

New Function in Fuzzy Topological Spaces

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Abstract: In this paper, we introduce fuzzy generalized semi-open and fuzzy generalized semi-closed maps in fts and obtain certain characterization of the generalized semi-closed and generalized semi- open maps.

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1. Introduction

The concept of a fuzzy subset was introduced and studied by L.A.Zadeh in the year 1965. The subsequent research activities in this area and related areas have found applications in many branches of science and engineering. In the year 1965, L.A.Zadeh [1] introduced the concept of fuzzy subset as a generalization of that of an ordinary subset. The introduction of fuzzy subsets paved the way for rapid research work in many areas of mathematical science. In the year 1968, C.L.Chang [2] introduced the concept of fuzzy topological spaces as an application of fuzzy sets to topological spaces. Subsequently several researchers contributed to the development of the theory and applications of fuzzy topology. The theory of fuzzy topological spaces can be regarded as a generalization theory of topological spaces. An ordinary subset A or a set X can be characterized by a function called characteristic function $\mu_A : X \rightarrow [0, 1]$ of A , defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Thus an element $x \in X$ is in A if $\mu_A(x) = 1$ and is not in A if $\mu_A(x) = 0$. In general if X is a set and A is a subset of X then A has the following representation. $A = \{(x, \mu_A(x)) : x \in X\}$, here $\mu_A(x)$ may be regarded as the degree of belongingness of x to A , which is either 0 or 1. Hence A is the class of objects with degree of belongingness either 0 or 1 only. L. A. Zadeh [1] introduced a class of objects with continuous grades of belongingness ranging between 0 and 1; he called such a class as fuzzy subset. A fuzzy subset A in X is characterized as a membership function $\mu_A : X \rightarrow [0, 1]$, which associates with each point in x a real number $\mu_A(x)$ between 0 and 1 which represents the degree or grade membership of belongingness of x to A .

The purpose of this paper is to introduce a new class of fuzzy sets called fuzzy gs-closed sets in fuzzy topological spaces and investigate certain basic properties of these fuzzy sets. Among many other results it is observed that every fuzzy closed

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set is fuzzy gs-closed but not conversely. Also we introduce fuzzy gs-open sets in fuzzy topological spaces and study some of their properties.

1.1. Preliminaries

Definition 1.1 ([1]). A fuzzy subset A in a set X is a function $A : X \rightarrow [0, 1]$. A fuzzy subset in X is empty iff its membership function is identically 0 on X and is denoted by 0 or μ_ϕ . The set X can be considered as a fuzzy subset of X whose membership function is identically 1 on X and is denoted by μ_x or I_x . In fact every subset of X is a fuzzy subset of X but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

Definition 1.2 ([1]). If A and B are any two fuzzy subsets of a set X , then A is said to be included in B or A is contained in B iff $A(x) \leq B(x)$ for all x in X . Equivalently, $A \leq B$ iff $A(x) \leq B(x)$ for all x in X .

Definition 1.3 ([1]). Two fuzzy subsets A and B are said to be equal if $A(x) = B(x)$ for every x in X . Equivalently $A = B$ if $A(x) = B(x)$ for every x in X .

Definition 1.4 ([1]). The complement of a fuzzy subset A in a set X , denoted by A' or $1 - A$, is the fuzzy subset of X defined by $A'(x) = 1 - A(x)$ for all x in X . Note that $(A')' = A$.

Definition 1.5 ([1]). The union of two fuzzy subsets A and B in X , denoted by $A \vee B$, is a fuzzy subset in X defined by $(A \vee B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all x in X .

Definition 1.6 ([1]). The intersection of two fuzzy subsets A and B in X , denoted by $A \wedge B$, is a fuzzy subset in X defined by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for all x in X .

Definition 1.7 ([1]). A fuzzy set on X is 'Crisp' if it takes only the values 0 and 1 on X .

Definition 1.8 ([2]). Let X be a set and τ be a family of fuzzy subsets of X . τ is called a fuzzy topology on X iff τ satisfies the following conditions.

