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Generalizing Inequalities Using Power Series Approach

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Abstract: In 1903 Nesbitt introduced a famous inequality: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ for any positive real numbers a, b and c. Among all its proofs, Mortici provided a unique approach applying the convergence of power series together with the power means inequality. Adopting this technique, we first generalize several Nesbitt type inequalities to n variable versions. We then combine the knowledge of power series, Young's inequality, and the rearrangement inequality, and deduce some new inequalities.

MSC: 26D15

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1. Introduction

In 1903 Nesbitt introduced a famous inequality in [6]: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ for any positive real numbers a, b and c. Since then this inequality has been studied and applied in many papers. There are many ways to prove this inequality. At the time our paper was composed, nine different proofs for it have been organized in its Wikipedia page [10]. In 2012 Mortici introduced yet another proof in [5]. In the proof Mortici converted the left side fractions into convergent power series and ingeniously found the lower bound of the limit applying the power means inequality. Very soon this method was then adopted to analyze many other cyclic inequalities, like [3] and [9]. When studying these papers, we noticed that some old results of Nesbitt type in [8] can all be proved using this method, and not surprisingly the new proofs are much simpler. Therefore in this paper, we will first recall these old results and introduce new proofs using this power series approach. We then will improve a theorem introduced by Lai in [4]. After generalizing two more inequalities according to Mortici's results, we will then introduce another new result inspired by Mortici's paper.

We would like to start with some background knowledge that will be used in this paper. The first property is of course the convergence of the power series. It can be found in many Calculus textbooks. Here we refer to [7] for the following.

Theorem 1.1 (The Convergence of Power Series)**.** *For a real variable $x\in(-1,1)$, we have*

$$\sum_{i=0}^{\infty}x^i=\frac{1}{1-x}\quad \textit{and}\quad \sum_{i=1}^{\infty}ix^i=\frac{x}{(1-x)^2}.$$

We will also be using several other inequalities in this paper, so we summarize them here. Interested readers may refer to [1] or [2] for their proofs.

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Theorem 1.2 (Power Means Inequality). *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, and p be a real number. If $p \neq 0$, the power mean of $x_1, x_2, \cdots, x_n$ with exponent p is defined by $M_p(x_1, x_2, \cdots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$. If $p = 0$, $M_0(x_1, x_2, \cdots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$, the geometric mean.*
If $p > q$, then

$$M_p(x_1, x_2, \cdots, x_n) \geq M_q(x_1, x_2, \cdots, x_n).$$

The equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 1.3 (Rearrangement Inequality). *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(x_1, x_2, \cdots, x_n)$ of $(a_1, a_2, \cdots, a_n)$ we have the following:*

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq x_1b_1 + x_2b_2 + \cdots + x_nb_n \geq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n.$$

Young's inequality is another famous result mentioned and applied in many Analysis books. In most books, including inequality focused ones like [1], it is introduced as a two variable version. In our paper we need to apply an n variable version, so here we provide a simple proof along with the statement.

Theorem 1.4 (Generalized Young's Inequality). *Let $a_1, a_2, \cdots, a_n$ be positive real numbers and $b_1, b_2, \cdots, b_n > 1$ be real numbers such that $\frac{1}{b_1} + \cdots + \frac{1}{b_n} = 1$. Then*

$$\sum_{i=1}^{n} \frac{a_i^{b_i}}{b_i} \geq \prod_{i=1}^{n} a_i.$$

Equality occurs if and only if all $a_i^{b_i}$ are equal.

Proof. Since $f(x) = \ln x$ is a concave function, we have

$$\ln\left(\frac{a_1^{b_1}}{b_1} + \cdots + \frac{a_n^{b_n}}{b_n}\right) \geq \frac{1}{b_1}\ln a_1^{b_1} + \cdots + \frac{1}{b_n}\ln a_n^{b_n} = \ln a_1 + \cdots + \ln a_n = \ln(a_1a_2\cdots a_n).$$

After exponentiating both ends of the above, we then have the desired inequality. □

2. Main Result

We first summarize Mortici's proof of Nesbitt's inequality.

