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Some Common Fixed Point Results in $b-$metric Spaces Endowed with a Graph

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Abstract

In this paper, we study the conditions for the existence of a unique common fixed point of generalized $(\psi, \varphi)-$ contractive mappings in the framework of $b-$metric spaces endowed with a graph. We give some examples to support our results.

Keywords: Common fixed point; weakly compatible; $b-$metric; graph.
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1. Introduction

Fixed point theory is widely used in mathematics and physics. To advance it, scholars have integrated it with graph theory. However, directed graphs complicate the structure of metric spaces (and b-metric spaces) endowed with them, making fixed point research challenging. Extending existing fixed point theorems and iterative algorithms to these spaces is thus theoretically significant.

In 1906, Frechet [4] first introduced the concept of metric spaces. Czerwik [3] generalized this to b-metric spaces and proved the Banach contraction mapping principle in this type spaces. Some fixed point results for b-metric spaces have also been extensively studied [1,13]. In 2007, Jachymski and Jozwik [5] incorporated graph structures into metric spaces and generalized partial ordered metric space results. Subsequent work (e.g., Bojor [2], Shukla [11] et al., Nantaporn [9] et al., Sushanta [12]) explored fixed points in graph metric spaces and graph$b$-metric spaces, expanding the theory [6]. Building on result of Liu [8], this paper investigates the existence and uniqueness of common fixed points for ψ-contraction and (ψ, φ)-contraction in graph b-metric spaces.

2. Preliminaries

Firstly, we recall some definitions and lemmas in $b-$metric space.

Definition 2.1 ([3]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to [0, +\infty)$ is said to be a b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:*

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(i) $d(x,y) = 0$ *if and only if* $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq s(d(x,z) + d(y,z))$.

It is obvious that the class of $b-$metric spaces is effectively larger than that of metric spaces since any metric space is a $b-$metric space with $s = 1$.

Definition 2.2. *Let* (X,d) *be a* $b-$*metric space with parameter* $s \geq 1$ *and* $\{x_n\}$ *be a sequence in* X. *Then*

(i) $\{x_n\}$ *converges to* $x \in X$ *if and only if there exists* $x \in X$ *such that* $d(x_n, x) \to 0$ *as* $n \to \infty$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x(n \to \infty)$.

(ii) $\{x_n\}$ *is a Cauchy sequence if and only if* $d(x_n, x_m) \to 0$ *when* $n, m \to \infty$.

(iii) (X,d) *is complete if and only if every Cauchy sequence in* X *is convergent.*

Each convergent sequence in a $b-$metric space has a unique limit and it is also a Cauchy sequence. Moreover, in general, a $b-$metric is not continuous.

Definition 2.3 ([7])**.** *A pair of self mappings* f *and* g *defined on a nonempty set* X *are said to be weakly compatible if for all* $t \in X$ *the equality* $ft = gt$ *implies* $fgt = gft$.

We next review some basic notions in graph theory.

Let (X,d) be a $b-$metric space. We assume that G is a reflexive digraph where the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges, i.e., $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. Let $\widetilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges, and $E(\widetilde{G}) = E(G) \bigcup E(G^{-1})$. If x, y are vertices of the digraph G, then a path in G from x to y of length $n (n \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\widetilde{G}$ is connected. The graph G_0 is defined by $E(G_0) = X \times X$.

Definition 2.4. *Let* A, B, S, *and* T *be self mappings in a* $b-$*metric space* (X,d) *endowed with a graph* G. T *and* S *weakly preserve edges in* $E(\widetilde{G})$, *if* $(x,y) \in E(\widetilde{G})$ *implies* $(Tx, Sy) \in E(\widetilde{G})$ *and* $(Sx, Ty) \in E(\widetilde{G})$. A *and* B *weakly preserve inverse edges in* $E(\widetilde{G})$, *if* $(Ax, By) \in E(\widetilde{G})$ *and* $(Bx, Ay) \in E(\widetilde{G})$ *imply* $(x,y) \in E(\widetilde{G})$.

Throughout this paper,

$\Phi_1 = \{\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing, and $\psi(t) = 0$ if and only if $t = 0\}$,

$\Phi_2 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is lower semi-continuous and nonincreasing, and $\varphi(t) = 0$ if and only if $t = 0\}$,

$\Phi_3 = \{\psi : \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semi-continuous, and $\lim_{n\to\infty} a_n = 0$ for each sequence $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $a_{n+1} \leq \psi(a_n), \forall n \in \mathbb{N}\}$

Definition 2.5. *Let A, B, S, and T be self mappings in a $b-$metric space (X, d) endowed with a graph G. The contractive condition is called a $\psi-$weakly contractive condition if the following inequation holds:*

$$s^3 d(Tx, Sy) \leq \psi(M(x, y)), \tag{1}$$

where

$$\begin{aligned} M(x,y) = \max\{& d(Ax, By), d(Ax, Tx), d(By, Sy), \frac{1}{2s}(d(Ax, Sy) + d(Tx, By)), \\ & \frac{1}{2}(d(Ax, By) + d(Ax, Tx)), \frac{d(By, Sy)}{1 + d(Tx, Sy)} d(Ax, Tx), \\ & \frac{1 + d(Tx, By) + d(Ax, Sy)}{1 + s(d(Ax, Tx) + d(By, Sy))} d(Ax, Tx), \\ & \frac{1 + d(Tx, By) + d(Ax, Sy)}{1 + s(d(Ax, By) + d(Tx, Sy))} d(By, Sy), \frac{1 + d(Ax, Tx)}{1 + d(Ax, By)} d(By, Sy), \\ & \frac{(1 + d(Tx, By) + d(Ax, Sy))^2 + d(Tx, By) d(Ax, Sy)}{(1 + s(d(Ax, By) + d(Tx, Sy)))^2} d(Ax, Tx), \\ & \frac{(1 + d(Tx, By) + d(Ax, Sy))^2 + d(Tx, By) d(Ax, Sy)}{(1 + s(d(Ax, Tx) + d(By, Sy)))^2} d(By, Sy)\}, \end{aligned} \tag{2}$$

for all $x, y \in X$ with $(x, y) \in E(\widetilde{G})$, $\psi \in \Phi_3$.