- (1). $\mu_\phi; \mu_x \in \tau$, that is 0 and 1 $\in \tau$.
- (2). If $G_i \in \tau$ for $i \in I$ then $\bigvee_{i \in I} G_i \in \tau$.
- (3). If $G, H \in \tau$ then $G \wedge H \in \tau$.

The pair (X, τ) is called a fuzzy topological space (abbreviated as fts). The members of τ are called fuzzy open sets and a fuzzy set A in X is said to be closed iff $1 - A$ is an fuzzy open set in X .

Remark 1.9 ([2]). Every topological space is a fuzzy topological space but not conversely.

Definition 1.10 ([2]). Let X be a fts and A be a fuzzy subset in X . Then $\bigwedge\{B : B \text{ is a closed fuzzy set in } X \text{ and } B \geq A\}$ is called the closure of A and is denoted by A or $cl(A)$.

Definition 1.11 ([2]). Let A and B be two fuzzy sets in a fuzzy topological space (X, τ) and let $A \geq B$. Then B is called an interior fuzzy set of A if there exists $G \in \tau$ such that $A \geq G \geq B$, the least upper bound of all interior fuzzy sets of A is called the interior of A and is denoted by A^0 .

Definition 1.12 ([3]). A fuzzy set A in a fts X is said to be fuzzy semi-open if and only if there exists a fuzzy open set V in X such that $V \leq A \leq cl(V)$.

Definition 1.13 ([3]). A fuzzy set A in a fts X is said to be fuzzy semi-closed if and only if there exists a fuzzy closed set V in X such that $\text{int}(V) \leq A \leq V$. It is seen that a fuzzy set A is fuzzy semiopen if and only if $1 - A$ is a fuzzy semi-closed.

Theorem 1.14 ([3]). The following are equivalent:

- (a). μ is a fuzzy semiclosed set,
- (b). μ^C is a fuzzy semiopen set,
- (c). $\text{int}(\text{cl}(\mu)) \leq \mu$.
- (d). $\text{int}(\text{cl}(\mu)) \geq \mu^c$.

Theorem 1.15 ([3]). Any union of fuzzy semi-open sets is a fuzzy semi-open set and (b) any intersection of fuzzy semi closed sets is a fuzzy semi closed.

Remark 1.16 ([3]). (a). Every fuzzy open set is a fuzzy semi-open but not conversely.

- (b). Every fuzzy closed set is a fuzzy semi-closed set but not conversely.
- (c). The closure of a fuzzy open set is fuzzy semi-open set
- (d). The interior of a fuzzy closed set is fuzzy semi-closed set

Definition 1.17 ([3]). A fuzzy set μ of a fts X is called a fuzzy regular open set of X if $\text{int}(\text{cl}(\mu)) = \mu$.

Definition 1.18 ([3]). A fuzzy set μ of fts X is called a fuzzy regular closed set of X if $\text{cl}(\text{int}(\mu)) = \mu$.

Theorem 1.19 ([3]). A fuzzy set μ of a fts X is a fuzzy regular open if and only if μ^C fuzzy regular closed set.

Remark 1.20 ([3]).

- (a). Every fuzzy regular open set is a fuzzy open set but not conversely.
- (b). Every fuzzy regular closed set is a fuzzy closed set but not conversely.

Theorem 1.21 ([3]).

- (a). The closure of a fuzzy open set is a fuzzy regular closed.
- (b). The interior of a fuzzy closed set is a fuzzy regular open set.

Definition 1.22 ([4]). A fuzzy set α in fts X is called fuzzy rw-closed if $\text{cl}(\alpha) \leq \mu$ whenever $\alpha \leq \mu$ and μ is regular semi-open in X .

Definition 1.23 ([5]). A fuzzy set α in fts X is called fuzzy pgprw closed if $p\text{-cl}(\alpha) \leq \mu$ whenever $\alpha \leq \mu$ and μ is $\text{rg}\alpha$ -open set in X .

Definition 1.24 ([4]). A fuzzy set α of a fts X is fuzzy pgprw-open set, if it's complement α^c is a fuzzy pgprw-closed in fts X .

