}$$

where α and β are smooth functions. If $\beta = 0$, then the manifold M is an Einstein manifold. Blair, Konfogiorgos and Papantoniou [4] introduced the (k, μ)-nullity distribution of a contact metric manifold M that is defined by

$$N(k, \mu) : p \to N_p(k, \mu),$$
$$N_p(k, \mu) = \{W \in T_pM : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in T(M)$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ)-contact metric manifold. If $\mu = 0$, the (k, μ)-nullity distribution reduces to k-nullity distribution [9]. The k-nullity distribution $N(k)$ of a Riemannian manifold is defined by [9]:

$$N(k) : p \to N_p(k) = \{W \in T_p(M) : R(X, Y)W = k(g(Y, W)X - g(X, W)Y)\},$$

k being a constant.

If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [2]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [5]. In an $N(k)$-contact metric manifold, the following relations hold:

$$h^2 = (k - 1)\varphi^2, \quad \text{where } h = \frac{1}{2}\mathcal{L}_\xi \varphi, \tag{7}$$

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{8}$$

$$S(X, Y) = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y) + [2nk - 2(n - 1)]\eta(X)\eta(Y), \quad n \geq 1, \tag{9}$$

$$S(\varphi X, \varphi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n - 1)g(hX, Y), \tag{10}$$

$$S(X, \xi) = 2nk\eta(X), \tag{11}$$

$$r = 2n(2n - 2 + k), \tag{12}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \varphi Y), \tag{13}$$

where R and S are the curvature tensor and Ricci tensor respectively with respect to Levi-Civita connection. In a $(2n + 1)$-dimensional almost contact metric manifold, if $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ is a local orthonormal basis of tangent space of the manifold, then $\{\varphi e_1, \varphi e_2, \ldots, \varphi e_{2n}, \xi\}$ is a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi e_i) = 2n, \tag{14}$$

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i) = r - 2nk, \tag{15}$$

$$\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, Z)S(Y, \varphi e_i) = S(Y, Z) - 2nk\eta(Y)\eta(Z), \tag{16}$$

$$\sum_{i=1}^{2n} g(R(e_i,Y)Z,e_i) = \sum_{i=1}^{2n} g(R(\varphi e_i,Y)Z,\varphi e_i) = S(Y,Z) - kg(\varphi Y,\varphi Z), \tag{17}$$

for all $Y, Z \in T(M)$. Given a non-Sasakian (k,μ)-contact manifold M, E. Boeckx [**?**] introduced an invariant

$$I_M = \frac{1-\mu/2}{\sqrt{1-k}}$$

and showed that for two non-Sasakian (k,μ)-manifolds M_1 and M_2, we have $I_{M_1} = I_{M_2}$ if and only if up to D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus, we see that from all non-Sasakian (k,μ)-manifolds of dimension $(2n+1)$ and for every possible value of the invariant I_M, one (k,μ)-manifold M can be obtained. If $I_M > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I_M = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie-algebra construction for any odd dimension and value of $I_M \leq -1$.

Example 2.1 ([2])**.** *Using this invariant, D. E. Blair, J-S Kim and M. M. Tripathi [2] constructed an example of a $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold, $n > 1$. The example is given as follows:*
Since the Boeckx invariant for $(1-\frac{1}{n},0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting D-homothetic deformation will be a $(1-\frac{1}{n},0)$-manifold. That is, for $k = c(2-c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{k+a^2-1}{a^2}, \quad 0 = \frac{\mu+2a-2}{a}$$

for a and c. The result is

$$c = \frac{\sqrt{n}\pm 1}{n-1}, \quad a = 1+c.$$

Taking c and a to these values we obtain an $N(1-\frac{1}{n})$-contact metric manifold. The above example will be used in theorems of Section 4.

3. Concircularly Semisymmetric $N(k)$-Contact Manifolds

Definition 3.1. *An $N(k)$-contact metric manifold is said to be concircularly semi-symmetric if*

$$R(X,Y).\widetilde{Z} = 0$$

for every vector fields X, Y.