Theorem 2.1 (Nesbitt's Inequality). *Let a, b, c be positive real numbers. Then*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof. (Mortici) Allow $s = a+b+c$, $\frac{a}{s} = a'$, $\frac{b}{s} = b'$, and $\frac{c}{s} = c'$. Then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a'}{b'+c'} + \frac{b'}{c'+a'} + \frac{c'}{a'+b'} = \frac{a'}{1-a'} + \frac{b'}{1-b'} + \frac{c'}{1-c'}.$$

Since $a', b', c' < 1$,

$$\frac{a'}{1-a'} + \frac{b'}{1-b'} + \frac{c'}{1-c'} = \sum_{i=1}^{\infty}(a')^i + \sum_{i=1}^{\infty}(b')^i + \sum_{i=1}^{\infty}(c')^i = \sum_{i=1}^{\infty}\left[(a')^i + (b')^i + (c')^i\right].$$

Applying power means inequality,

$$\sum_{i=1}^{\infty}\left[(a')^i + (b')^i + (c')^i\right] \geq \sum_{i=1}^{\infty} 3\left[\frac{a'+b'+c'}{3}\right]^i = \sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i = \frac{3}{2}.$$

□

In Nesbitt's inequality, if $a = b = c$, the three fractions at the left side all take value $\frac{1}{2}$. That is when the equality occurs. If we consider the case of n variables with n fractions at the left side, it is very reasonable to guess the lower bound $\frac{n}{n-1}$. The next theorem was introduced by Wei and Wu in [8] as Theorem 2, which is the exact generalization of Nesbitt's inequality of n variable case.

Theorem 2.2 (Wei and Wu)**.** *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, where $n \geq 2$. Then*

$$\frac{x_1}{x_2 + x_3 + \cdots + x_n} + \frac{x_2}{x_1 + x_3 + x_4 + \cdots + x_n} + \cdots + \frac{x_n}{x_1 + x_2 + \cdots + x_{n-1}} \geq \frac{n}{n-1}.$$

For the original proof of this inequality in [8], Wei and Wu applied Chebyshev's inequality. However, readers may already notice that, we can use the same power series approach to prove this old result. Moreover, Theorems 3, 4, and 5 in [8] can all be proved using the same technique. Here we generalize their Theorem 5 to an n variable case and provide a proof using power series approach.

Theorem 2.3. *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, $x_1 + x_2 + \cdots + x_n = s$, and $r \geq 1$. Then*

$$\frac{x_1^r}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^r}{x_1 + x_3 + x_4 + \cdots + x_n} + \cdots + \frac{x_n^r}{x_1 + x_2 + \cdots + x_{n-1}} \geq \frac{n}{n-1}\left(\frac{s}{n}\right)^{r-1}.$$

Proof. Allow $x_i' = \frac{x_i}{s}$ for all i. We then have

$$\begin{aligned}
\frac{x_1^r}{s - x_1} + \frac{x_2^r}{s - x_2} + \cdots + \frac{x_n^r}{s - x_n} &= s^{r-1}\left(\frac{(x_1')^r}{1 - x_1'} + \frac{(x_2')^r}{1 - x_2'} + \cdots + \frac{(x_n')^r}{1 - x_n'}\right) \\
&= s^{r-1}\left(\sum_{i=0}^{\infty}(x_1')^{i+r} + \cdots + \sum_{i=0}^{\infty}(x_n')^{i+r}\right) \\
&= s^{r-1}\sum_{i=0}^{\infty}\left((x_1')^{i+r} + \cdots + (x_n')^{i+r}\right) \\
&\geq s^{r-1}\sum_{i=0}^{\infty}\left(n \cdot \left(\frac{x_1' + \cdots + x_n'}{n}\right)^{i+r}\right) \\
&= s^{r-1}\sum_{i=0}^{\infty}\frac{1}{n^{i+r-1}} \\
&= \frac{n}{n-1}\left(\frac{s}{n}\right)^{r-1}.
\end{aligned}$$

□

Trying to generalize the Nesbitt's inequality to the case of more than one item summing at the numerators, Lai proved the next result in [4] using Radon's inequality.

Theorem 2.4 (Lai)**.** *Let $x_1, x_2, \cdots, x_n$ be positive real numbers. For positive integer $k < n$, let $S(k)_1, S(k)_2, \cdots, S(k)_{C(n,k)}$ be the sums of k elements in $x_1, x_2, \cdots, x_n$ for all $C(n,k)$ combinations respectively, and let $S(n) = x_1 + \cdots + x_n$. Then*

$$\frac{S(k)_1}{S(n) - S(k)_1} + \frac{S(k)_2}{S(n) - S(k)_2} + \cdots + \frac{S(k)_{C(n,k)}}{S(n) - S(k)_{C(n,k)}} \geq \frac{k \cdot C(n,k)}{n-k}.$$