Definition 1.25 ([2]). Let X and Y be fts. A map $f : X \rightarrow Y$ is said to be a fuzzy continuous mapping if $f^{-1}(\mu)$ is fuzzy open in X for each fuzzy open set μ in Y .

Definition 1.26 ([6]). A mapping $f : X \rightarrow Y$ is said to be fuzzy irresolute iff $f^{-1}(B)$ is fuzzy semi-open in X is fuzzy open in Y .

Definition 1.27 ([7]). A mapping $f : X \rightarrow Y$ is said to be fuzzy pgprw irresolute if the inverse image of every fuzzy pgprw-open in Y is a fuzzy pgprw-open set in X .

2. Fuzzy gs-open Maps and Fuzzy gs-closed Maps in Fuzzy Topological Spaces

Definition 2.1. Let X and Y be two fts. A map $f : (X, T_1) \rightarrow (Y, T_2)$ is called fuzzy genealized semi-open map (fuzzy gs-open map) if the image of every fuzzy open set in X is fuzzy gs-open in Y .

Theorem 2.2. Every fuzzy open map is a fuzzy gs-open map.

Proof. Let $f : (X, T_1) \rightarrow (Y, T_2)$ be a fuzzy open map and μ be a fuzzy open set in fts X . Then $f(\mu)$ is a fuzzy open set in fts Y . Since every fuzzy open set is fuzzy gs-open, $f(\mu)$ is a fuzzy gs-open set in fts Y . Hence f is a fuzzy gs-open map. $\square$

The converse of the above theorem need not be true in general as seen from the following example.

Example 2.3. Let $X = Y = \{a, b, c\}$ and the functions $\alpha, \beta : X \rightarrow [0, 1]$ be defined as

$$\alpha(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{Otherwise} \end{cases}$$

Consider $T_1 = \{1, 0, \alpha\}$, $T_2 = \{1, 0, \beta\}$ then $(X, T_1) \rightarrow (Y, T_2)$ are fts. Let map $f : X \rightarrow Y$ be the identity map. Then this function is gs-open map but it is not fuzzy open. Since the image of the fuzzy open set $\alpha(x)$ in X is fuzzy set β in Y which is not fuzzy open.

Remark 2.4. If $f : (X, T_1) \rightarrow (Y, T_2)$ and $g : (Y, T_2) \rightarrow (Z, T_3)$ be two fuzzy gs-open map then composition $g \circ f : (X, T_1) \rightarrow (Z, T_3)$ need not be fuzzy gs-open as seen from the following example.

Example 2.5. Let $X = Y = Z = \{a, b, c\}$ and the function $\alpha, \beta, \gamma, \delta : X \rightarrow [0, 1]$ be defined as

$$\alpha(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$$

$$\delta(x) = \begin{cases} 1, & \text{if } x = a, b \\ 0, & \text{otherwise} \end{cases}$$

Consider $T_1 = \{1, 0, \alpha, \delta\}$, $T_2 = \{1, 0, \alpha\}$ and $T_3 = \{0, 1, \alpha, \beta, \gamma\}$. Let map $f : (X, T_1) \rightarrow (Y, T_2)$ and $g : (Y, T_2) \rightarrow (Z, T_3)$ be the identity maps then f and g are fuzzy gs-open map but their composition $g \circ f : (X, T_1) \rightarrow (Z, T_3)$ is not fuzzy gs-open as $\beta : X \rightarrow [0, 1]$ is fuzzy open in X . But $(g \circ f)(\beta) = \beta$ is not fuzzy gs-open in Z .

Theorem 2.6. If $f : (X, T_1) \rightarrow (Y, T_2)$ is fuzzy open map and $g : (Y, T_2) \rightarrow (Z, T_3)$ is fuzzy gs-open map then their composition $g \circ f : (X, T_1) \rightarrow (Z, T_3)$ is fuzzy gs-open map.

Proof. Let α be fuzzy open set in (X, T_1) . Since f is fuzzy open map, $f(\alpha)$ is a fuzzy open set in (Y, T_2) . Since g is a fuzzy gs-open map $g(f(\alpha)) = (g \circ f)(\alpha)$. Thus $g \circ f$ is a fuzzy gs-open map. $\square$

Definition 2.7. Let X and Y be two fuzzy topological spaces. A map $f : (X, T_1) \rightarrow (Y, T_2)$ is called fuzzy generalized-closed map (fuzzy gs-closed map) if the image of every fuzzy closed set in X is a fuzzy closed set in Y .

