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# Common Fixed Point Theorems in Modular Multi-Metric Spaces

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**Abstract**

In this paper, we study the existence and uniqueness of common fixed points for compatible self-mappings in modular multi-metric spaces, which generalize both modular and multi-metric spaces. We introduce contractive conditions suitable for this setting and establish a common fixed point theorem that extends classical results such as Banach's contraction principle and Jungck's theorem. An illustrative example is provided to demonstrate the applicability of the main result. The results contribute to the theory of fixed points in spaces with multiple modular structures and offer potential applications in nonlinear analysis and optimization.

**Keywords:** Common fixed point; Modular multi-metric space; Compatible mappings; Contractive mappings; Nonlinear analysis.
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## 1. Introduction

Fixed point theory is a fundamental tool in mathematical analysis, with applications in differential equations, optimization, and dynamic systems. Classical results, such as Banach's contraction principle, provide conditions under which self-mappings have unique fixed points in metric spaces. Modular spaces extend normed spaces by replacing the norm with a modular function, providing greater flexibility in convergence and completeness. Multi-metric spaces allow multiple metrics on a single set, enabling the analysis of systems with multiple distance measures. A modular multi-metric space combines these concepts, providing a richer framework to study mappings that are simultaneously contractive under several modular metrics. In this paper, we establish a common fixed point theorem for compatible mappings in such spaces, extending several classical fixed point results.

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## 2. Preliminaries

**Definition 2.1** ([1,2,13,15]). *A modular multi-metric space is a pair $(X, \{\rho_i\}_{i=1}^m)$, where $X$ is a nonempty set and each $\rho_i : X \times X \to [0, \infty)$ is a modular metric. The space is said to be complete if every Cauchy sequence with respect to each $\rho_i$ converges to a point in $X$.*

**Definition 2.2** ([2]). *A function $\rho : X \times X \to [0, \infty)$ is called a modular metric if for all $x, y, z \in X$:*

1. $\rho(x, y) = 0$ *if and only if* $x = y$,
2. $\rho(x, y) = \rho(y, x)$,
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

**Definition 2.3** ([3,4]). *Let $X$ be a nonempty set and let $\{\rho_i\}_{i=1}^m$ be a finite family of functions $\rho_i : X \times X \to [0, \infty)$, called* modular $b$-metrics*, each satisfying the following:*

*(i)* $\rho_i(x, y) = 0 \iff x = y$.

*(ii) (Symmetry)* $\rho_i(x, y) = \rho_i(y, x)$.

*(iii) (Relaxed triangle inequality) There exists $b \geq 1$ such that for all $x, y, z \in X$, $\rho_i(x, z) \leq b\big[\rho_i(x, y) + \rho_i(y, z)\big]$.*

*The pair $(X, \{\rho_i\}_{i=1}^m)$ is called a* modular $b$-metric space.

**Definition 2.4** ([5,6]). *A sequence $\{x_n\}$ in a modular multi-metric space $(X, \{\rho_i\}_{i=1}^m)$ is called Cauchy if for every $\varepsilon > 0$ and every $i = 1, \ldots, m$, there exists $N$ such that for all $m, n \geq N$, $\rho_i(x_m, x_n) < \varepsilon$.*

**Definition 2.5** ([5,6]). *It is said to be convergent to $x \in X$ if $\rho_i(x_n, x) \to 0 \quad \forall\ i$.*

**Definition 2.6** ([7,8]). *A mapping $T : X \to X$ is called contractive if there exists $k \in [0, 1)$ such that $\rho_i(Tx, Ty) \leq k\rho_i(x, y) \quad \forall\, x, y \in X, \forall\, i$.*

**Definition 2.7** ([25]). *Two mappings $T, S : X \to X$ are said to be compatible if for any sequence $\{x_n\} \subset X$ with $Tx_n \to p$ and $Sx_n \to p$, it holds that $\rho_i(TSx_n, STx_n) \to 0, \quad \forall\, i$.*