Definition 2.6. *Let A, B, S, and T be self mappings in a $b-$metric space (X, d) endowed with a graph G. The contractive condition is called a $(\psi, \varphi)-$weakly contractive condition if the following inequation holds:*

$$\psi(s^3 d(Tx, Sy)) \leq \psi(M(x, y)) - \varphi(N(x, y)), \tag{3}$$

where $M(x, y)$ satisfies (2) and

$$\begin{aligned} N(x,y) = \max\{& d(Ax, By), d(Ax, Tx), d(By, Sy), d(Tx, Sy), \frac{1}{2s}(d(Ax, Sy) + d(Tx, By)), \\ & \frac{1 + d(Tx, By) + d(Ax, Sy)}{1 + s(d(Ax, Tx) + d(By, Sy))} d(By, Sy), \frac{1 + d(Ax, Tx)}{1 + d(Ax, By)} d(By, Sy)\}, \end{aligned} \tag{4}$$

for all $x, y \in X$ with $(x, y) \in E(\widetilde{G})$, $\psi \in \Phi_1$, $\varphi \in \Phi_2$ and $\varphi(t) \leq \psi(t)$.

Lemma 2.7 ([10])**.** *Let $\psi \in \Phi_3$. Then $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.*

Lemma 2.8 ([8])**.** *Let A, B, S, and T be self mappings in a $b-$metric space (X, d) endowed with a graph G satisfying*

$$\psi(s^3 d(Tx, Sy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \forall x, y \in X, \tag{5}$$

where $(\psi, \varphi) \in \Phi_1 \times \Phi_2$. *Assume that* $I : \mathbb{R}^+ \to \mathbb{R}^+$ *is the identity mapping and*

$$\psi_1(t) = (\psi + I)^{-1}(\psi + I - \varphi)(t), \forall t \in \mathbb{R}^+.$$

Then $\psi_1 \in \Phi_3$ *and*

$$s^3 d(Tx, Sy) \le \psi_1(M(x,y)), \forall x, y \in X.$$

Remark 2.9. *It follows from Lemma 2.8 that (5) relative to four mappings A, B, S and T implies (1) relative to four mappings A, B, S and T.*

3. Main Results

In this section, we assume that (X, d) is a $b-$metric space with the coefficient $s \ge 1$, and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges. Let A, B, S, T: $X \to X$ be such that $T(X) \subseteq B(X)$, $S(X) \subseteq A(X)$. If $x_0 \in X$ is arbitrary, then there exists elements $x_1, x_2 \in X$ such that $Bx_1 = Tx_0$, $Ax_2 = Sx_1$, since $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$. Proceeding in this way, we can construct a sequence $\{y_n\}$ such that $y_{2n+1} = Bx_{2n+1} = Tx_{2n}$, $y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}$, $n \in \mathbb{N}$. By C_n we denote the set of all elements x_0 of X such that $(x_0, x_1) \in E(\widetilde{G})$.

Theorem 3.1. *Let* (X, d) *be a* $b-$*metric space endowed with a graph G and the mappings A, B, S, T:* $X \to X$ *satisfy* $\psi-$*weakly contractive condition. {A, T} and {B, S} are weakly compatible.* $T(X) \subseteq B(X)$ *and* $S(X) \subseteq A(X)$. *One of* $A(X)$, $B(X)$, $S(X)$ *and* $T(X)$ *is complete. If the following conditions hold:*

(i) T and S weakly preserve edges in $E(\widetilde{G})$, *A and B weakly preserve inverse edges in* $E(\widetilde{G})$,

(ii) $C_n \neq \emptyset$,

(iii) If $\{y_n\}$ *is a sequence in X such that* $y_n \to z$ *and one of* $z = A\overline{x}$, $z = B\overline{x}$, $z = S\overline{x}$, $z = T\overline{x}$ *is satisfied,* $(Tx_{2n}, Sx_{2n+1}) \in E(\widetilde{G})$, *for all* $n \in \mathbb{N}$, *then there exists a subsequence* $\{y_{n_i}\}$ *of* $\{y_n\}$ *such that*

(1) $(T\overline{x}, y_{n_i}) \in E(\widetilde{G})$ *and* $(y_{n_i}, S\overline{x}) \in E(\widetilde{G})$,

(2) $(Tz, S\overline{x}) \in E(\widetilde{G})$,

(iv) if x, y *are common fixed points of A, B, S and T in X, then* $(x, y) \in E(\widetilde{G})$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Because of $C_n \neq \emptyset$, we choose an $x_0 \in C_n$ and keep it fixed. Since $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$, there exists a sequence $\{y_n\}$ such that $y_{2n+1} = Bx_{2n+1} = Tx_{2n}$, $y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}$, $n \in \mathbb{N}$ and $(x_0, x_1) \in E(\widetilde{G})$. Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Now we prove

