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Eccentric Degree Connectivity Index of Graphs

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Abstract

In this paper, a novel eccentric degree based topological index termed as eccentric degree connectivity index is conceptualized. For a connected graph G with a vertex set V, the eccentric degree connectivity index denoted by ξ^{ed} is defined as $\xi^{ed} = \sum\limits_{v \in V} ed(v)e(v)$. We obtain it's value for some classes of graphs and graph operations namely, join, corona, corona join, $r-$crown graph of fan graph and $r-$crown graph of wheel graph. We also derive some upper and lower bounds.

Keywords: Eccentricity; Eccentric degree; Eccentric degree connectivity index.

2020 Mathematics Subject Classification: 05C09, 05C12, 05C38, 05C76, 05C92.

1. Introduction

Topological indices are graph theoretical invariants formed to obtain the relationships between physical properties and structures of chemical compounds. Various topological indices are used for the study of quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR). Some of the topological indices based on degree and eccentricity are Weiner index, Zagreb indices, Ŕandic index, eccentric connectivity index, etc.

Let $G = (V, E)$ be a simple, finite and connected graph. A graph $G = (V, E)$ with $|V| = p$ and $|E| = q$ is called a (p, q) graph. The degree of $v_i \in V$ denoted by d_i is the number of edges incident with v_i. The distance between $u, v \in V(G)$ is length of the shortest path between them. The eccentricity of $u \in V(G)$ is $e(u) = max\{d(u, v) : v \in V(G)\}$[5]. A graph is said to be equieccentric graph if each vertex has the same eccentricity. Radius $r(G)$ is the minimum eccentricity of the vertices whereas diameter $diam(G)$ is the maximum eccentricity of the vertices. A vertex v is a central vertex if $e(v) = r(G)$, center of G is the set of all central vertices and is denoted by $Z(G)$ [5]. The double star graph $S_{m,n}$ is the graph constructed from $K_{1,m-1}$ and $K_{1,n-1}$ by joining their centers with an edge [7]. A fan graph F_n is obtained by joining all vertices of P_n, $n \geq 2$ to a further vertex, called the centre. Thus F_n contains

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$n+1$ vertices [9]. A wheel graph W_n, $n \geq 4$ is a graph obtained by joining all vertices of cycle C_{n-1} to a further vertex called centre [10]. In this paper we introduce a topological index namely eccentric degree connectivity index which is based on eccentric degree and eccentricity. We obtain it's value for some standard graphs and graph operations. Also, derive some upper and lower bounds for it.

2. Eccentric Degree Connectivity Index of Graphs

In a connected graph G with vertex set V, the eccentric degree $ed(u)$ of a vertex $u \in V$ is the number of vertices at a distance equal to $e(v)$. The vertices at a distance equal to $e(u)$ from u are called eccentric vertices of u, hence eccentric degree of u is the number of eccentric vertices of u in G[11]. Also, the eccentric connectivity index of G denoted by $\xi^C(G)$ is defined as $\xi^C(G) = \sum\limits_{v \in V} d(v)e(v)$[6]. Motivated by the definitions of eccentric degree and eccentric connectivity index, we introduce the eccentric degree connectivity index as follows.

Definition 2.1. *For a connected graph G with a vertex set V, the eccentric degree connectivity index denoted by ξ^{ed} is defined as $\xi^{ed}(G) = \sum\limits_{v \in V} ed(v)e(v)$.*

Now, we obtain eccentric connectivity index of some standard graphs.