Since there are k elements at the numerator and $(n-k)$ elements at the denominator in each fraction of the left side, we can see where the factor $\frac{k}{n-k}$ comes from at the right side. Totally, there are $C(n,k)$ fractions in the sum of the left side, so that contributes to the factor $C(n,k)$ at the right side lower bound. However, the sum of fractions at the left side of the above result is symmetric, not cyclically arranged throughout $x_1, \cdots, x_n$ like the original Nesbitt's inequality. Besides, if we have some big n and k, the value of $C(n,k)$, hence the number of fractions needed, will also be big in order to reach the lower bound. With the help of the power series technique, we can finally improve the above theorem to a cyclic version, which is more in line with the original Nesbitt's inequality, and more applicable in related areas. Before introducing the next result, we would like to warn the readers about the notations in advance. Since it is an improvement of the previous result, we adopted most of the notations. However, the notation $S(k)_i$ used for a symmetric sum in the previous result is now used to indicate a cyclic sum in the next theorem.

Theorem 2.5. *Let $x_1, x_2, \cdots, x_n$ be positive real numbers. For a positive integer $k < n$, let $S(k)_i = x_i + \cdots + x_{i+k-1}$ where $x_{n+j} = x_j$, and let $S(n) = x_1 + \cdots + x_n$. Then*

$$\frac{S(k)_1}{S(n) - S(k)_1} + \cdots + \frac{S(k)_n}{S(n) - S(k)_n} \geq \frac{k \cdot n}{n-k}.$$

Proof. Similar to other proofs we had already, we let $S'(k)_i = \frac{S(k)_i}{S(n)}$. Then

$$\begin{aligned}
\frac{S(k)_1}{S(n) - S(k)_1} + \cdots + \frac{S(k)_n}{S(n) - S(k)_n} &= \frac{S'(k)_1}{1 - S'(k)_1} + \cdots + \frac{S'(k)_n}{1 - S'(k)_n} \\
&= \sum_{i=1}^{\infty} \left(S'(k)_1\right)^i + \cdots + \sum_{i=1}^{\infty} \left(S'(k)_n\right)^i \\
&= \sum_{i=1}^{\infty} \left[\left(S'(k)_1\right)^i + \cdots + \left(S'(k)_n\right)^i\right] \\
&\geq \sum_{i=1}^{\infty} n \left(\frac{S'(k)_1 + \cdots + S'(k)_n}{n}\right)^i \\
&= \sum_{i=1}^{\infty} n \left(\frac{k}{n}\right)^i \\
&= \frac{k \cdot n}{n-k}.
\end{aligned}$$

□

Next we move on to some results requiring other inequalities. As a matter of fact, Mortici already practiced this idea in [5]. For example, the next result was introduced by Mortici, applying Young's inequality.

Theorem 2.6 (Mortici)**.** *For $a, b \in (0,1)$ and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\frac{q}{1-a^p} + \frac{p}{1-b^q} \geq \frac{pq}{1-ab}$$

and

$$\frac{a^p}{p\left(1-a^p\right)^2} + \frac{b^q}{q\left(1-b^q\right)^2} \geq \frac{ab}{(1-ab)^2}.$$

Since we already proved a generalized Young's inequality in Section 1, we then can generalize Mortici's inequalities to an n variable version.

Theorem 2.7. *For $a_1, \cdots, a_n \in (0,1)$ and $b_1, \cdots, b_n > 0$ with $\frac{1}{b_1} + \cdots + \frac{1}{b_n} = 1$, we have*

$$\frac{1}{b_1\left(1 - a_1^{b_1}\right)} + \cdots + \frac{1}{b_n\left(1 - a_n^{b_n}\right)} \geq \frac{1}{1 - a_1 a_2 \cdots a_n}$$

and

$$\frac{a_1^{b_1}}{b_1\left(1 - a_1^{b_1}\right)^2} + \cdots + \frac{a_n^{b_n}}{b_n\left(1 - a_n^{b_n}\right)^2} \geq \frac{a_1 a_2 \cdots a_n}{\left(1 - a_1 a_2 \cdots a_n\right)^2}.$$

Proof. Applying the generalized Young's inequality we have

$$\begin{aligned}
\frac{1}{b_1\left(1-a_1^{b_1}\right)}+\cdots+\frac{1}{b_n\left(1-a_n^{b_n}\right)} &= \frac{1}{b_1}\cdot\sum_{i=0}^{\infty}\left(a_1^{b_1}\right)^i+\cdots+\frac{1}{b_n}\cdot\sum_{i=0}^{\infty}\left(a_n^{b_n}\right)^i \\
&= \sum_{i=0}^{\infty}\left(\frac{\left(a_1^i\right)^{b_1}}{b_1}+\cdots+\frac{\left(a_n^i\right)^{b_n}}{b_n}\right) \\
&\geq \sum_{i=0}^{\infty}\left(a_1a_2\cdots a_n\right)^i \\
&= \frac{1}{1-a_1a_2\cdots a_n}.
\end{aligned}$$