$$\lim_{n\to\infty} d_n = 0. \tag{6}$$

Because of $(x_0, x_1) \in E(\widetilde{G})$ and T and S weakly preserve edges in $E(\widetilde{G})$, we gain $(Tx_0, Sx_1) \in E(\widetilde{G})$, i.e.,$(Bx_1, Ax_2) \in E(\widetilde{G})$. And since A and B weakly preserve inverse edges in $E(\widetilde{G})$, we get $(x_1, x_2) \in E(\widetilde{G})$. In general, $(x_n, x_{n+1}) \in E(\widetilde{G})$, $(Sx_{2n-1}, Tx_{2n}) \in E(\widetilde{G})$ and $(Tx_{2n}, Sx_{2n+1}) \in E(\widetilde{G})$. Using (1) and (2), we derive

$$s^3 d_{2n} = s^3 d(y_{2n}, y_{2n+1}) = s^3 d(Tx_{2n}, Sx_{2n-1}) \leq \psi(M(x_{2n}, x_{2n-1})) \tag{7}$$

and

$$\begin{aligned} M(x_{2n}, x_{2n-1}) = \max\{ & d_{2n-1}, d_{2n}, d_{2n-1}, \frac{1}{2s}(d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})), \\ & \frac{1}{2}(d_{2n-1} + d_{2n}), \frac{d_{2n-1}}{1 + d_{2n}} d_{2n}, \frac{1 + d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n})}{1 + s(d_{2n} + d_{2n-1})} d_{2n}, \\ & \frac{1 + d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n})}{1 + s(d_{2n-1} + d_{2n})} d_{2n-1}, \frac{1 + d_{2n}}{1 + d_{2n-1}} d_{2n-1}, \\ & \frac{(1 + d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n}))^2 + d(y_{2n+1}, y_{2n-1}) d(y_{2n}, y_{2n})}{(1 + s(d_{2n-1} + d_{2n}))^2} d_{2n}, \\ & \frac{(1 + d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n}))^2 + d(y_{2n+1}, y_{2n-1}) d(y_{2n}, y_{2n})}{(1 + s(d_{2n} + d_{2n-1}))^2} d_{2n-1}\} \\ = \max\{ & d_{2n-1}, d_{2n}, \frac{1 + d_{2n}}{1 + d_{2n-1}} d_{2n-1}\} \end{aligned} \tag{8}$$

Suppose that $d_{2n_0-1} < d_{2n_0}$ for some $n_0 \in \mathbb{N}$. It follows that

$$d_{2n_0}(1 + d_{2n_0-1}) = d_{2n_0} + d_{2n_0} d_{2n_0-1} > d_{2n_0-1} + d_{2n_0} d_{2n_0-1} = d_{2n_0-1}(1 + d_{2n_0}),$$

that is,

$$d_{2n_0} > d_{2n_0-1} \frac{1 + d_{2n_0}}{1 + d_{2n_0-1}}, \tag{9}$$

which implies $M(x_{2n_0}, x_{2n_0-1}) = d_{2n_0}$. By means of (7), $s \geq 1$, $\psi \in \Phi_3$, and Lemma 2.7, we conclude

$$s^3 d_{2n_0} \leq \psi(M(x_{2n_0}, x_{2n_0-1})) = \psi(d_{2n_0}) < d_{2n_0},$$

which is a contradiction. Consequently, we deduce

$$d_{2n} \leq d_{2n-1} = M(x_{2n}, x_{2n-1}), \ \forall n \in \mathbb{N}. \tag{10}$$

Similarly we have

$$d_{2n+1} \leq d_{2n} = M(x_{2n}, x_{2n+1}), \ \forall n \in \mathbb{N}. \tag{11}$$

It follows from (10) and (11) that

$$d_{n+1} \leq d_n, \ \forall n \in \mathbb{N},$$

which means that the sequence $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. So, there exists $r \geq 0$ with $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. It follows from (7), (10), $\psi \in \Phi_3$, $s \geq 1$, and Lemma 2.7 that

$$s^3 r = \limsup_{n\to\infty} s^3 d_{2n} \leq \limsup_{n\to\infty} \psi(M(x_{2n}, x_{2n-1})) = \limsup_{n\to\infty} \psi(d_{2n-1}) < \limsup_{n\to\infty} d_{2n-1} = r,$$

which is inconsistent. Hence $r = 0$, that is, (6) holds.

In order to prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we need only to show that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbb{N}}$ of $\{y_{2n}\}_{n\in\mathbb{N}}$ such that

$$2n(k) > 2m(k) > 2k, \quad d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon, \ \forall k \in \mathbb{N}, \tag{12}$$

where $2n(k)$ is the smallest index satisfying (12). It follows that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon, \ \forall k \in \mathbb{N}. \tag{13}$$

Taking advantage of (12), (13), and the triangle inequality, we get

$$\begin{aligned} \varepsilon \leq d(y_{2m(k)}, y_{2n(k)}) &\leq sd(y_{2m(k)}, y_{2n(k)-2}) + s^2 d(y_{2n(k)-2}, y_{2n(k)-1}) + s^2 d(y_{2n(k)-1}, y_{2n(k)}) \\ &< s\varepsilon + s^2 d_{2n(k)-2} + s^2 d_{2n(k)-1}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{14}$$

Letting $k \to \infty$ in (14), we have

$$\varepsilon \leq \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) \leq s\varepsilon. \tag{15}$$

Moreover,

$$\begin{aligned} d(y_{2m(k)}, y_{2n(k)}) &\leq sd(y_{2m(k)}, y_{2m(k)+1}) + sd(y_{2m(k)+1}, y_{2n(k)}), \\ d(y_{2m(k)+1}, y_{2n(k)}) &\leq sd(y_{2m(k)+1}, y_{2m(k)}) + sd(y_{2m(k)}, y_{2n(k)}), \end{aligned} \tag{16}$$

putting $k \to \infty$ in (16) and using (6) and (15), we deduce

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) \leq s^2\varepsilon. \tag{17}$$