Proposition 2.2. *For any complete graph $G = K_n$, $\xi^{ed}(G) = n(n-1)$.*

Proof. For every vertex $v \in V(K_n)$, $ed(v) = n-1$, and

$$e(v) = \begin{cases} 0, & if \quad n = 1 \\ 1, & if \quad n \geq 2. \end{cases}$$

Hence,

$$\xi^{ed}(G) = \sum_{v \in V} (n-1)(1) = n(n-1).$$

□

Proposition 2.3. *For any path P_n, $\xi^{ed}(P_n) = \begin{cases} \frac{3(n^2-1)}{4}, & \text{if } n \text{ is odd,} \\ \frac{n(3n-2)}{4} & \text{if } n \text{ is even.} \end{cases}$*

Proof. Let $G = P_n$ be a path and $V(G) = \{v_1, v_2, \cdots, v_n\}$. We consider the following cases.

Case 1. n is odd. Then $ed(v_i) = \begin{cases} 1 & if \quad i \neq \frac{n+1}{2}, \\ 2 & if \quad i = \frac{n+1}{2}. \end{cases}$

Also, $e(v_1) = e(v_n) = n-1$, $e(v_2) = e(v_{n-1}) = n-2$, $\cdots$, $e(v_{\frac{n-1}{2}}) = e(v_{\frac{n+3}{2}}) = \frac{n+1}{2}$ and $e(v_{\frac{n+1}{2}}) = \frac{n-1}{2}$. So, $\xi^{ed}(G) = 2[(n-1)(1) + (n-2)(1) + \cdots + (\frac{n+1}{2})(1)] + (\frac{n-1}{2})(2) = \frac{3(n^2-1)}{4}$.

Case 2. n is even. Then $ed(v_i) = 1$, for each i, $e(v_1) = e(v_n) = n-1$, $e(v_2) = e(v_{n-1}) = n-2$, $\cdots$, $e(v_{\frac{n}{2}}) = e(v_{\frac{n+3}{2}}) = \frac{n}{2}$. Hence $\xi^{ed}(G) = 2[(n-1)(1) + (n-2)(1) + \cdots + (\frac{n}{2})(1)] = \frac{n(3n-2)}{4}$. □

Proposition 2.4. *For any cycle C_n,* $\xi^{ed}(C_n) = \begin{cases} n(n-1), & \text{if } n \text{ is odd} \\ \frac{n^2}{2} & \text{if } n \text{ is even.} \end{cases}$

Proof. Let $G = C_n$. If n is odd, then $ed(v) = 2$ and $e(v) = \frac{n-1}{2}$ for every $v \in V(G)$, hence $\xi^{ed}(G) = n(2)(\frac{n-1}{2}) = n(n-1)$. If n is even, then $ed(v) = 1$ and $e(v) = \frac{n}{2}$ for every $v \in V(G)$, hence $\xi^{ed}(G) = n(1)(\frac{n}{2}) = \frac{n^2}{2}$. □

Proposition 2.5. *For a wheel graph $W_n = C_{n-1} + K_1$, $n \geq 5$, $\xi^{ed}(W_{n-1}) = (n-1)(2n-7)$.*

Proof. In wheel graph W_n, for every $v \in V(W_n)$,

$$ed(v) = \begin{cases} n-1, & \text{if } v \text{ is a central vertex;} \\ n-4, & \text{if } v \text{ is not a central vertex.} \end{cases}$$

$$e(v) = \begin{cases} 1, & \text{if } v \text{ is a central vertex;} \\ 2, & \text{if } v \text{ is not a central vertex.} \end{cases}$$

Hence, $\xi^{ed}(W_n) = (n-1)(1) + (n-1)(n-4)(2) = (n-1)(2n-7)$. □

Proposition 2.6. *For a complete bipartite graph $K_{m,n}$,*

$$\xi^{ed}(K_{m,n}) = \begin{cases} n(2n-1), & \text{if } m = 1 \text{ and } n \geq 2; \\ 2(m^2+n^2) - 2(m+n), & \text{if } m \geq 2, n \geq 2. \end{cases}$$

Proof. For $G = K_{m,n}$, we consider the following cases.