In this section we deal with concircularly semi-symmetric $N(k)$-contact manifolds. Suppose

$$R(X,Y).\widetilde{Z} = 0. \tag{18}$$

From the above equation, we have

$$\begin{aligned}0 &= (R(X,Y).\widetilde{Z})(U,V)W \\ &= R(X,Y)\widetilde{Z}(U,V)W - \widetilde{Z}(R(X,Y)U,V)W - \widetilde{Z}(U,R(X,Y)V)W - \widetilde{Z}(U,V)R(X,Y)W.\end{aligned}$$

Put $X = \xi$, the above equation gives

$$0 = R(\xi,Y)\widetilde{Z}(U,V)W - \widetilde{Z}(R(\xi,Y)U,V)W - \widetilde{Z}(U,R(\xi,Y)V)W - \widetilde{Z}(U,V)R(\xi,Y)W.$$

In view of (8), the equation can be written as

$$\begin{aligned}0 = k[&\widetilde{Z}(U,V,W,Y)\xi - \eta(\widetilde{Z}(U,V)W)Y + \eta(U)\widetilde{Z}(Y,V)W \\ &- g(Y,U)\widetilde{Z}(\xi,V)W + \eta(V)\widetilde{Z}(U,Y)W - g(Y,V)\widetilde{Z}(U,\xi)W + \eta(W)\widetilde{Z}(U,V)Y],\end{aligned}$$

where $\widetilde{Z}(U,V,W,Y) = g(\widetilde{Z}(U,V)W,Y)$. Taking inner product with ξ, we get

$$\begin{aligned}0 = k[&\widetilde{Z}(U,V,W,Y) - \eta(\widetilde{Z}(U,V)W)\eta(Y) + \eta(U)\eta(\widetilde{Z}(Y,V)W) \\ &- g(Y,U)\eta(\widetilde{Z}(\xi,V)W) + \eta(V)\eta(\widetilde{Z}(U,Y)W) \\ &- g(Y,V)\eta(\widetilde{Z}(U,\xi)W) + \eta(W)\eta(\widetilde{Z}(U,V)Y) \\ &- g(Y,W)\eta(\widetilde{Z}(U,V)\xi)].\end{aligned}$$

Putting $Y = U$ yields

$$0 = k[\widetilde{Z}(U,V,W,U) - g(U,U)\eta(\widetilde{Z}(\xi,V)W) - g(U,V)\eta(\widetilde{Z}(U,\xi)W) + \eta(W)\eta(\widetilde{Z}(U,V)U)].$$

Let $\{e_1, e_2, \ldots, e_{2n}, \xi\}$ be a local orthonormal basis of tangent space of the manifold. Putting $U = e_i$ and summing up from 1 to $2n$, then by virtue of (1), (8), (11) and (14)–(17), the above equation reduces to

$$S(V,W) = 2nk\, g(V,W). \tag{19}$$

Hence, we can state the following theorem:

Theorem 3.2. *If M is a $(2n+1)$-dimensional concircularly semi-symmetric $N(k)$-contact metric manifold, then the manifold is an Einstein manifold.*

4. $N(k)$-Contact Metric Manifold Satisfying Certain Curvature Conditions $\widetilde{Z}.S = 0$, $Q.\widetilde{Z} = 0$, $\widetilde{Z}.P = 0$

We consider an $N(k)$-contact metric manifold M^{2n+1} satisfying the curvature condition $\widetilde{Z}.S = 0$. Suppose

$$(\widetilde{Z}(X,Y).S)(U,V) = 0 \tag{20}$$

implies

$$S(\widetilde{Z}(X,Y)U,V) + S(U,\widetilde{Z}(X,Y)V) = 0. \tag{21}$$

Put $X = V = \xi$ in (21), we have

$$S(\widetilde{Z}(\xi,Y)U,\xi) + S(U,\widetilde{Z}(\xi,Y)\xi) = 0. \tag{22}$$

Now, from (1),

$$\widetilde{Z}(\xi,Y)U = \left(k - \frac{r}{2n(2n+1)}\right)[g(Y,U)\xi - \eta(U)Y], \tag{23}$$

$$\widetilde{Z}(\xi,Y)\xi = \left(k - \frac{r}{2n(2n+1)}\right)[\eta(Y)\xi - Y]. \tag{24}$$