Similarly, we can show that

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) \leq s^2\varepsilon \tag{18}$$

and

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}) \leq s^3\varepsilon. \tag{19}$$

So by (2) and connectivity of G, taking the limit supremum as $k \to \infty$ in $M(x_{2m(k)}, x_{2n(k)-1})$ and using (6), (15), (17), (18) and (19), we gain

$$\begin{aligned}
\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1}) = \limsup_{k\to\infty} \max\{ & d(y_{2m(k)}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)}), \\
& \frac{1}{2s}(d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})), \\
& \frac{1}{2}(d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2m(k)}, y_{2m(k)+1})), \\
& \frac{d(y_{2n(k)-1}, y_{2n(k)})}{1 + d(y_{2m(k)+1}, y_{2n(k)})} d(y_{2m(k)}, y_{2m(k)+1}), \\
& \frac{1 + d(y_{2m(k)+1}, y_{2n(k)-1}) + d(y_{2m(k)}, y_{2n(k)})}{1 + s(d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2n(k)-1}, y_{2n(k)}))} d(y_{2m(k)}, y_{2m(k)+1}), \\
& \frac{1 + d(y_{2m(k)+1}, y_{2n(k)-1}) + d(y_{2m(k)}, y_{2n(k)})}{1 + s(d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2m(k)+1}, y_{2n(k)}))} d(y_{2n(k)-1}, y_{2n(k)}), \\
& \frac{1 + d(y_{2m(k)}, y_{2m(k)+1})}{1 + d(y_{2m(k)}, y_{2n(k)-1})} d(y_{2n(k)-1}, y_{2n(k)}), \\
& \frac{(1 + d(y_{2m(k)+1}, y_{2n(k)-1}) + d(y_{2m(k)}, y_{2n(k)}))^2}{(1 + s(d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2m(k)+1}, y_{2n(k)})))^2} d(y_{2m(k)}, y_{2m(k)+1}) \\
& + \frac{d(y_{2m(k)+1}, y_{2n(k)-1}) d(y_{2m(k)}, y_{2n(k)})}{(1 + s(d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2m(k)+1}, y_{2n(k)})))^2} d(y_{2m(k)}, y_{2m(k)+1}), \\
& \frac{(1 + d(y_{2m(k)+1}, y_{2n(k)-1}) + d(y_{2m(k)}, y_{2n(k)}))^2}{(1 + s(d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2n(k)-1}, y_{2n(k)})))^2} d(y_{2n(k)-1}, y_{2n(k)}) \\
& + \frac{d(y_{2m(k)+1}, y_{2n(k)-1}) d(y_{2m(k)}, y_{2n(k)})}{(1 + s(d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2n(k)-1}, y_{2n(k)})))^2} d(y_{2n(k)-1}, y_{2n(k)}) \} \\
\leq \max\{s^2\varepsilon, \frac{1}{2s}(s\varepsilon + s^3\varepsilon), \frac{1}{2}s^2\varepsilon\} = s^2\varepsilon.
\end{aligned}$$

Hence, by Lemma 2.7, we obtain

$$\begin{aligned}
s^2\varepsilon = s^3(\frac{\varepsilon}{s}) \leq s^3 \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) &= s^3 \limsup_{k\to\infty} d(Tx_{2m(k)}, Sx_{2n(k)-1}) \\
&\leq \limsup_{k\to\infty} \psi(M(x_{2m(k)}, x_{2n(k)-1})) \leq \psi(s^2\varepsilon) < s^2\varepsilon,
\end{aligned}$$

which is a contradiction. So $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Assume that $A(X)$ is complete. Observe that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Consequently there exists $z \in A(X)$ and $v \in X$ with $\lim_{n\to\infty} y_{2n} = z = Av$. It is easy to see

$$z = lim_{n\to\infty} y_n = lim_{n\to\infty} Tx_{2n} = lim_{n\to\infty} Bx_{2n+1} = lim_{n\to\infty} Sx_{2n-1} = lim_{n\to\infty} Ax_{2n}. \tag{20}$$

Suppose that $Tv \neq z$. By (iii), (2) and (20), there exists a subsequence $\{y_{2n_i}\}$ of $\{y_{2n}\}$ such that

$$\begin{aligned}\limsup_{i\to\infty} M(v, x_{2n_i+1}) \leq \max\{&0, d(z,Tv), 0, \frac{1}{2s}sd(Tv,z), \frac{1}{2}d(z,Tv), 0,\\ &\frac{1+sd(Tv,z)}{1+s(d(z,Tv)+\limsup_{i\to\infty} d(Bx_{2n_i+1}, Sx_{2n_i+1}))}d(z,Tv),\\ &0,0,\frac{(1+s\limsup_{i\to\infty} d(Tv,Sx_{2n_i+1}))^2}{(1+s\limsup_{i\to\infty} d(Tv,Sx_{2n_i+1}))^2}d(z,Tv),0\}\\ = d(Tv,z),&\end{aligned} \tag{21}$$

which together with (1), (20), $\psi \in \Phi_3$, $s \geq 1$, and Lemma 2.7 give that

$$\begin{aligned} s^2 d(Tv,z) &\leq s^3 \limsup_{i\to\infty} d(Tv, y_{2n_i+2}) = s^3 \limsup_{i\to\infty} d(Tv, Sx_{2n_i+1})\\ &\leq \limsup_{i\to\infty} \psi(M(v, x_{2n_i+1})) \leq \psi(d(Tv,z)) < d(Tv,z),\end{aligned} \tag{22}$$

which is impossible. Hence $Tv = z = Av$. It follows from $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$ that there exists a point $w \in X$ with $z = Bw = Tv$. Suppose that $Sw \neq z$. In light of (2), (20) and (iii), we have