Case 1. $m = 1$ and $n \geq 2$. Then $ed(v) = \begin{cases} n, & \text{if } v \text{ is a central vertex;} \\ n-1, & \text{if } v \text{ is a pendant vertex.} \end{cases}$

Also, $e(v) = \begin{cases} 1, & \text{if } v \text{ is a central vertex;} \\ 2, & \text{if } v \text{ is a pendant vertex.} \end{cases}$

Hence, $\xi^{ed}(K_{m,n}) = (n)(1) + n(n-1)(2) = n(2n-1)$.

Case 2. $m \geq 2$ and $n \geq 2$. If $V(K_{m,n}) = V_1 \cup V_2$ is a bipartition of $V(K_{m,n})$ with $|V_1| = m$ and $|V_2| = n$, then $e(v) = 2$ for every $v \in V(K_{m,n})$ and $ed(v) = \begin{cases} m-1, & \text{if } v \in V_1; \\ n-1, & \text{if } v \in V_2. \end{cases}$

Hence, $\xi^{ed}(K_{m,n}) = m(m-1)(2) + n(n-1)(2) = 2(m^2+n^2) - 2(m+n)$. □

Proposition 2.7. *Let $S_{m,n}$, $m, n \geq 2$ be a double star graph. Then $\xi^{ed}(S_{n,m}) = 2(3mn - 2m - 2n + 1)$.*

Proof. Consider $V(S_{m,n}) = \{u, w, u_1, u_2, \cdots, u_{m-1}, w_1, w_2, \cdots, w_{n-1}\}$, where, u and w are central vertices, $u_1, u_2, \cdots, u_{m-1}$ being adjacent to u and $w_1, w_2, \cdots, \cdots, w_{n-1}$ being adjacent to w. Then $ed(u) = ed(u_i) = n-1$ for $1 \leq i \leq m-1$, $ed(w) = ed(w_j) = m-1$ for $1 \leq j \leq n-1$. Also, $e(u) = e(w) = 2$ and $e(u_i) = e(w_j) = 3$ for $1 \leq i \leq m-1$, $1 \leq j \leq n-1$. Hence,

$$\xi^{ed}(S_{n,m}) = (n-1)(2) + (m-1)(2) + \sum_{i=1}^{m-1}(n-1)(3) + \sum_{i=1}^{n-1}(m-1)(3)$$

$$= 2(n-1) + 2(m-1) + 3(n-1)(m-1) + 3(m-1)(n-1)$$
$$= 2(3mn - 2m - 2n + 1).$$

□

3. Eccentric Degree Connectivity Index of Graph Operations

In this section we obtain eccentric degree connectivity index of graph operations namely, join, corona, corona join, $r-$crown graph of fan graph and $r-$crown graph of wheel graph.

Definition 3.1 ([5]). *Let G_1 and G_2 be disjoint graphs. The join $G_1 + G_2$ is the graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1)$ and $v \in V(G_2)\}$.*

Proposition 3.2 ([11]). *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be disjoint connected graphs having no full degree vertices. Then $ed(G_1 + G_2) = n_1^2 + n_2^2 - (n_1 + n_2) - 2(m_1 + m_2)$.*

Proposition 3.3. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be disjoint connected graphs having no full degree vertices. Then $\xi^{ed}(G_1 + G_2) = 2[n_1^2 + n_2^2 - (n_1 + n_2) - 2(m_1 + m_2)]$.*

Proof. For every vertex u of $G_1 + G_2$, $d_{G_1+G_2}(u)$ vertices are at a distance one from u and the remaining $n_1 + n_2 - 1 - d_{G_1+G_2}(u)$ vertices are at a distance two from u. Since G_1 and G_2 do not have full degree vertices, it follows that $e_{G_1+G_2}(u) = 2$ for every $u \in V(G_1 + G_2)$. Also, by the proof of Proposition 3.2, for every $u \in V(G_1 + G_2)$,

$$ed_{G_1+G_2}(u) = \begin{cases} n_1 - 1 - d_{G_1}(u), \text{ if } u \in V(G_1); \\ n_2 - 1 - d_{G_2}(u), \text{ if } u \in V(G_2). \end{cases}$$