**Definition 2.8** ([9]). *A point $x^* \in X$ is called a fixed point of a self-mapping $T : X \to X$ if $Tx^* = x^*$. A point $x^*$ is a common fixed point of $T$ and $S$ if $Tx^* = Sx^* = x^*$.*

**Definition 2.9** ([10]). *A pair of mappings $T, S : X \to X$ is said to satisfy the contractive condition if there exists $k \in [0, 1)$ such that $\rho_i(Tx, Sy) \leq k \max\{\rho_i(x, y), \rho_i(x, Tx), \rho_i(y, Sy)\}, \forall x, y \in X,\ i = 1, \ldots, m$.*

**Definition 2.10** ([11]). *Mappings $T, S : X \to X$ are called* self-mappings *of the modular $b$-metric space $(X, \{\rho_i\})$.*

**Definition 2.11** ([11]). *Two self-mappings $T$ and $S$ are said to be* weakly compatible *if $Tx = Sx \quad \Rightarrow TSx = STx$.*

**Definition 2.12** ([12]). *Given an arbitrary point* $x_0 \in X$*, define a sequence* $\{x_n\}$ *by* $x_{2n+1} = Tx_{2n}, x_{2n+2} = Sx_{2n+1}, n = 0, 1, 2, \ldots$.

This alternating iteration will be used to show the existence of a common fixed point.

## 3. Main Result

**Theorem 3.1** (Common Fixed Point Theorem [17] in Modular Multi-Metric Spaces). *Let* $(X, \{\rho_i\}_{i=1}^m)$ *be a complete modular multi-metric space. Let* $T, S : X \to X$ *be two self-mappings satisfying the following conditions:*

1. *There exists* $k \in [0, 1)$ *such that for all* $x, y \in X$ *and all* $i = 1, 2, \ldots, m$,

$$\rho_i(Tx, Sy) \le k \max\{\rho_i(x, y), \rho_i(x, Tx), \rho_i(y, Sy)\}. \tag{1}$$

2. *The mappings* $T$ *and* $S$ *are compatible* [25], *i.e.*

$$\lim_{n\to\infty} \rho_i(TSx_n, STx_n) = 0$$

*whenever* $Tx_n \to p$ *and* $Sx_n \to p$ *for some* $p \in X$.

*Then* $T$ *and* $S$ *have a unique common fixed point* [25] $x^* \in X$*, i.e.* $Tx^* = Sx^* = x^*$.

*Proof.* Let $x_0 \in X$ be arbitrary. Define two sequences recursively by $x_{2n+1} = Tx_{2n}$, $x_{2n+2} = Sx_{2n+1}$, $n = 0, 1, 2, \ldots$. We will show that $\{x_n\}$ is a Cauchy sequence in each modular metric $\rho_i$. From (1),

$$\rho_i(x_{2n+1}, x_{2n+2}) = \rho_i(Tx_{2n}, Sx_{2n+1}) \le k \max\{\rho_i(x_{2n}, x_{2n+1}), \rho_i(x_{2n}, Tx_{2n}), \rho_i(x_{2n+1}, Sx_{2n+1})\}.$$

But $\rho_i(x_{2n}, Tx_{2n}) = \rho_i(x_{2n}, x_{2n+1})$ and $\rho_i(x_{2n+1}, Sx_{2n+1}) = \rho_i(x_{2n+1}, x_{2n+2})$. Therefore,

$$\rho_i(x_{2n+1}, x_{2n+2}) \le k \max\{\rho_i(x_{2n}, x_{2n+1}), \rho_i(x_{2n+1}, x_{2n+2})\}.$$

If $\rho_i(x_{2n+1}, x_{2n+2}) > \rho_i(x_{2n}, x_{2n+1})$, the inequality implies $\rho_i(x_{2n+1}, x_{2n+2}) \le k\rho_i(x_{2n+1}, x_{2n+2})$, which is impossible unless $\rho_i(x_{2n+1}, x_{2n+2}) = 0$. Thus, for all $n$,

$$\rho_i(x_{2n+1}, x_{2n+2}) \le k\rho_i(x_{2n}, x_{2n+1}).$$

By iteration, $\rho_i(x_{n+1}, x_n) \le k^n \rho_i(x_1, x_0)$. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, \rho_i)$ for each $i$, and since the space is complete, there exists $x^* \in X$ [1] such that $x_n \to x^*$ in all $\rho_i$. Taking limits in the recursive definition, we get:

$$Tx^* = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = x^*,$$

and

$$Sx^* = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = x^*.$$

Hence, $Tx^* = Sx^* = x^*$.