$$\begin{aligned}\limsup_{i\to\infty} M(x_{2n_i}, w) \leq \max\{&0,0,d(z,Sw), \frac{1}{2s}sd(z,Sw),0,0,0,\\ &\frac{1+s\limsup_{i\to\infty} d(Tx_{2n_i}, Sw)}{1+s\limsup_{i\to\infty} d(Tx_{2n_i}, Sw)}d(z,Sw), d(z,Sw),\\ &0,\frac{(1+sd(z,Sw))^2}{(1+s(\limsup_{i\to\infty} d(Ax_{2n_i}, Tx_{2n_i})+d(z,Sw)))^2}d(z,Sw)\}\\ = d(z,Sw),&\end{aligned} \tag{23}$$

which together with (1), $\psi \in \Phi_3$, $s \geq 1$, and Lemma 2.7 yield

$$\begin{aligned} s^2 d(z,Sw) &\leq s^3 \limsup_{i\to\infty} d(y_{2n_i+1}, Sw) = s^3 \limsup_{i\to\infty} d(Tx_{2n_i}, Sw)\\ &\leq \limsup_{i\to\infty} \psi(M(x_{2n_i}, w)) \leq \psi(d(z,Sw)) < d(z,Sw),\end{aligned} \tag{24}$$

which is inconsistent, so $Sw = z$. Because $\{A, T\}$ and $\{B, S\}$ are weakly compatible, $Az = ATv = TAv = Tz$ and $Bz = BSw = SBw = Sz$ hold. Suppose that $Tz \neq Sz$. By (iii) and $z = Av = Tv = Sw = Bw$, we deduce $(Tv, y_{n_i}) \in E(\widetilde{G})$ or $(y_{n_i}, Sw) \in E(\widetilde{G})$. Note that the connectivity of G implies $(Tv, Sw) \in E(\widetilde{G})$, i.e., $(z,z) \in E(\widetilde{G})$. Then because T and S weakly preserve edges in $E(\widetilde{G})$, we gain $(Tz, Sz) \in E(\widetilde{G})$. It follows from (2), (1), $\psi \in \Phi_3$, $s \geq 1$ and Lemma 2.7 that

$$M(z,z) = \max\{d(Tz,Sz),0,0,\frac{1}{s}d(Tz,Sz),\frac{1}{2}d(Tz,Sz),0,0,0,0,0,0\} = d(Tz,Sz)$$

and

$$s^3 d(Tz,Sz) \leq \psi(M(z,z)) = \psi(d(Tz,Sz)) < d(Tz,Sz),$$

a contradiction, hence $Az = Tz = Sz = Bz$. Suppose that $Tz \neq z$. It follows from (iii) and (2) that

$$M(z,w) = \max\{d(Tz,z), 0, 0, \frac{1}{s}d(Tz,z), \frac{1}{2}d(Tz,z), 0,0,0,0,0,0\} = d(Tz,z),$$

which together with (1), $\psi \in \Phi_3$, $s \geq 1$ and Lemma 2.7 imply

$$s^3 d(Tz,z) = s^3 d(Tz,Sw) \leq \psi(M(z,w)) = \psi(d(Tz,z)) < d(Tz,z),$$

which is impossible and hence $z = Tz = Sz = Az = Bz$, that is , z is a common fixed point of A, B, S, and T. Suppose that A, B, S, and T have another common fixed point $u \in X \setminus \{z\}$. It follows from (iv), (2), (1), $\psi \in \Phi_3$, $s \geq 1$ and Lemma 2.7 that

$$M(u,z) = \max\{d(u,z), 0, 0, \frac{1}{s}d(u,z), \frac{1}{2}d(u,z), 0,0,0,0,0,0\} = d(u,z)$$

and

$$s^3 d(u,z) = s^3 d(Tu,Sz) \leq \psi(M(u,z)) = \psi(d(u,z)) < d(u,z),$$

which is a contradiction and hence z is a unique common fixed point of A, B, S, and T in X if $A(X)$ is complete.

Similarly we conclude that A, B, S, and T have a unique common fixed point in X if one of $B(X)$, $S(X)$ and $T(X)$ is complete. This completes the proof. □

Utilizing Theorem 3.1, Lemma 2.8, and Remark 2.9, we get the following result.

Theorem 3.2. *Let A, B, S and T be self mappings in a b-metric space (X,d) endowed with a graph G. Suppose that mappings A, B, S, T satisfy (5), {A, T} and {B, S} are weakly compatible, $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$, and one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is complete. If the conditions (i), (ii) and (iii) hold, A, B, S and T have a unique common fixed point in X.*

In the Theorem 3.2, the arguments of the functions $\varphi(t)$ and $\psi(t)$ on the right-hand side of the contraction condition are identical. Considering the case where the arguments are different, we obtain the following theorem.

Theorem 3.3. *Let (X,d) be a $b-$metric space endowed with a graph G and the mappings A, B, S, T: $X \to X$ satisfy $(\psi,\varphi)-$weakly contractive condition. Suppose that {A, T} and {B, S} are weakly compatible, $T(X) \subseteq$*

$B(X)$ and $S(X) \subseteq A(X)$ and one of $A(X)$, $B(X)$, $S(X)$ and $T(X)$ is complete. If the conditions (i), (ii) and (iii) hold, then A, B, S and T have a unique common fixed point in X.

