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Application of m-Ranking to Node Analysis in Complex Networks

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Abstract

Identifying influential nodes in complex networks has gained considerable attention because of its significant theoretical importance and broad range of applications. The m-Ranking method incorporates both the degree of nodes and the weights of edges to rank nodes with varying importance levels. This approach can also be applied to unweighted networks by simply setting the weight parameter $\alpha = 1$. Furthermore, we illustrate the effectiveness of the proposed method using a real-world air traffic network, where the rankings produced are shown to be more meaningful compared to those obtained through existing methods.

Keywords: Graphs; Networks; Centrality Measures; m-Ranking of Nodes.

1. Introduction

Complex systems can often be represented as networks of interacting elements and their relationships. These networks are typically modeled as graphs, where nodes denote the elements of the system, and the links capture their interactions. Such representations are widely used to explore the internal structure and behavior of social, physical, economic, climatic, and financial systems. Networks provide valuable insight into the dynamics of the underlying systems, which naturally evolve over time. As a system changes, these transformations are reflected in its network structure. For example, nodes that act as highly influential hubs at one stage may lose their prominence later, while initially less active nodes may emerge as hubs through evolving interactions within the network.

The nodes with the highest degree of connectivity in a network are often the most influential and significant. In contact-based models of disease transmission, such hub nodes play a pivotal role, as an infection originating from them can lead to rapid and widespread diffusion across the network. Hence, targeting and vaccinating these highly connected nodes becomes a key strategy to control epidemics [5]. A similar phenomenon is observed in the study of how rumors, information, or cultural practices spread within a community. Information travels much faster when it comes from

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people who are strongly connected to others. Therefore, to understand the dynamics of information dissemination, it is essential to examine the role and position of the central actors driving the process. In these scenarios, the importance of nodes in shaping the behavior and outcomes of networks is undeniable.

Importance of nodes in a network is the focus of a series of papers [1,6,8,9,11]. Some nodes which were less important in the beginning may become most important and some other nodes which enjoyed key position in the network may lose its status. Some individual nodes play very vital role in the dynamics of the networks. For example hubs - nodes having degree near and around maximum degree- in a network are known as popular. They attract the attention of other members and tend to increase their connections further [4,7]. Ordinary members who have relatively less connections may prefer to establish link with the hubs because of the awareness that such activity may increase their connections in the network and that would help to improve their status.

Understanding how the importance of nodes evolves has been a central problem in the study of complex networks. Traditionally, measures such as degree, closeness, and betweenness have been widely employed to identify influential nodes. In recent years, several additional centrality measures have been developed, with PageRank [3] and LeaderRank [11] serving as prominent examples.

Five important node ranking methods are defined here which are particularly important in development of the paper. The simplest among them is the Degree Decomposition [12–14]. It is the ranking of the nodes of a network based on the degree of each node. Nodes with greatest degree ranked first and so on. Clearly,degree decomposition is a local metric.

In 2010, Kitsak et al. introduced an efficient node-ranking method known as k-shell decomposition [10] for analyzing large-scale networks. The method partitions all nodes into different shells through an iterative pruning process. First, all nodes with degree k = 1 are removed and assigned to shell $ks = 1$. The procedure is then repeated for nodes with degree $k = 2$, which are assigned to shell $ks = 2$. This iterative process continues until every node is allocated to a corresponding k-shell. Nodes located in the innermost core are considered the most influential within the network.Variations of this method have been studied by [15–17].

Mixed Degree Decomposition (MDD) is a decomposition method proposed by An Zeng et al. [1], which takes into account both residual degree (number of links connecting to the remaining nodes) and exhausted degree (number of links connecting to the removed nodes) for decomposition. For a node i, we denote exhausted degree as $k_i^{(e)}$ and residual degree as $k_i^{(r)}$. In MDD method, the nodes are removed according the mixed degree

$$k^{(m)} = k^{(r)} + \lambda k^{(e)}$$

where λ is a parameter between 0 and 1. This method ranks nodes finer than k-shell decomposition method. Another important ranking is based on *Neighborhood coreness*. There are two neighbourhood coreness methods, which are studied by Joonhyun Bae et al.[8]. It is based on on the assumption that a spreader node with more connections to nodes located in the core of the network is more powerful. With this assumption,the neighborhood coreness is defined as

$$C_{nc}(v) = \sum_{w \in N(v)} ks(w)$$

Here $N(v)$ is the neighborhood set of the node v and $ks(w)$ is the k - shell value of the node w. The extended coreness (C_{nc+}) [8] of a node v is defined as

$$C_{nc}(v) = \sum_{w \in N(v)} C_{nc}(w)$$

In the above formula $C_{nc}(w)$ is the neighborhood coreness of w, that we have already defined. Here, C_{nc+} is more refined than C_{nc}. But both C_{nc} and C_{nc+} depends only on the k - shell value. However, designing an effective method to identify the node importance is still an open issue.