**Uniqueness:** Suppose $y^*$ is another common fixed point [25]. Then

$$\rho_i(x^*, y^*) = \rho_i(Tx^*, Sy^*) \le k\max\{\rho_i(x^*, y^*), \rho_i(x^*, Tx^*), \rho_i(y^*, Sy^*)\} = k\rho_i(x^*, y^*).$$

This implies $(1-k)\rho_i(x^*, y^*) \le 0$, so $\rho_i(x^*, y^*) = 0$ for all $i$, hence $x^* = y^*$. Therefore, the common fixed point is unique [25]. □

**Lemma 3.2** (Cauchy Sequence Lemma in Modular Multi-Metric Spaces)**.** *Let $(X, \{\rho_i\}_{i=1}^m)$ be a modular multi-metric space and let $T, S : X \to X$ be two self-mappings satisfying* [9]

$$\rho_i(Tx, Sy) \le k\max\{\rho_i(x, y), \rho_i(x, Tx), \rho_i(y, Sy)\}, \quad 0 \le k < 1,$$

*for all $x, y \in X$ and $i = 1, 2, \ldots, m$. Define the sequence $\{x_n\}$ recursively by*

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \quad n = 0, 1, 2, \ldots$$

*Then $\{x_n\}$ is a Cauchy sequence in each modular metric* [2] *$\rho_i$.*

*Proof.* From the given inequality, we have

$$\rho_i(x_{2n+1}, x_{2n+2}) = \rho_i(Tx_{2n}, Sx_{2n+1}) \le k\max\{\rho_i(x_{2n}, x_{2n+1}), \rho_i(x_{2n}, Tx_{2n}), \rho_i(x_{2n+1}, Sx_{2n+1})\}.$$

Since $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, this simplifies to

$$\rho_i(x_{2n+1}, x_{2n+2}) \le k\max\{\rho_i(x_{2n}, x_{2n+1}), \rho_i(x_{2n+1}, x_{2n+2})\}.$$

If $\rho_i(x_{2n+1}, x_{2n+2}) > \rho_i(x_{2n}, x_{2n+1})$, we would obtain $\rho_i(x_{2n+1}, x_{2n+2}) \le k\rho_i(x_{2n+1}, x_{2n+2})$, which is possible only if $\rho_i(x_{2n+1}, x_{2n+2}) = 0$. Therefore,

$$\rho_i(x_{2n+1}, x_{2n+2}) \le k\rho_i(x_{2n}, x_{2n+1}).$$

By iteration, $\rho_i(x_{n+1}, x_n) \le k^n\rho_i(x_1, x_0)$. Thus, for $m > n$,

$$\rho_i(x_m, x_n) \le \sum_{r=n}^{m-1} \rho_i(x_{r+1}, x_r) \le \frac{k^n}{1-k}\rho_i(x_1, x_0),$$

which tends to 0 as $n \to \infty$. Hence, $\{x_n\}$ is Cauchy in each $(X, \rho_i)$. □

**Corollary 3.3** (Banach Type Fixed Point Theorem in Modular Multi-Metric Spaces)**.** *Let $(X, \{\rho_i\}_{i=1}^m)$ be* *a complete modular multi-metric space and let $T : X \to X$ be a self-mapping satisfying* [9]

$$\rho_i(Tx, Ty) \leq k \max\{\rho_i(x, y), \rho_i(x, Tx), \rho_i(y, Ty)\}, \quad 0 \leq k < 1,$$

*for all $x, y \in X$ and $i = 1, 2, \ldots, m$. Then $T$ has a unique fixed point* [25] *$x^* \in X$.*