Proof. Because of $C_n \neq \emptyset$, we choose an $x_0 \in C_n$ and keep it fixed. Since $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$, there exists a sequence $\{y_n\}$ such that $y_{2n+1} = Bx_{2n+1} = Tx_{2n}$, $y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}$, $n \in \mathbb{N}$ and $(x_0, x_1) \in E(\widetilde{G})$. Put $d_n = d(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. Now we prove (6). Similar to Theorem 3.1, we have $(x_n, x_{n+1}) \in E(\widetilde{G})$, $(Sx_{2n-1}, Tx_{2n}) \in E(\widetilde{G})$ and $(Tx_{2n}, Sx_{2n+1}) \in E(\widetilde{G})$. Using (3), (2) and (4), we derive

$$\psi(s^3 d_{2n}) = \psi(s^3 d(y_{2n}, y_{2n+1})) = \psi(s^3 d(Tx_{2n}, Sx_{2n-1})) \leq \psi(M(x_{2n}, x_{2n-1})) - \varphi(N(x_{2n}, x_{2n-1})) \tag{25}$$

and $M(x_{2n}, x_{2n-1})$ satisfies (8),

$$\begin{aligned} N(x_{2n}, x_{2n-1}) = \max\{ & d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \\ & \frac{1}{2s}(d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})), \\ & \frac{1 + d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n})}{1 + s(d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}))} d(y_{2n-1}, y_{2n}), \\ & \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})} d(y_{2n-1}, y_{2n}) \} \\ = \max\{ & d_{2n-1}, d_{2n}\}. \end{aligned}$$

Suppose that $d_{2n_0-1} < d_{2n_0}$ for some $n_0 \in \mathbb{N}$. It follows (9) that $M(x_{2n_0}, x_{2n_0-1}) = d_{2n_0}$ and $N(x_{2n_0}, x_{2n_0-1}) = d_{2n_0}$. By means of (25), $s \geq 1$, $\psi \in \Phi_1$, $\varphi \in \Phi_2$, we conclude

$$\psi(d_{2n_0}) \leq \psi(s^3 d_{2n_0}) \leq \psi(M(x_{2n_0}, x_{2n_0-1})) - \varphi(N(x_{2n_0}, x_{2n_0-1})) = \psi(d_{2n_0}) - \varphi(d_{2n_0}),$$

which implies $\varphi(d_{2n_0}) = 0$. That is $d_{2n_0} = 0$. Similar to the proof of Theorem 3.1, we obtain

$$d_{n+1} \leq d_n, \ \forall n \in \mathbb{N},$$

which means that the sequence $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. So, there exists $r \geq 0$ with $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. It follows from (25), $\psi \in \Phi_1$, $\varphi \in \Phi_2$, $s \geq 1$ that

$$\begin{aligned} \psi(r) \leq \psi(s^3 r) &= \limsup_{n\to\infty} \psi(s^3 d_{2n}) \leq \limsup_{n\to\infty} \psi(M(x_{2n}, x_{2n-1})) - \liminf_{n\to\infty} \varphi(N(x_{2n}, x_{2n-1})) \\ &= \limsup_{n\to\infty} \psi(d_{2n-1}) - \liminf_{n\to\infty} \varphi(d_{2n-1}) = \psi(r) - \varphi(r), \end{aligned}$$

which explains $\varphi(r) = 0$. Hence $r = 0$, that is, (6) holds.

In order to prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we need only to show that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbb{N}}$ of $\{y_{2n}\}_{n\in\mathbb{N}}$ such that (12) holds. Similar to the proof of

Theorem 3.1, we can obtain

$$\begin{aligned}\varepsilon \le \liminf_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) \le \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) \le s\varepsilon,\\ \frac{\varepsilon}{s} \le \liminf_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) \le \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)}) \le s^2\varepsilon,\\ \frac{\varepsilon}{s} \le \liminf_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) \le \limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)-1}) \le s^2\varepsilon,\\ \frac{\varepsilon}{s^2} \le \liminf_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}) \le \limsup_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)-1}) \le s^3\varepsilon.\end{aligned} \tag{26}$$

So by (2), (4) and connectivity of G, taking the limit supremum as $k \to \infty$ in $M(x_{2m(k)}, x_{2n(k)-1})$, taking the limit infimum as $k \to \infty$ in $N(x_{2m(k)}, x_{2n(k)-1})$ and using (6) and (26), we gain

$$\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1}) \le s^2\varepsilon, \qquad \frac{\varepsilon}{s} \le \liminf_{k\to\infty} N(x_{2m(k)}, x_{2n(k)-1}) \le s^2\varepsilon. \tag{27}$$

Hence, by (3), we obtain

$$\begin{aligned}\psi(s^2\varepsilon) = \psi(s^3(\frac{\varepsilon}{s})) &\le \limsup_{k\to\infty} \psi(s^3 d(y_{2m(k)+1}, y_{2n(k)})) = \limsup_{k\to\infty} \psi(s^3 d(Tx_{2m(k)}, Sx_{2n(k)-1}))\\ &\le \limsup_{k\to\infty} \psi(M(x_{2m(k)}, x_{2n(k)-1})) - \liminf_{k\to\infty} \varphi(N(x_{2m(k)}, x_{2n(k)-1}))\\ &\le \psi(s^2\varepsilon) - \liminf_{k\to\infty} \varphi(N(x_{2m(k)}, x_{2n(k)-1})),\end{aligned}$$

which implies

$$\liminf_{k\to\infty} \varphi(N(x_{2m(k)}, x_{2n(k)-1})) = 0,$$

a contradiction to (27). So $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Consequently there exists $z \in A(X)$ and $v \in X$ with $\lim_{n\to\infty} y_{2n} = z = Av$. It is easy to get (20). Suppose that $Tv \neq z$. By (iii), (2), (4) and (20), there exists a subsequence $\{y_{2n_i}\}$ of $\{y_{2n}\}$ such that (21) and

$$\liminf_{i\to\infty} N(v, x_{2n_i+1}) \le \max\{0, d(z, Tv), 0, sd(Tv, z), \frac{1}{2}d(Tv, z), 0, 0\} = sd(Tv, z),$$