Using (1), (4), (8), (11), (23) and (24), we have

$$S(\widetilde{Z}(\xi,Y)U,\xi) = \left(k - \frac{r}{2n(2n+1)}\right)[2nk\,g(Y,U) - 2nk\,\eta(Y)\eta(U)], \tag{25}$$

$$S(U,\widetilde{Z}(\xi,Y)\xi) = \left(k - \frac{r}{2n(2n+1)}\right)[2nk\,\eta(Y)\eta(U) - S(U,Y)]. \tag{26}$$

Put the value of (25) and (26) in (22), we have

$$\left(k - \frac{r}{2n(2n+1)}\right)[2nk\,g(U,Y) - S(U,Y)] = 0. \tag{27}$$

From (27), we get

$$S(U,Y) = 2nk\,g(U,Y), \quad \text{if } k - \frac{r}{2n(2n+1)} \neq 0. \tag{28}$$

Hence, we can state the following theorem:

Theorem 4.1. *A $(2n+1)$-dimensional N(k)-contact metric manifold satisfying the curvature condition $\widetilde{Z}.S = 0$ is an Einstein manifold provided*

$$k - \frac{r}{2n(2n+1)} \neq 0.$$

Proof. Suppose

$$Q.\widetilde{Z} = 0, \tag{29}$$

then,

$$Q(\widetilde{Z}(X,Y)W) - \widetilde{Z}(QX,Y)W - \widetilde{Z}(X,QY)W - \widetilde{Z}(X,Y)QW = 0. \tag{30}$$

Put $W = \xi$ in (30), we have

$$Q(\widetilde{Z}(X,Y)\xi) - \widetilde{Z}(QX,Y)\xi - \widetilde{Z}(X,QY)\xi - \widetilde{Z}(X,Y)Q\xi = 0. \tag{31}$$

Now, from (1),

$$\widetilde{Z}(X,Y)\xi = \left(k - \frac{r}{2n(2n+1)}\right)[\eta(Y)X - \eta(X)Y]. \tag{32}$$

Using equations (1), (4), (8), (11) and (32), we have

$$\begin{aligned} Q(\widetilde{Z}(X,Y)\xi) = &\left(k - \frac{r}{2n(2n+1)}\right)[2(n-1)X\eta(Y) + 2(n-1)hX\eta(Y) \\ &- 2(n-1)Y\eta(X) - 2(n-1)hY\eta(X)], \end{aligned} \tag{33}$$

$$\begin{aligned} \widetilde{Z}(QX,Y)\xi = &\left(k - \frac{r}{2n(2n+1)}\right)[2(n-1)X\eta(Y) + 2(n-1)hX\eta(Y) \\ &+ \{2(1-n) + 2nk\}\eta(X)\eta(Y)\xi - 2nk\eta(X)Y], \end{aligned} \tag{34}$$

$$\begin{aligned} \widetilde{Z}(X,QY)\xi = &\left(k - \frac{r}{2n(2n+1)}\right)[2nk\eta(Y)X - 2(n-1)Y\eta(X) \\ &- 2(n-1)hY\eta(X) - \{2(1-n) + 2nk\}\eta(X)\eta(Y)\xi], \end{aligned} \tag{35}$$

$$\widetilde{Z}(X,Y)Q\xi = 2nk\left(k - \frac{r}{2n(2n+1)}\right)[\eta(Y)X - \eta(X)Y]. \tag{36}$$

Put the value of (33)–(36) in (31), we have

$$4nk\left(k - \frac{r}{2n(2n+1)}\right)[\eta(X)Y - \eta(Y)X] = 0. \tag{37}$$

Now, $[\eta(Y)X - \eta(X)Y] \neq 0$ in a contact metric manifold, in general. Therefore, from (37) gives either $k = 0$ or

$$k = \frac{r}{2n(2n+1)}. \tag{38}$$

Using (12) in (38) we get

$$k = 1 - \frac{1}{n}. \tag{39}$$

Hence, from (39) M is locally isometric to Example 1 for $n > 1$ and flat for $n = 1$. □

In view of the above discussion, we have the following theorem:

Theorem 4.2. *A $(2n+1)$-dimensional $N(k)$-contact metric manifold M satisfying the curvature condition $Q.\widetilde{Z} = 0$ is either flat or locally isometric to Example 2.1.*