*Proof.* Let $S = T$ in Theorem 3.1. The compatibility condition is trivially satisfied since $TS = ST = T^2$. By Lemma 3.2, define the sequence $x_{n+1} = Tx_n$. Then,

$$\rho_i(x_{n+1}, x_{n+2}) = \rho_i(Tx_n, Tx_{n+1}) \leq k \max\{\rho_i(x_n, x_{n+1}), \rho_i(x_n, Tx_n), \rho_i(x_{n+1}, Tx_{n+1})\}.$$

Since $x_{n+1} = Tx_n$, this gives $\rho_i(x_{n+1}, x_{n+2}) \leq k\rho_i(x_n, x_{n+1})$, so that $\{x_n\}$ is Cauchy in each $\rho_i$. By completeness, there exists $x^* \in X$ such that $x_n \to x^*$ in all $\rho_i$. Taking limits in $x_{n+1} = Tx_n$, we get $Tx^* = x^*$. If $y^*$ is another fixed point of $T$, then

$$\rho_i(x^*, y^*) = \rho_i(Tx^*, Ty^*) \leq k \max\{\rho_i(x^*, y^*), \rho_i(x^*, Tx^*), \rho_i(y^*, Ty^*)\} = k\rho_i(x^*, y^*).$$

This implies $\rho_i(x^*, y^*) = 0$ for all $i$, hence $x^* = y^*$. Therefore, the fixed point is unique. □

**Corollary 3.4** (Common Fixed Point for Commuting Mappings)**.** *Let $(X, \{\rho_i\}_{i=1}^m)$ be a complete modular multi-metric space. Let $T, S : X \to X$ be two commuting self-mappings (i.e., $TS = ST$) satisfying* [9]

$$\rho_i(Tx, Sy) \leq k \max\{\rho_i(x, y), \rho_i(x, Tx), \rho_i(y, Sy)\}, \quad 0 \leq k < 1,$$

*for all $x, y \in X$ and $i = 1, 2, \ldots, m$. Then $T$ and $S$ have a unique common fixed point* [25] *$x^* \in X$.*

*Proof.* Since $T$ and $S$ commute, we have $TSx = STx$ for all $x \in X$. Thus, the compatibility condition of Theorem 1 is automatically satisfied. By Theorem 3.1, there exists a unique $x^* \in X$ such that

$$Tx^* = Sx^* = x^*.$$

Hence, $x^*$ is the unique common fixed point [25] of $T$ and $S$. □

**Theorem 3.5** (Common Fixed Point Theorem for Compatible Mappings in Modular $b$-Metric Spaces)**.** *Let $(X, \{\rho_i\}_{i=1}^m)$ be a complete modular b-metric space. Let $T, S : X \to X$ be two self-mappings satisfying* [9] *the following conditions:*

1. *There exists a constant $k \in [0, 1)$ such that for all $x, y \in X$ and all $i = 1, 2, \ldots, m$,*

$$\rho_i(Tx, Sy) \leq k[\rho_i(x, y) + \rho_i(x, Tx) + \rho_i(y, Sy)].$$

2. *The mappings $T$ and $S$ are weakly compatible* [11]*, i.e., $Tx = Sx \implies TSx = STx$.*

*Then $T$ and $S$ have a unique common fixed point* [25] $x^* \in X$*, that is,* $Tx^* = Sx^* = x^*$*.*

*Proof.* Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \quad n = 0, 1, 2, \ldots$$

We show that $\{x_n\}$ is a Cauchy sequence in each modular $\rho_i$. From condition (1),

$$\rho_i(x_{2n+1}, x_{2n+2}) = \rho_i(Tx_{2n}, Sx_{2n+1}) \leq k[\rho_i(x_{2n}, x_{2n+1}) + \rho_i(x_{2n}, Tx_{2n}) + \rho_i(x_{2n+1}, Sx_{2n+1})].$$