In practice, most of the real networks being used in modeling problems are weighted. There is no good centrality measure to rank nodes of a weighted network.So, we propose the new m - ranking method for weighted complex networks. We also compare ranking of nodes obtained by the m - ranking method with the node ranking obtained by all the methods described above. Next section contains the details of the formula used in the determination of rank of the nodes. It is followed by an algorithm for the calculations. We describe the method on an example graph and the ranking of nodes is given in a table. Then we compare the ranking of nodes in the example network obtained by the methods mentioned in the first section and the new method. We also do similar comparison on a real network and the results are tabulated in the fourth section.

2. The m-Ranking Method for Weighted Complex Networks

In this paper we propose a new method to rank the nodes of a weighed network. The ranking is based on the following formula, which helps us to find the total power of a node i. It is denoted by $T(i)$.

$$T(i) = \{\alpha[d_i^{(0)} + \frac{1}{\beta}\sum d_i^{(1)} + \frac{1}{\beta^2}\sum d_i^{(2)} + \cdots] + (1-\alpha)[\sum W_i^{(0)} + \frac{1}{\beta}\sum W_i^{(1)} + \frac{1}{\beta^2}\sum W_i^{(2)} + \cdots]\} \quad (1)$$

In the formula $\alpha = \frac{p}{q}$ is a parameter between 0 and 1, β>1 another parameter. It is good practice to choose β an integer close to the average degree of the nodes of the graph. The first series contains at most D+1 terms and second series contains at most D terms, where D is the diameter of the graph. Here $d_i^{(0)}$ is the degree of the node i, $\sum d_i^{(j)}$ is the sum of the degrees of the nodes at a distance j from i. Similarly, $\sum W_i^{(0)}$, is the sum of the weights of incident edges to i and $\sum W_i^{(j)}$ is the sum of weights of edges j away from node i. Since we are considering degree of all nodes and weights of all links, usually total power of nodes will be different. Therefore, there are very little chance for two nodes have equal rank. When β value is very large this method gives usual degree centrality. The following algorithm helps us to compute total power of each node in a network, given the weighted adjacency matrix of the underlying weighted graph.

Algorithm: m-ranking

Input: Weighted adjacency matrix M and adjacency matrix N of the graph $G = (V, E)$. α and β are constants such that $0 \leq \alpha \leq 1$ and $\beta > 1$.

Output: The vector T containing the power of vertices

Initialize: $n =$ Order of the matrix N

$Q = 1$

Matrix $F = Floyd - Warshall \ \ (N)$

For $i = 1$ to n do

$dm = Dijkstra(M, i)$

$c = maximum \ \ of \ \ dm$

for $p = 0$ to c do

for $r = 1$ to n do

if $F[i][r] = p$ then

$s =$ Number of nonzero values in the i^{th} row of Matrix M

$d[i][p] = s$

$X =$ Weighted Adjacency Matrix M

$k = 0$

$p =$ dequeue(Q)

while Q is not empty do

$s = 0$

For $q = 1$ to n do

if $X[p][q] > 0$ then

$s = s + X[p][q]$

enqueue(q)

$X[p][q] = X[q][p] = 0$

$W[i][k] = s$

$k = k + 1$

$h1 = 0$

$t1 = 1$

for $p = 0$ to c do

$h1 = h1 + d[i][p] * t1$

$t1 = t1/\beta$

$h1 = \alpha * h1$

$h2 = 0$

$t2 = 1$

for $p = 0$ to $k - 1$ do

$h2 = h2 + W[i][p] * t2$

$t2 = t2/\beta$

$h2 = (1 - \alpha) * h2$

$T[i] = h1 + h2$

Next we consider the graph given in figure 1 to calculate $T(i)$ and hence the rank of the nodes. Since this is an unweighted graph we take $\alpha = 1$ and let $\beta = 2$. Using the formula 2. 1, total power for the nodes are calculated and the values are tabulated in table 1. Since nodes {2,3}, {4,5}, {11,12} have equal status they are ranked equally. All other nodes have different scores. Thus we rank all the nodes with different rank.