Proof. Consider a $(2n+1)$-dimensional N(k)-contact metric manifold satisfying $\widetilde{Z}.P = 0$. Therefore, we have

$$(\widetilde{Z}(X,Y).P)(U,V)W = 0,$$

which implies

$$\widetilde{Z}(X,Y)P(U,V)W - P(\widetilde{Z}(X,Y)U,V)W - P(U,\widetilde{Z}(X,Y)V)W - P(U,V)\widetilde{Z}(X,Y)W = 0. \tag{40}$$

Put $X = V = \xi$ in (40),

$$\widetilde{Z}(\xi,Y)P(U,\xi)W - P(\widetilde{Z}(\xi,Y)U,\xi)W - P(U,\widetilde{Z}(\xi,Y)\xi)W - P(U,\xi)\widetilde{Z}(\xi,Y)W = 0. \tag{41}$$

Now,

$$\begin{aligned}
\widetilde{Z}(\xi,Y)P(U,\xi)W &= R(\xi,Y)P(U,\xi)W - \frac{r}{2n(2n+1)}[g(Y,P(U,\xi)W)\xi - g(\xi,P(U,\xi)W)Y] \\
&= \left(k - \frac{r}{2n(2n+1)}\right)[g(Y,P(U,\xi)W)\xi - \eta(P(U,\xi)W)Y], \qquad (42)\\
P(\widetilde{Z}(\xi,Y)U,\xi)W &= \left(k - \frac{r}{2n(2n+1)}\right)[g(Y,U)P(\xi,\xi)W - \eta(U)P(Y,\xi)W], \qquad (43)\\
P(U,\widetilde{Z}(\xi,Y)\xi)W &= \left(k - \frac{r}{2n(2n+1)}\right)[\eta(Y)P(U,\xi)W - P(U,Y)W], \qquad (44)\\
P(U,\xi)\widetilde{Z}(\xi,Y)W &= \left(k - \frac{r}{2n(2n+1)}\right)[g(Y,W)P(U,\xi)\xi - \eta(W)P(U,\xi)Y]. \qquad (45)
\end{aligned}$$

Using equations (42)–(45) in (41), yields

$$\begin{aligned}
&\left(k - \frac{r}{2n(2n+1)}\right)[g(Y,P(U,\xi)W)\xi - \eta(P(U,\xi)W)Y - g(Y,U)P(\xi,\xi)W \\
&\quad + \eta(U)P(Y,\xi)W - \eta(Y)P(U,\xi)W + P(U,Y)W - g(Y,W)P(U,\xi)\xi + \eta(W)P(U,\xi)Y] = 0. \qquad (46)
\end{aligned}$$

Therefore, either $k = \frac{r}{2n(2n+1)}$, or

$$\begin{aligned}
&g(Y,P(U,\xi)W)\xi - \eta(P(U,\xi)W)Y - g(Y,U)P(\xi,\xi)W + \eta(U)P(Y,\xi)W \\
&\quad - \eta(Y)P(U,\xi)W + P(U,Y)W - g(Y,W)P(U,\xi)\xi + \eta(W)P(U,\xi)Y = 0. \qquad (47)
\end{aligned}$$

Using (2), (4), (8) and (11) in (47), and taking inner product with ξ we have

$$\eta(W)[S(U,Y) - 2nk\,g(U,Y)] = 0 \tag{48}$$

for all vector fields Y, U, W. Since $\eta(W) \neq 0$, equation (48) yields

$$S(U,Y) = 2nk\,g(U,Y).$$

Thus, $\widetilde{Z}.P = 0$ implies $r = 2n(2n+1)k$ or Einstein manifold. Comparing the above value of r from (12), we have

$$k = 1 - \frac{1}{n},$$

and hence M is locally isometric to the manifold of Example 2.1 for $n > 1$ and flat for $n = 1$. □

In view of the above discussion we have the following:

Theorem 4.3. *If a $(2n+1)$-dimensional non-Sasakian N(k)-contact metric manifold M satisfies $\widetilde{Z}.P = 0$, then either M is an Einstein manifold or M is locally isometric to Example 1 for $n > 1$ and flat for $n = 1$.*

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