Since $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$, we have

$$\rho_i(x_{2n+1}, x_{2n+2}) \leq k[\rho_i(x_{2n}, x_{2n+1}) + \rho_i(x_{2n}, x_{2n+1}) + \rho_i(x_{2n+1}, x_{2n+2})],$$

which implies

$$(1-k)\rho_i(x_{2n+1}, x_{2n+2}) \leq 2k\rho_i(x_{2n}, x_{2n+1}).$$

Hence,

$$\rho_i(x_{2n+1}, x_{2n+2}) \leq \frac{2k}{1-k}\rho_i(x_{2n}, x_{2n+1}).$$

By iteration,

$$\rho_i(x_{n+1}, x_n) \leq \left(\frac{2k}{1-k}\right)^n \rho_i(x_1, x_0).$$

Since $0 \leq \frac{2k}{1-k} < 1$, the sequence $\{x_n\}$ is Cauchy. Completeness of $X$ ensures that $x_n \to x^*$ in all $\rho_i$. Taking limits in the recursive definitions,

$$Tx^* = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} x_{2n+1} = x^*,$$

and similarly,

$$Sx^* = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = x^*.$$

Hence, $Tx^* = Sx^* = x^*$.

**Uniqueness.** Suppose $y^*$ is another common fixed point [25] of $T$ and $S$. Then

$$\rho_i(x^*, y^*) = \rho_i(Tx^*, Sy^*) \leq k[\rho_i(x^*, y^*) + \rho_i(x^*, Tx^*) + \rho_i(y^*, Sy^*)] = k\rho_i(x^*, y^*).$$

This gives $(1-k)\rho_i(x^*, y^*) \leq 0$, implying $\rho_i(x^*, y^*) = 0$ for all $i$. Thus, $x^* = y^*$. Therefore, the common fixed point is unique [25]. □

**Example 3.6.** *Consider $X = [0, \infty)$ with the modular b-metric*

$$\rho(x, y) = \frac{|x-y|}{1+|x-y|}.$$

Define mappings $T, S : X \to X$ by

$$Tx = \frac{x}{2}, \qquad Sx = \frac{x}{3}.$$

For all $x, y \in X$, we have

$$\rho(Tx, Sy) = \frac{\left|\frac{x}{2} - \frac{y}{3}\right|}{1 + \left|\frac{x}{2} - \frac{y}{3}\right|} \le \frac{\frac{1}{2}|x-y|}{1 + \frac{1}{2}|x-y|} \le \tfrac{1}{2}\rho(x,y).$$

Hence, the condition of the corollary holds with $k = \frac{1}{2} < 1$. Therefore, by Theorem 3.6, $T$ and $S$ have a unique common fixed point [25] $x^* = 0$, since $Tx^* = Sx^* = 0 = x^*$.

**Example 3.7.** *Let $X = [0,1]$ with modulars defined by $\rho_1(x,y) = |x-y|$, $\rho_2(x,y) = |x-y|^2$. Then $(X, \{\rho_1, \rho_2\})$ is a complete modular multi-metric space.*

Define the mappings:

$$Tx = \frac{x}{4}, \quad Sx = \frac{x}{2}.$$

For all $x, y \in X$,

$$\rho_1(Tx, Sy) = \left|\frac{x}{4} - \frac{y}{2}\right| = \frac{1}{4}|x - 2y| \le \frac{1}{2}|x-y| = \frac{1}{2}\rho_1(x,y),$$

and similarly,

$$\rho_2(Tx, Sy) = \left|\frac{x}{4} - \frac{y}{2}\right|^2 \le \frac{1}{4}|x-y|^2 = \frac{1}{2}\rho_2(x,y).$$

Hence, condition (1) holds with $k = \frac{1}{2}$. The mappings $T$ and $S$ are compatible, and the only common fixed point satisfying [25], $Tx = Sx = x$ is $x^* = 0$. Therefore, by the theorem, $T$ and $S$ have the unique common fixed point [25], $x^* = 0$.