So,

$$\begin{aligned}\xi^{ed}(G_1 + G_2) &= \sum_{u \in V(G_1+G_2)} ed_{(G_1+G_2)}(u) e_{(G_1+G_2)}(u) \\ &= \sum_{u \in V(G_1)} ed_{(G_1+G_2)}(u) e_{(G_1+G_2)}(u) + \sum_{u \in V(G_2)} ed_{(G_1+G_2)}(u) e_{(G_1+G_2)}(u) \\ &= \sum_{u \in V(G_1)} (n_1 - 1 - d_{G_1}(u))(2) + \sum_{u \in V(G_2)} (n_2 - 1 - d_{G_2}(u))(2) \\ &= 2[n_1(n_1 - 1) - 2m_1 + n_2(n_2 - 1) - 2m_2] \\ &= 2[n_1^2 + n_2^2 - (n_1 + n_2) - 2(m_1 + m_2)].\end{aligned}$$

□

Definition 3.4 ([5]). *Let G_1 and G_2 be disjoint graphs. The corona $G_1 \circ G_2$ is the graph G with $|V(G)| = |V(G_1)| + n_1|V(G_2)|$ which is obtained by taking one copy of G_1 and n_1 copies of G_2, and joining the i^{th} vertex of G_1 to each vertex of i^{th} copy of G_2.*

Definition 3.5 ([3]). *Let G_1 and G_2 be disjoint graphs. The corona join graph of G_1 and G_2 denoted by $G_1 \overset{+}{\circ} G_2$ is obtained by taking one copy of G_1, n_1 copies of G_2 and by joining each vertex of the i^{th} copy of G_2 with all vertices of G_1.*

Proposition 3.6. *Let $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$ be disjoint connected graphs having no full degree vertices. Then*

(1). $\xi^{ed}(G) = n_2(n_2+1)\xi^{ed}(G_1) + n_2(2n_2+1)ed(G_1)$

(2). $\xi^{ed}(G) = 2[(n_1n_2-1)^2 + (n_1-1)^2 - 2(m_1+n_1m_2) + n_1(n_2+1) - 2]$

Proof.

(1). Let $G = G_1 \circ G_2$, with $V(G) = X \cup Y$, where $X = \{u_i : u_i \in V(G_1) \text{ and } 1 \leq i \leq n_1\}$ and $Y = \{w_{kj} : w_j \in V(G_2) \text{ and } 1 \leq k \leq n_1, 1 \leq j \leq n_2\}$. Then $e_G(u_i) = e_{G_1}(u_i) + 1$, $e_G(w_{kj}) = e_{G_1}(u_k) + 2$, $ed_G(u_i) = n_2 ed_{G_1}(u_i)$ and $ed_G(w_{kj}) = n_2 ed_{G_1}(u_k)$. Hence

$$
\begin{aligned}
\xi^{ed}(G) &= \sum_{u \in V(G)} ed_G(u) e_G(u) \\
&= \sum_{i=1}^{n_1} ed_G(u) e_G(u) + \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} ed_G(w_{kj}) e_G(w_{kj}) \\
&= \sum_{i=1}^{n_1} n_2 ed_{G_1}(u_i)(e_{G_1}(u_i)+1) + \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} n_2 ed_{G_1}(u_k)(e_{G_1}(u_k)+2) \\
&= n_2 \sum_{i=1}^{n_1} ed_{G_1}(u_i) e_{G_1}(u_i) + n_2 \sum_{i=1}^{n_1} ed_{G_1}(u_i) + n_2^2 \sum_{k=1}^{n_1} ed_{G_1}(u_k) e_{G_1}(u_k) + 2n_2^2 \sum_{k=1}^{n_1} ed_{G_1}(u_k) \\
&= n_2 \xi^{ed}(G_1) + n_2 ed(G_1) + n_2^2 \xi^{ed}(G_1) + 2n_2^2 ed(G_1) \\
&= n_2(n_2+1)\xi^{ed}(G_1) + n_2(2n_2+1) ed(G_1).
\end{aligned}
$$