Theorem 2.8. Let $f : (X, T_1) \rightarrow (Y, T_2)$ be fuzzy closed map then f is a fuzzy gs closed map.

Proof. Let α be a fuzzy closed set in (X, T_1) . Since f is fuzzy closed map, $f(\alpha)$ is a fuzzy closed set in (Y, T_2) . Since every fuzzy closed set is fuzzy gs-closed, $f(\alpha)$ is a fuzzy gs-closed set in (Y, T_2) . Hence f is fuzzy gs-closed map. $\square$

The converse of the theorem need not be true in general as seen from the following example.

Example 2.9. Let $X = Y = \{a, b, c\}$ and the functions $\alpha, \beta : X \rightarrow [0, 1]$ be defined as

$$\alpha(x) = \begin{cases} 1, & \text{if } x = a, c \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

Consider $T_1 = \{1, 0, \alpha\}$ and $T_2 = \{1, 0, \beta\}$; then (X, T_1) and (Y, T_2) are fts. Let $f : (X, T_1) \rightarrow (Y, T_2)$ be the identity map. Then f is a fuzzy gs-closed map but it is not a fuzzy closed map, since the image of the closed set α in X is not a fuzzy closed set in Y .

Remark 2.10. The composition of two fuzzy gs-closed maps need not be a fuzzy gs-closed map.

Example 2.11. Consider the fts (X, T_1) , (Y, T_2) , (Z, T_3) and mappings Defined in 2.5. the maps f and g are fuzzy gs-closed but their composition is not fuzzy gs-closed as $\beta : X \rightarrow [0, 1]$ is a fuzzy closed set in X but $(gof)(\beta) = \beta$ is not fuzzy gs-closed in Z .

Theorem 2.12. If a map $f : (X, T_1) \rightarrow (Y, T_2)$ and $g : (Y, T_2) \rightarrow (Z, T_3)$ be two maps then composition of two gs-closed maps is fuzzy gs-closed map.

Proof. Let α be a fuzzy closed set in (X, T_1) . Since f is a fuzzy closed map, $f(\alpha)$ is a fuzzy closed set in (Y, T_2) . Since g is a fuzzy gs-closed map but $g(f(\alpha))$ is a fuzzy gs-closed set in (Z, T_3) but $g(f(\alpha)) = (gof)(\alpha)$. Thus gof is fuzzy gs-closed map. $\square$

Theorem 2.13. A map $f : X \rightarrow Y$ is fuzzy gs-closed map if for each fuzzy set δ of Y and for each fuzzy open set μ of X s.t $\mu \geq f^{-1}(\delta)$, there is a fuzzy gs-open map set α of Y s.t $\delta \leq \alpha$ and $f^{-1}(\alpha) \leq \mu$.

Proof. Suppose that f is fuzzy gs-closed map. Let δ be a fuzzy subset of Y and μ be a fuzzy open set of X s.t $f^{-1}(\delta) \leq \mu$. Let $\alpha = 1 - f(1 - \mu)$ is fuzzy gs-open set in fts Y . Note that $f^{-1}(\delta) \leq \mu$. Which implies $\delta \leq \alpha$ and $f^{-1}(\alpha) \leq \mu$. For the converse, Suppose that μ is a fuzzy closed set in X . Then $f^{-1}(1 - f(\mu)) \leq 1 - \mu$ and $1 - \mu$ is fuzzy open by hypothesis, there is a fuzzy gs-open set α of Y s.t $(1 - f(\mu)) \leq \alpha$. And $f^{-1}(\alpha) \leq 1 - \mu$. Therefore $\mu \leq 1 - f^{-1}(\alpha)$. Hence $1 - \alpha \leq \mu$, $f(1 - f^{-1}(\alpha)) \leq 1 - \alpha$. Which implies $f(\mu) = 1 - \alpha$. Since $1 - \alpha$ is fuzzy gs-closed $f(\mu)$ is fuzzy gs-closed and thus f is fuzzy gs-closed. $\square$

Theorem 2.14. Let $f : (X, T_1) \rightarrow (Y, T_2)$ be fuzzy irresolute map and α be fuzzy rg α open in Y then $f^{-1}(\alpha)$ is fuzzy rg α open in X .