**Example 3.8.** *Let $X = [0,2]$ with modulares defined by $\rho_1(x,y) = |x-y|$, $\rho_2(x,y) = |x-y|^3$. Then $(X, \{\rho_1, \rho_2\})$ is a complete modular multi-metric space.*

Define the mappings:

$$Tx = \frac{x}{3}, \quad Sx = \frac{x}{6}.$$

For all $x, y \in X$,

$$\rho_1(Tx, Sy) = \left|\frac{x}{3} - \frac{y}{6}\right| = \frac{1}{6}|2x - y| \le \frac{1}{2}|x-y| = \frac{1}{2}\rho_1(x,y),$$

and similarly,

$$\rho_2(Tx, Sy) = \left|\frac{x}{3} - \frac{y}{6}\right|^3 \le \frac{1}{8}|x-y|^3 = \frac{1}{2}\rho_2(x,y).$$

Hence, condition (1) holds with $k = \frac{1}{2}$. The mappings $T$ and $S$ are compatible, and the only common fixed point satisfying [25], $Tx = Sx = x$ is $x^* = 0$. Therefore, $T$ and $S$ have the unique common fixed point [25], $x^* = 0$.

**Example 3.9.** *Let $X = [0,1]$ with*

$$\rho_1(x,y) = |x-y|, \quad \rho_2(x,y) = |x-y|^{1/2}.$$

*Then* $(X, \{\rho_1, \rho_2\})$ *is a complete modular multi-metric space.*

Define

$$Tx = \frac{x}{5}, \quad Sx = \frac{x}{3}.$$

Then for all $x, y \in X$,

$$\rho_1(Tx, Sy) = \left|\frac{x}{5} - \frac{y}{3}\right| = \frac{1}{15}|3x - 5y| \le \frac{2}{3}|x - y| = \frac{2}{3}\rho_1(x, y),$$

and similarly,

$$\rho_2(Tx, Sy) = \left|\frac{x}{5} - \frac{y}{3}\right|^{1/2} \le \left(\frac{2}{3}\right)^{1/2} |x - y|^{1/2} = \left(\frac{2}{3}\right)^{1/2} \rho_2(x, y).$$

Hence, condition (1) holds with $k = \frac{2}{3}$. The only common fixed point satisfying [25], $Tx = Sx = x$ is $x^* = 0$.

**Example 3.10.** *Let* $X = [0, 1]$ *with modulares*

$$\rho_1(x, y) = |x - y|, \quad \rho_2(x, y) = (x - y)^2 + |x - y|.$$

Define

$$Tx = \frac{x}{2}, \quad Sx = \frac{x}{3}.$$

Then for all $x, y \in X$,

$$\rho_1(Tx, Sy) = \left|\frac{x}{2} - \frac{y}{3}\right| = \frac{1}{6}|3x - 2y| \le \frac{1}{2}|x - y| = \frac{1}{2}\rho_1(x, y),$$

and

$$\rho_2(Tx, Sy) = \left(\frac{x}{2} - \frac{y}{3}\right)^2 + \left|\frac{x}{2} - \frac{y}{3}\right| \le \frac{1}{2}\left[(x - y)^2 + |x - y|\right] = \frac{1}{2}\rho_2(x, y).$$

So, condition (1) holds with $k = \frac{1}{2}$. The only common fixed point satisfying [25], $Tx = Sx = x$ is $x^* = 0$. Therefore, $T$ and $S$ have the unique common fixed point [25], $x^* = 0$.

**Example 3.11** (Linear Maps in a Modular $b$-Metric Space)**.** *Let* $X = \mathbb{R}$ *and consider a family of modulars* $\rho_i(x, y) = |x - y|^{p_i}$, $p_i \in (0, 1]$, $i = 1, \ldots, m$.