(2). Let $G = G_1 \overset{+}{\circ} G_2$ with $V(G) = X \cup Y$, where $X = \{u_i : u_i \in V(G_1) \text{ and } 1 \leq i \leq n_1\}$ and $Y = \{w_{kj} : w_j \in V(G_2) \text{ and } 1 \leq k \leq n_1, 1 \leq j \leq n_2\}$. For $1 \leq i \leq n_1, 1 \leq k \leq n_1, 1 \leq j \leq n_2$, we have $e_G(u_i) = e_G(w_{kj}) = 2$, $ed_G(u_i) = n_1 - d_{G_1}(u_i) - 1$ and $ed_G(w_{kj}) = n_1n_2 - d_{G_2}(w_j) - 1$. Thus,

$$
\begin{aligned}
\xi^{ed}(G) &= \sum_{u \in V(G)} ed_G(u) e_G(u) \\
&= \sum_{i=1}^{n_1} ed_G(u) e_G(u) + \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} ed_G(w_{kj}) e_G(w_{kj}) \\
&= \sum_{i=1}^{n_1} (n_1 - d_{G_1}(u_i) - 1)(2) + \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} (n_1n_2 - d_{G_2}(w_j) - 1)(2) \\
&= 2[n_1^2 - 2m_1 - n_1] + 2n_1[n_1n_2^2 - 2m_2 - n_2] \\
&= 2[(n_1n_2-1)^2 + (n_1-1)^2 - 2(m_1+n_1m_2) + n_1(n_2+1) - 2].
\end{aligned}
$$

□

Definition 3.7 ([12]). *The $r-$crown graph $I_r(G)$ of a graph G is obtained by splicing r hang edges at every vertex of G.*

Proposition 3.8. *The eccentric degree connectivity index of $I_r(F_n)$, $n \geq 4$ is given by $\xi^{ed}(I_r(F_n)) = r^2(4n^2 - 9n + 8) + r(3n^2 - 7n + 6)$.*

Proof. Let $G = I_r(F_n)$, $n \geq 4$, the $r-$crown graph of fan graph as shown in Figure 1. Let $P_n =< v_1, v_2, \cdots, v_n >$ with $v_i^1, v_i^2, \cdots, v_i^r$ being the r hanging vertices of v_i, $1 \leq i \leq n$ and v be a vertex in F_n adjacent to all vertices of P_n with $v^1, v^2, \cdots, v^r$ being the r hanging vertices of v. By definition of $r-$crown graph, we have $e(v) = 2$, $e(v_i) = 3$, $1 \leq i \leq n$, $e(v^i) = 3$, $1 \leq i \leq r$ and $e(v_i^j) = 4$, where $1 \leq i \leq n, 1 \leq j \leq r$. Also, we have $ed(v^i) = nr$, $1 \leq i \leq r$,

$$ed(v_i^j) = \begin{cases} (n-2)r, & \text{if } i = 1 \text{ or } n \text{ and } 1 \leq j \leq r; \\ (n-3)r, & \text{if } 2 \leq i \leq n-1, 1 \leq j \leq r. \end{cases}$$