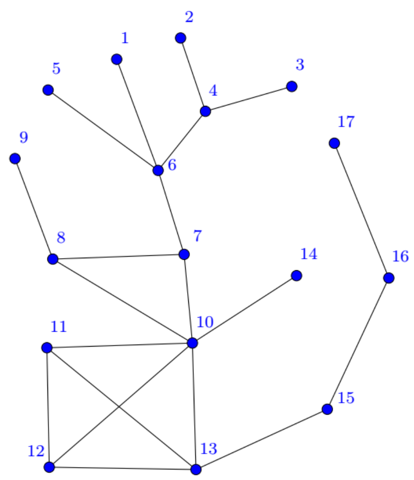

Figure 1: An example network

Vertex i	T(i)	**Rank**	•	Vertx i	T(i)	**Rank**
1	6.97	**11**	•	10	17.31	**1**
2	5.35	**13**	•	11	13.06	**4**
3	5.35	**13**	•	12	13.06	**2**
4	9.22	**9**	•	13	14.37	**3**
5	6.97	**11**	•	14	9.40	**8**
6	12.45	**6**	•	15	9.62	**7**
7	14.40	**2**	•	16	6.68	**12**
8	12.90	**5**	•	17	4.08	**14**
9	7.20	**10**	•			

Table 1: $T(i)$ values and ranks of nodes in the network

3. Test on Schematic Network

Here, efficiency and effectiveness of the new method (m - ranking method) is evaluated. The m-ranking method is compared with the five well known methods, the degree centrality(d), k - shell decomposition (k-s), mixed degree decomposition(MDD), neighborhood coreness (C_{nc}) and the extended neighborhood coreness (C_{nc+}). Efficiency and effectiveness are analyzed using the two measures called monotonicity [8] and correctness [2]. Monotonicity (M) measures the resolution of a ranking system. Its value ranges from 0 to 1. As the number of different ranks decreases, monotonicity comes close to 0. If each node get a different rank, monotonicity is 1. It is computed using the following formula, which was introduced by Joonhyun [8].

$$M(R) = (1 - \frac{\sum n_r(n_r - 1)}{n(n-1)})^2 \tag{2}$$

In the equation R denote the ranking vector of network nodes and n_r denote the number of nodes with same rank. Correctness of two ranks obtained by two different methods is quantified using Kendalls τ as the rank correlation coefficient [2]. Let $(a_1, b_1), (a_2, b_2), ...(a_n, b_n)$ be a set of joint ranks from two different ranking methods. Any pair (a_i, b_i) is concordant with (a_j, b_j) if either $a_i > a_j$ and $b_i > b_j$ or if $a_i < a_j$ and $b_i < b_j$. They are said to be discordant if $a_i > a_j$ and $b_i < b_j$ or if $a_i < a_j$ and $b_i > b_j$. If $a_i = a_j$ or $b_i = b_j$, then the pairs are neither concordant or discordant. Then the rank correlation is defined as

$$\tau = \frac{n_c - n_d}{0.5n(n-1)}$$

In the following table we summarize ranks obtained by the nodes in the network given in Figure 1. This network contains 17 nodes and 20 edges. From the table it is clear that the degree decomposition ranks the nodes in five groups. The $k - s$ method ranks them in only three groups. But the C_{nc+} method ranks them in 12 groups. The m-ranking method is seen to be best with 14 groups, which is the highest among all the methods.

Rank	d	KS	MDD	C_{nc}	C_{nc+}	**m-Rank**
1	10	10,11,12,13	10	10	10	**10**
2	6,13	7,8	12,13	13	13	**7**
3	4,7,8,11,12	others	6	11,12	11,12	**13**
4	15,16		7	7,8	7	**11, 12**
5	others		15	15	8	**8**
6			11	4,6,14	14	**6**
7			8	9,16	15	**15**
8			4	1,2,3,5,17	6	**14**
9			16		9	**4**
10			others		4,16	**9**
11					1,2,3,5	**1,5**
12					17	**16**
13						**2,3**
14						**17**

Table 2: Node order of the network in fig 1 by different measures

When we compute M(R) values for the six different methods, separately using the formula 3.1, we get the following. The m - ranking method gets the highest value among all methods, which indicates that this method is most efficient in ranking the nodes in the network.

Sl No.	Method	M-values
1	d	0.5733
2	k_s	0.29
3	MDD	0.702
4	C_{nc}	0.77
5	C_{nc+}	0.880
6	m-rank	0.97

Table 3: m - values of the six different ranks calculated using the formula 3.1

Compairing the m - ranking method with all other methods, we compute the corresponding τ - values. These values are tabulated below. Among the correlation coefficients calculated, the value for the pair C_{nc+} and m - ranking is the highest.It means the method proposed in this paper has strong positive correlation with the latest and modified method, that is the C_{nc+} method.