Define two mappings

$$Tx = \tfrac{1}{2}x, \qquad Sx = \tfrac{1}{4}x.$$

We compute

$$\rho_i(Tx, Sy) = \left|\tfrac{1}{2}x - \tfrac{1}{4}y\right|^{p_i}$$

and

$$\rho_i(x, Tx) = \left|x - \tfrac{1}{2}x\right|^{p_i} = \frac{|x|^{p_i}}{2^{p_i}}, \qquad \rho_i(y, Sy) = \left|y - \tfrac{1}{4}y\right|^{p_i} = |y|^{p_i} \left(\tfrac{3}{4}\right)^{p_i}.$$

Choosing $k = \frac{1}{2}$, the inequality $\rho_i(Tx, Sy) \leq k\big(\rho_i(x,y) + \rho_i(x,Tx) + \rho_i(y,Sy)\big)$ holds for all $x, y \in X$. Thus the contractive - type inequality hold. If $Tx = Sx$, then $\frac{1}{2}x = \frac{1}{4}x$, hence $x = 0$. At this point,

$$TSx = T(0) = 0, \qquad STx = S(0) = 0.$$

Thus $T$ and $S$ are weakly compatible. Solving $Tx = x$ gives $x = 0$, and solving $Sx = x$ also gives $x = 0$. Hence the unique common fixed point [25], is $x^* = 0$.

**Example 3.12** (Nonlinear Maps on $[0,2]$). *Let $X = [0,2]$ and define the modular $\rho(x,y) = |x-y|^2$.*

Define two mappings

$$Tx = \sqrt{x}, \qquad Sx = \frac{x+1}{2}.$$

We have

$$\rho(Tx, Sy) = \left|\sqrt{x} - \frac{y+1}{2}\right|^2,$$

$$\rho(x, Tx) = |x - \sqrt{x}|^2, \qquad \rho(y, Sy) = \left|\frac{y-1}{2}\right|^2.$$

For $k = \frac{1}{2}$, one checks that $\rho(Tx, Sy) \leq k\big(\rho(x,y) + \rho(x,Tx) + \rho(y,Sy)\big)$ for all $x, y \in [0,2]$. Thus the contractive inequality holds. If $Tx = Sx$, then

$$\sqrt{x} = \frac{x+1}{2},$$

which gives $x = 1$. At $x = 1$, $TSx = T(1) = 1$, $STx = S(1) = 1$. Thus the mappings are weakly compatible. Solving $Tx = x$ yields $x = 0$ or $1$, whereas $Sx = x$ implies $x = 1$. Thus the unique common fixed point [25], is $x^* = 1$.

**Example 3.13** (Piecewise Maps on $[0,1]$). *Let $X = [0,1]$ and define the modular $\rho(x,y) = |x-y|^{1/2}$.*

Define

$$Tx = \begin{cases} \frac{x}{3}, & x \leq \frac{1}{2}, \\ \frac{1}{2}, & x > \frac{1}{2}, \end{cases} \qquad Sx = \begin{cases} \frac{x}{4}, & x \leq \frac{1}{2}, \\ \frac{1}{2}, & x > \frac{1}{2}. \end{cases}$$

For $x, y \leq \frac{1}{2}$,

$$\rho(Tx, Sy) = \left|\frac{x}{3} - \frac{y}{4}\right|^{1/2} \leq \tfrac{1}{2}\left(\rho(x,y) + \rho(x,Tx) + \rho(y,Sy)\right).$$

For $x, y > \frac{1}{2}$, we have $Tx = Sy = \frac{1}{2}$, so

$$\rho(Tx, Sy) = 0,$$

and the inequality holds trivially. To ensure compatibility, define $Tx = Sx = \frac{1}{2}$ for all $x > 1/2$. Then

$$TSx = STx = \frac{1}{2},$$

and weak compatibility holds. To find common fixed points solve:

$$Tx = x, \qquad Sx = x.$$

Both functions satisfy $Tx = Sx = \frac{1}{2}$ only at

$$x^* = \frac{1}{2}.$$

Thus, $x^* = \frac{1}{2}$ is the unique common fixed .

## 4. Conclusion

We have established a common fixed point theorem in modular multi-metric spaces for compatible contractive mappings. This result generalizes classical fixed point theorems and provides a flexible framework for studying mappings in spaces with multiple modular metrics. Applications may include nonlinear equations, iterative processes, and optimization problems in multi-criteria settings.

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