So,

$$\begin{aligned}
\xi^{ed}(G) &= \sum_{v \in V(G)} ed_G(v) e_G(v) \\
&= ed(v)e(v) + \sum_{i=1}^{r} ed_G(v^i)e_G(v^i) + \sum_{i=1}^{n} ed_G(v_i)e_G(v_i) + \sum_{i=1}^{n}\sum_{j=1}^{r} ed_G(v_i^j)e_G(v_i^j) \\
&= nr(2) + \sum_{i=1}^{r} nr(3) + ed(v_1)e(v_1) + ed(v_n)e(v_n) + \sum_{i=2}^{n-1} ed(v_i)e(v_i) + \sum_{j=1}^{r} ed(v_1^j)e(v_1^j) \\
&+ \sum_{j=1}^{r} ed(v_n^j)e(v_n^j) + \sum_{i=2}^{n-1}\sum_{j=1}^{r} ed(v_i^j)e(v_i^j) \\
&= 2nr + 3nr^2 + (n-2)r(3) + (n-2)r(3) + \sum_{i=2}^{n-1}(n-3)r(3) + \sum_{j=1}^{r}(n-2)r(4) + \sum_{j=1}^{r}(n-2)r(4) \\
&+ \sum_{i=2}^{n-1}\sum_{j=1}^{r}(n-3)r(4) \\
&= 2nr + 3nr^2 + 3nr - 6r + 3nr - 6r + 3(n-3)(n-2)r + 4(n-2)r^2 \\
&+ 4(n-2)r^2 + 4(n-3)(n-2)r^2 \\
&= r^2(4n^2 - 9n + 8) + r(3n^2 - 7n + 6).
\end{aligned}$$

□

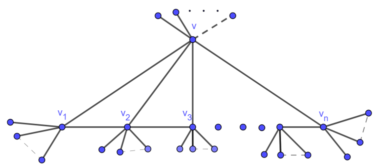

Figure 1: $r-$crown graph of fan graph

Proposition 3.9. *The eccentric degree connectivity index of $I_r(W_n)$, $n \geq 5$ is given by $\xi^{ed}(I_r(W_n)) = r^2(4n^2 - 17n + 13) + r(3n^2 - 13n + 10)$.*

Proof. Let $G = I_r(W_n)$, $n \geq 5$, the $r-$crown graph of wheel graph as shown in Figure 2. Let $C_{n-1} =< v_1, v_2, \cdots, v_{n-1} >$ with $v_i^1, v_i^2, \cdots, v_i^r$ being r hanging vertices of v_i,, $1 \leq i \leq n-1$ and v be a vertex in W_n adjacent to all vertices of C_{n-1} with $v^1, v^2, \cdots, v^r$ being r hanging vertices of v. By definition of $r-$ crown graph we have $e(v) = 2$, $e(v^i) = 3$, $1 \leq i \leq r$, $e(v_i) = 3$ $1 \leq i \leq n-1$ and $e(v_i^j) = 4$,$1 \leq i \leq n-$,$1 \leq j \leq r$. Also, we have $ed(v) = (n-1)r, ed(v^i) = (n-1)r, ed(v_i) = (n-4)r$, $1 \leq i \leq n-1$,$ed(v_i^j) = (n-4)r$, $1 \leq i \leq n-1$, $1 \leq j \leq r$. So,

$$\begin{aligned}
\xi^{ed}(G) &= \sum_{v \in V(G)} ed_G(v) e_G(v) \\
&= ed(v)e(v) + \sum_{i=1}^{r} ed_G(v^i)e_G(v^i) + \sum_{i=1}^{n-1} ed_G(v_i)e_G(v_i) + \sum_{i=1}^{n-1}\sum_{j=1}^{r} ed_G(v_i^j)e_G(v_i^j) \\
&= (n-1)r(2) + \sum_{i=1}^{r}(n-1)r(3) + \sum_{i=1}^{n-1}(n-4)r(3) + \sum_{i=1}^{n-1}\sum_{j=1}^{r}(n-4)r(4) \\
&= 2(n-1)r + 3(n-1)r^2 + 3(n-4)(n-1)r + 4(n-4)(n-1)r^2 \\
&= r^2(4n^2 - 17n + 13) + r(3n^2 - 13n + 10).
\end{aligned}$$

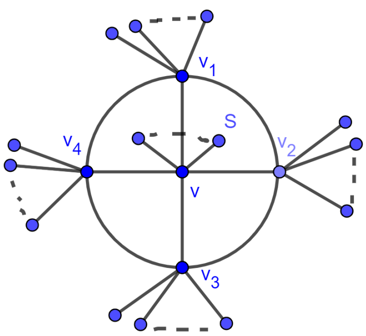

Figure 2: $r-$crown graph of wheel graph

□

4. Bounds for Eccentric Degree Connectivity Index

In this section, we obtain upper and lower bounds for eccentric degree connectivity index in terms of order, radius and diameter.