Sl No.	Comparison with m-R Method	τ values
1	$\tau(d, m-R)$	0.485
2	$\tau(k_s, m-R)$	0.544
3	$\tau(MDD, m-R)$	0.65
4	$\tau(C_{nc}, m-R)$	0.780
5	$\tau(C_{nc+}, m-R)$	0.830

Table 4: τ value comparison

4. Real Weighted Network Example: The Air Transportation Network

We have not applied the m - ranking method for any real network yet. So, we proceed to apply it in a real air transportation network. Spicejet is a low cost air line company in India. It has 1014 trips to 33 airports in India as on 26th March 2016. This network contains 33 Airports (nodes) and 140 routes (links) connecting these nodes. Each edge has a weight w_{ij} , the number of flights between two airports i and j. The data is available in [**?**]. The network is given in figure 2. The weights are not given in the figure in-order to make the figure readable. We calculate $T(i)$ value for each node using equation 2.1 and it is the basis for ranking of nodes given in table 5. The highest T value is for node 11 (Delhi) with $T(11) = 379.37$ and the next highest value is for node 23 (Mumbai) with $T(23) = 367.87$. The least T value is for node 29 (Tuticorin) with $T(29) = 149.25$. The for-most thing is that each node gets a unique rank in the ranking system. So this is the best method available to rank the nodes in a network according to the nodes importance. The node getting first rank is the most influential figure in a network. The importance of other nodes decreases as they go down in the ranking system.

In the next section, we bring in the limelight the various degree of computational complexity associated with all the methods discussed in the paper.

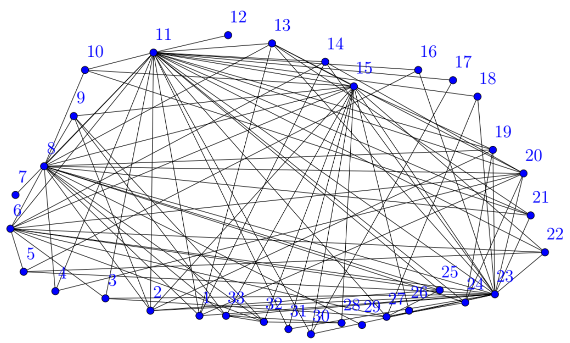

Figure 2: A real network of air traffic of Spice Jet airlines between important cities in India

5. The Computational Complexity and Characteristics of Different Measures

Among the six methods explained above, d is a local metric. ks, MDD are global metrics. C_{nc}, C_{nc+} and m-ranking methods are hybrid methods considering local influence and global influence simultaneously. In general, hybrid metrics can perform better than the local or global metrics. In the new method proposed in this paper computational complexity is $O(n^2)$. In table 6 computational complexity for different methods are given.

Rank	Total Power	Name of Airport with node number	Rank	Total Power	Name of Airport with node number
1	379.37	Delhi(11)	18	247.55	Dehradun(10)
2	367.87	Mumbai(23)	19	246.37	Udaipur(30)
3	362	Chennai(8)	20	245.5	Varanasi(31)
4	333.25	Benguluru(6)	21	240.25	Bagdogra(4)
5	331.37	Hydrabad(15)	22	239.25	Jammu(18)
6	312.25	Kolkata(20)	23	234.12	Vijayawada(32)
7	310.37	Goa(13)	24	233.12	Coimbatore(9)
8	298.12	Kochi(19)	25	231.5	Mangalre(22)
9	295.75	Ahemadabad(2)	26	228.62	Jabalpur(16)
10	283.5	Agarthala(1)	27	214.25	Belgaum(5)
11	274.75	Madhurai(21)	28	200.75	Jaipur(17)
12	267.5	srinagar(27)	29	214.25	Tirupati(28)
13	266	Guahati(14)	30	182.75	Chandigarh(7)
14	260	Pune(25)	31	174.37	Dharmasala(12)
15	254.37	Port Blair(24)	32	149.25	Rajamundry(26)
16	252.5	Amritzar(3)	33	149.25	Tuticorin(29)
17	252	Visakapatanam(33)			

Table 5: Ranking of airports by total power $T(i)$ value

Ranking measure	Category	Computational complexity
d	local metric	O(n)
KS	Global metric	O(n)
MDD	Global metric	O(n)
C_{nc}	Global metric	O(n)
C_{nc+}	Hybrid metric	$O(n^2)$
m-Rank	**Hybrid metric**	$O(n^2)$

Table 6: Computational complexity of various methods

6. Conclusions

In this paper, we propose a new method called m - ranking to find node ranking in a weighted network. There are not many methods to find node importance of weighted complex networks. Here we are using all node degree and all edge weights with different weights to find a nodes rank. The m-Ranking method is a hybrid method. Its complexity is of $O(n^2)$. If β is small many nodes will have equal rank and if β is very large, this method tends to degree centrality.

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