For the second inequality, we have

$$\begin{aligned}
\frac{a_1^{b_1}}{b_1\left(1-a_1^{b_1}\right)^2}+\cdots+\frac{a_n^{b_n}}{b_n\left(1-a_n^{b_n}\right)^2} &= \frac{1}{b_1}\cdot\sum_{i=1}^{\infty}i\left(a_1^{b_1}\right)^i+\cdots+\frac{1}{b_n}\cdot\sum_{i=1}^{\infty}i\left(a_n^{b_n}\right)^i \\
&= \sum_{i=1}^{\infty}i\left(\frac{\left(a_1^i\right)^{b_1}}{b_1}+\cdots+\frac{\left(a_n^i\right)^{b_n}}{b_n}\right) \\
&\geq \sum_{i=1}^{\infty}i\left(a_1a_2\cdots a_n\right)^i \\
&= \frac{a_1a_2\cdots a_n}{\left(1-a_1a_2\cdots a_n\right)^2}.
\end{aligned}$$

□

The next theorem is a result applying the rearrangement inequality.

Theorem 2.8. *For $a_1 \leq a_2 \leq \cdots \leq a_n \in (0,1)$, let $b_1, \cdots, b_n$ be a rearrangement of sequence $\{a_i\}$ in a random order. For real numbers $k > m$, we have*

$$\frac{1}{1-a_1^k}+\cdots+\frac{1}{1-a_n^k} \geq \frac{1}{1-a_1^{k-m}b_1^m}+\cdots+\frac{1}{1-a_n^{k-m}b_n^m}$$

and

$$\frac{a_1^k}{(1-a_1^k)^2}+\cdots+\frac{a_n^k}{(1-a_n^k)^2} \geq \frac{a_1^{k-m}b_1^m}{(1-a_1^{k-m}b_1^m)^2}+\cdots+\frac{a_n^{k-m}b_n^m}{(1-a_n^{k-m}b_n^m)^2}.$$

Proof. From the rearrangement inequality we know

$$a_1^k+\cdots+a_n^k \geq a_1^{k-m}b_1^m+\cdots+a_n^{k-m}b_n^m.$$

Therefore,

$$\begin{aligned}
\frac{1}{1-a_1^k}+\cdots+\frac{1}{1-a_n^k} &= \sum_{i=0}^{\infty}a_1^{ki}+\cdots+\sum_{i=0}^{\infty}a_n^{ki} \\
&\geq \sum_{i=0}^{\infty}\left(a_1^{k-m}b_1^m\right)^i+\cdots+\sum_{i=0}^{\infty}\left(a_n^{k-m}b_n^m\right)^i \\
&= \frac{1}{1-a_1^{k-m}b_1^m}+\cdots+\frac{1}{1-a_n^{k-m}b_n^m}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{a_1^k}{(1-a_1^k)^2}+\cdots+\frac{a_n^k}{(1-a_n^k)^2} &= \sum_{i=1}^{\infty}ia_1^{ki}+\cdots+\sum_{i=1}^{\infty}ia_n^{ki} \\
&\geq \sum_{i=1}^{\infty}i\left(a_1^{k-m}b_1^m\right)^i+\cdots+\sum_{i=1}^{\infty}i\left(a_n^{k-m}b_n^m\right)^i \\
&= \frac{a_1^{k-m}b_1^m}{(1-a_1^{k-m}b_1^m)^2}+\cdots+\frac{a_n^{k-m}b_n^m}{(1-a_n^{k-m}b_n^m)^2}.
\end{aligned}$$

□

Theorems 2.7 and 2.8 provide some pathways to the insight of inequalities involving sums of fractions. These type of inequalities are often seen in Mathematics Olympiad or other Mathematics competitions. However there is a catch, which may make these results not always applicable. Because we need to make sure the power series in the proof are convergent, the sequence $\{a_i\}$ in these two theorems has to be restricted in the interval $(0,1)$. As for how to generalize them to the case in other intervals, or even the whole positive real numbers, it is still open.

It is worth noting that, in case readers are interested, Xu also practiced this idea of combining power series and other inequalities, in [9], and introduced some interesting results applying Muirhead inequality.

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