Proposition 4.1. *For any connected graph G of order n , $nrad(G) \leq \xi^{ed}(G) \leq n(n-1)diam(G)$. Equality holds if and only if $G = K_n, n \geq 2$.*

Proof. By definition, $\xi^{ed}(G) = \sum_{v \in V(G)} ed(v)e(v) \leq \sum_{v \in V(G)} ed(v)diam(G) \leq n(n-1)diam(G)$. Equality holds if and only if G is equieccentric graph of order atleast two and $e(v) = diam(G) = 1$ for every

$v \in V(G)$. That is for $G = K_n, n \geq 2$. Similarly, $\xi^{ed}(G) = \sum_{v \in V(G)} ed(v)e(v) \geq n(1)rad(G)$, with equality if and only if G is equieccentric graph of order atleast two and $e(v) = rad(G) = 1$. That is for $G = K_n, n \geq 2$. □

Proposition 4.2. *For any connected graph G of order n, $n \leq \xi^{ed}(G) \leq n(n-1)^2$, with equality for $G = K_n, n \geq 2$.*

Proof. The upper and lower bounds for eccentric degree and eccentricity of vertices are given by $1 \leq ed(v) \leq n-1$ and $1 \leq e(v) \leq n-1$. So, we have, $n(1)(1) \leq \xi^{ed}(G) \leq n(n-1)^2$. □

Proposition 4.3. *For a connected graph G of order $n \geq 3$, $\xi^{ed} \geq n+3$.*

Proof. Let $X = \{v \in V(G) : ed(v) = n-1\}$, $Y = \{v \in V(G) : 1 \leq ed(v) \leq n-2\}$ such that $|X| = a$ and $|Y| = b$. We consider the following cases.

Case 1. Let $X \neq \phi$ and $Y \neq \phi$, then $a \geq 1$ and $b \geq 2$, and we have

$$\begin{aligned}
\xi^{ed}(G) &= \sum_{u \in V(G)} ed_G(u)e_G(u) \\
&= \sum_{u \in X} ed_G(u)e_G(u) + \sum_{u \in Y} ed_G(u)e_G(u) \\
&\geq \sum_{u \in X} (n-1)1 + \sum_{u \in Y} (1)2 \\
&= a(n-1) + 2b \\
&\geq n-1+4 \\
&= n+3.
\end{aligned}$$

Hence, $\xi^{ed}(G) \geq n+3$ and equality holds for $G = P_3$.

Case 2. Let $X \neq \phi$ and $Y = \phi$, then $b = 0$ and so, $a = n$. Therefore

$$\begin{aligned}
\xi^{ed}(G) &= \sum_{u \in V(G)} ed_G(u)e_G(u) \\
&= \sum_{u \in X} ed_G(u)e_G(u) \\
&= n(n-1)(1) \\
&= n(n-1) \\
&\geq n+3, \forall n \geq 3
\end{aligned}$$

Equality holds for $G = K_3$.

Case 3. If $X = \phi$ and $Y \neq \phi$ then $a = 0$ and so, $b = n$. Therefore

$$\xi^{ed}(G) = \sum_{u \in V(G)} ed_G(u)e_G(u)$$

$$= \sum_{u \in Y} ed_G(u)e_G(u)$$
$$\geq \sum_{u \in Y} (1)2$$
$$= 2b$$
$$= 2n.$$

Equality holds for $G = C_4$. Hence $\xi^{ed}(G) \geq 2n \geq n + 3$, for $n \geq 3$. □

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