which together with (3), $\psi \in \Phi_1$, $\varphi \in \Phi_2$, give

$$\begin{aligned}\psi(d(Tv,z)) \le \psi(s^2 d(Tv,z)) &\le \limsup_{i\to\infty} \psi(s^3 d(Tv, y_{2n_i+2})) = \limsup_{i\to\infty} \psi(s^3 d(Tv, Sx_{2n_i+1}))\\ &\le \limsup_{i\to\infty} \psi(M(v, x_{2n_i+1})) - \liminf_{i\to\infty} \varphi(N(v, x_{2n_i+1})) \le \psi(d(Tv,z)) - \varphi(sd(Tv,z)),\end{aligned}$$

which implies $\varphi(sd(Tv,z)) = 0$. Hence $Tv = z = Av$. It follows from $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$ that there exists a point $w \in X$ with $z = Bw = Tv$. Suppose that $Sw \neq z$. In light of (2), (4), (20) and (iii), we have (23) and

$$\liminf_{i\to\infty} N(x_{2n_i}, w) \le \max\{0, 0, d(z, Sw), sd(z, Sw), \frac{1}{2}d(z, Sw), d(Bw, Sw), \liminf_{i\to\infty} \frac{1+sd(z, Sw)}{1+s(d(Ax_{2n_i}, Tx_{2n_i})+d(z, Sw))}d(z, Sw)\} = sd(z,Sw),$$

which together with (3), $\psi \in \Phi_1$, $\varphi \in \Phi_2$ yield

$$\psi(d(z, Sw)) \le \psi(s^2 d(z, Sw)) \le \limsup_{n_i\to\infty} \psi(s^3 d(y_{2n_i+1}, Sw)) = \limsup_{n_i\to\infty} \psi(s^3 d(Tx_{2n_i}, Sw)) \le \limsup_{n_i\to\infty} \psi(M(x_{2n_i}, w)) - \liminf_{n_i\to\infty} \varphi(N(x_{2n_i}, w)) \le \psi(d(z, Sw)) - \varphi(sd(z, Sw)),$$

which implies $\varphi(sd(z, Sw)) = 0$, so $Sw = z$. Because $\{A, T\}$ and $\{B, S\}$ are weakly compatible, $Az = ATv = TAv = Tz$ and $Bz = BSw = SBw = Sz$ hold. Suppose that $Tz \ne Sz$. By (iii) and $z = Av = Tv = Sw = Bw$, we deduce $(Tv, y_{n_i}) \in E(\widetilde{G})$ or $(y_{n_i}, Sw) \in E(\widetilde{G})$. Note that the connectivity of G implies $(Tv, Sw) \in E(\widetilde{G})$, i.e., $(z, z) \in E(\widetilde{G})$. Then because T and S weakly preserve edges in $E(\widetilde{G})$, we gain $(Tz, Sz) \in E(\widetilde{G})$. It follows from (2), (4), (3), $\psi \in \Phi_1$ and $\varphi \in \Phi_2$ that

$$\psi(d(Tz, Sz)) \le \psi(s^3 d(Tz, Sz)) \le \psi(M(z, z)) - \varphi(N(z, z)) = \psi(d(Tz, Sz)) - \varphi(d(Tz, Sz)),$$

which explains $\varphi(d(Tz, Sz)) = 0$, hence $Az = Tz = Sz = Bz$.
Suppose that $Tz \ne z$. It follows from (iii), (2), (4) and (3) that

$$\psi(d(Tz, z)) \le \psi(s^3 d(Tz, z)) = \psi(s^3 d(Tz, Sw)) \le \psi(M(z, w)) - \varphi(N(z, w)) = \psi(d(Tz, z)) - \varphi(d(Tz, z)),$$

which implies $\varphi(d(Tz, z)) = 0$ and hence $z = Tz = Sz = Az = Bz$, that is , z is a common fixed point of A, B, S, and T in X.
Suppose that A, B, S, and T have another common fixed point $u \in X \setminus \{z\}$. It follows from (2), (4), (3), $\psi \in \Phi_1$ and $\varphi \in \Phi_2$ that

$$\psi(d(u, z)) \le \psi(s^3 d(u, z)) = \psi(s^3 d(Tu, Sz)) \le \psi(M(u, z)) - \varphi(N(u, z)) = \psi(d(u, z)) - \varphi(d(u, z)),$$

which explains $\varphi(d(u, z)) = 0$. Hence $u = z$. So z is a unique common fixed point of A, B, S, and T.
Similarly we conclude that A, B, S, and T have a unique common fixed point in X if one of $B(X)$, $S(X)$ and $T(X)$ is complete. This completes the proof. □

Example 3.4. *Let $X = \mathbb{R}^+$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a $b-$metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \{(x, x) : x \in$*

$X\} \cup \{(0, \frac{1}{\sqrt[2n]{16}}), (0, \frac{17^m}{\sqrt[2n]{16}}) : m \in \mathbb{R}, n = 0, 1, 2, \cdots\}$. Let A, B, S and T:$X \to X$ be defined by

$$Ax = \sqrt{x}, \qquad Bx = 17x, \qquad Tx = 0, \quad \forall x \in X, \qquad Sx = \begin{cases} 0, & \forall x \in \mathbb{R}^+ \setminus \{\frac{1}{16}\}, \\ \frac{1}{16}, & x = \frac{1}{16}. \end{cases}$$