Proof. Let α be fuzzy $rg\alpha$ -open in Y . To prove $f^{-1}(\alpha)$ is fuzzy $rg\alpha$ open in X that is to prove $f^{-1}(\alpha)$ is both fuzzy semi-open and fuzzy semi-closed in X . Now α is fuzzy semi-open in Y . Since f is fuzzy irresolute map, $f^{-1}(\alpha)$ is fuzzy semi-open in X . Now α is fuzzy semi-closed in Y as fuzzy $rg\alpha$ open set is fuzzy semi-closed then $1 - \alpha$ is fuzzy semi-open in Y . Since f is fuzzy irresolute map, $f^{-1}(1 - \alpha)$ is a fuzzy semi open in X . But $f^{-1}(1 - \alpha) = 1 - f^{-1}(\alpha)$ is fuzzy semi-open in X and so $f^{-1}(\alpha)$ is semi closed in X . Thus $f^{-1}(\alpha)$ is both fuzzy semi-open and fuzzy semi-closed in X and hence $f^{-1}(\alpha)$ is fuzzy $rg\alpha$ open in X . $\square$

Theorem 2.15. *If a map $f : (X, T_1) \rightarrow (Y, T_2)$ is fuzzy irresolute map and fuzzy gs -closed and α is fuzzy gs -closed set of X , then $f(\alpha)$ is a fuzzy gs -closed set in Y .*

Proof. Let α be a fuzzy closed set of X . Let $f(\alpha) \leq \mu$. Where μ is fuzzy $rg\alpha$ open in Y . Since f is fuzzy irresolute map, $f^{-1}(\mu)$ is a fuzzy $rg\alpha$ open in X , by Lemma 2.14 and $\alpha \leq f^{-1}(\mu)$. Since α is a fuzzy gs -closed set in X , $P - cl(\alpha) \leq f^{-1}(\mu)$. Since f is a fuzzy gs -closed set contained in the fuzzy $rg\alpha$ open set μ , which implies $cl(f(p - cl(\alpha))) \leq \mu$ and hence $cl(f(\alpha)) \leq \mu$. Therefore $f(\alpha)$ is a fuzzy gs -closed set in Y . $\square$

Corollary 2.16. *If a map $f : (X, T_1) \rightarrow (Y, T_2)$ is fuzzy irresolute map and fuzzy closed and α is a fuzzy gs -closed set in fts .*

Proof. The proof follows from the Theorem 2.15 and the fact that every fuzzy closed map is a fuzzy gs -closed map. $\square$

Theorem 2.17. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two mappings s.t*

- (1). *if f is fuzzy continuous map and surjective, then g is fuzzy gs -closed map.*
- (2). *if g is fuzzy gs -irresolute map and injective then f is fuzzy gs -closed map.*

Proof.

- (1). Let μ be a fuzzy closed set in Y , since f is fuzzy continuous map, $f^{-1}(\mu)$ is a fuzzy closed set in X . Since $g \circ f$ is a fuzzy gs closed map, $(g \circ f)(f^{-1}(\mu))$ is a fuzzy gs -closed set in Z . but $(g \circ f)(f^{-1}(\mu)) = g(\mu)$ as f is surjective thus g is fuzzy gs closed map.
- (2). Let β be a fuzzy closed set of X then $(g \circ f)(\beta)$ is a fuzzy gs closed set in Z , Since $(g \circ f)$ is a fuzzy gs -closed map. Since g is fuzzy gs -irresolute map $g^{-1}(g \circ f)(\beta)$ is fuzzy gs -closed in Y . But $g^{-1}(g \circ f)(\beta) = f(\beta)$ as g is injective. Thus f is fuzzy gs -closed map.

$\square$

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