Now we use Theorem 3.1 to prove the existence of common fixed points of A, B, S, and T in X. Obviously, $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$. {A, T} and {B, S} are weakly compatible. $0 \in C_n$.
If $(x, y) \in E(\widetilde{G})$, then $Tx = Ty = 0$, Sy and Sx is equal to 0 or $\frac{1}{16}$ and so $(Tx, Sy) \in E(\widetilde{G}), (Sx, Ty) \in E(\widetilde{G})$.
For $(Ax, By) \in E(\widetilde{G})$ and $(Bx, Ay) \in E(\widetilde{G})$, we have $(x, y) = (0, 0), (\frac{1}{17^2}, \frac{1}{17^2}), (0, \frac{1}{\sqrt[2n]{16}})$ and $(\frac{1}{\sqrt[2n]{16}}, 0)$,$(0, \frac{17^m}{\sqrt[2n]{16}})$ and $(\frac{17^m}{\sqrt[2n]{16}}, 0)$. So $(x, y) \in E(\widetilde{G})$.
Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(t) = \begin{cases} \frac{\sqrt{t}}{2}, & \forall t \in [1, +\infty), \\ \frac{t}{2}, & \forall t \in [0, 1) \end{cases}$$

In order to verify (1), we consider two cases as follows:
Case 1. $x \in \mathbb{R}^+, y \in \mathbb{R}^+ \setminus \{\frac{1}{16}\}$. It is clear that

$$s^3 d(Tx, Sy) = 0 \leq \psi(M(x, y)).$$

Case 2. $x \in \mathbb{R}^+, y = \frac{1}{16}$. We have

$$s^3 d(Tx, Sy) = 8 \mid 0 - \frac{1}{16} \mid^2 = \frac{1}{32} \leq \psi(d(By, Sy)) = \frac{1}{2} \leq \psi(M(x, y)).$$

That is, (1) holds. Hence there is a sequence, $\{0\}$, satisfying the conditions of Theorem 3.1. So it follows from Theorem 3.1 that A, B, S, and T possess a unique common fixed point $0 \in X$.

Example 3.5. *Let $X = \left[\frac{18}{5}, +\infty\right)$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a $b-$metric space with the coefficient $s = 2$. Let G be a digraph such that $V(G) = X$ and $E(G) = \{(x, x), (\frac{2^n \cdot 3^l x}{16^m \cdot 18^k}, \frac{2^n \cdot 3^l y}{16^m \cdot 18^k}) : x, y \in X, k, l, m, n \in \mathbb{R}\}$. Let A, B, S and T:$X \to X$ be defined by*

$$Ax = \frac{x}{2}, \qquad Bx = \frac{x}{3}, \qquad Tx = \frac{x}{16}, \qquad Sx = \frac{x}{18}, \qquad \forall x \in X.$$

Now we use Theorem3.3 to prove the existence of common fixed points of A, B, S, and T in X. Obviously, $T(X) \subseteq B(X)$ and $S(X) \subseteq A(X)$. {A, T} and {B, S} are weakly compatible. $0 \in C_n$.
If $(x, y) \in E(\widetilde{G})$, we get $(Tx, Sy) = (x, x)$ or $(\frac{2^n \cdot 3^l x}{16^m \cdot 18^k}, \frac{2^n \cdot 3^l y}{16^m \cdot 18^k})$ and $(Sx, Ty) = (x, x)$ or $(\frac{2^n \cdot 3^l x}{16^m \cdot 18^k}, \frac{2^n \cdot 3^l y}{16^m \cdot 18^k})$. Then $(Tx, Sy) \in E(\widetilde{G})$, $(Sx, Ty) \in E(\widetilde{G})$.
For $(Ax, By) \in E(\widetilde{G})$ and $(Bx, Ay) \in E(\widetilde{G})$, we have $(x, y) = (x, x)$ or $(\frac{2^n \cdot 3^l x}{16^m \cdot 18^k}, \frac{2^n \cdot 3^l y}{16^m \cdot 18^k})$. So $(x, y) \in E(\widetilde{G})$.

Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *by*

$$\psi(t) = t, \quad \forall t \in \mathbb{R}^+ \qquad \varphi(t) = \begin{cases} 0, & t = 0, \\ 2^{-t}, & t \in (0, +\infty). \end{cases}$$

In order to verify (3), for all $x, y \in X$ we consider:

$$\begin{aligned}
\psi(s^3 d(Tx, Sy)) &= 8|\frac{x}{16} - \frac{y}{18}|^2 \leq \frac{1}{32}\max\{x^2, y^2\}, \\
\psi(M(x,y)) &\geq \max\{d(Ax, Tx), d(By, Sy)\} = \max\{|\frac{x}{2} - \frac{x}{16}|^2, |\frac{y}{3} - \frac{y}{18}|^2\} \geq (\frac{5}{18})^2 \max\{x^2, y^2\}, \\
N(x,y) &\geq \max\{d(Ax, Tx), d(By, Sy)\} = \max\{|\frac{x}{2} - \frac{x}{16}|^2, |\frac{y}{3} - \frac{y}{18}|^2\} \geq (\frac{5}{18})^2 \max\{x^2, y^2\}, \\
\varphi(N(x,y)) &\leq 2^{-(\frac{5}{18})^2 \max\{x^2, y^2\}} \leq \frac{1}{2} \cdot (\frac{5}{18})^2 \max\{x^2, y^2\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\psi(s^3 d(Tx, Sy)) &\leq \frac{1}{32}\max\{x^2, y^2\} \leq (\frac{5}{18})^2 \max\{x^2, y^2\} - \frac{1}{2} \cdot (\frac{5}{18})^2 \max\{x^2, y^2\} \\
&\leq \psi(M(x,y)) - \varphi(N(x,y))
\end{aligned}$$

That is, (3) holds. So it follows from Theorem3.3 that A, B, S, and T in X possess a unique common fixed point $0 \in X$.

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