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Fixed Point Theorem of Generalized Contraction for Multi-valued Mapping on Fuzzy Metric Spaces

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Abstract

We prove a Generalized type fixed point theorem for multi-valued mappings on G-complete fuzzy metric spaces. The proof uses the Hausdorff fuzzy metric space which was introduced by Rodriguez-Lopez and Romaguera [13]. We also generalized previous known results.

Keywords: fixed point theorem; fuzzy metric space; Generalized Type Contraction; Chatterjea Type contraction; Kannan-type contraction.
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1. Introduction and Preliminaries

Fixed point theory is highly significant in both mathematics and applied sciences, providing a broad spectrum of applications that ensure the existence and uniqueness of solutions in differential and integral equations [1,5]. There was a considerable necessity to simplify and unify these concepts and principles. In his doctoral thesis from 1906, M. Frechet [6] effectively addressed this issue by introducing the notion of a metric space, thus fulfilling this urgent requirement. Building on this concept, Banach [2] utilized it to formulate the famous fixed point theorem in 1922, representing a major advancement in the development of various extensions of metric spaces.

In 1969, Nadler [11] presented a multi-valued version of Banach's theorem for metric spaces, which included the Hausdorff distance. Lopez et al. [13] expanded this idea to encompass fuzzy metric spaces (FMS). They investigated a fuzzy Hausdorff distance on the collection of compact subsets within these spaces. In this framework, we apply the definition given in [13] to formulate a principle for multi-valued fuzzy contraction mappings.

It is important to note that the introduction of fuzzy sets was made by L. A. Zadeh [16] in 1965, marking a significant milestone. Fuzzy concepts have advanced in nearly every area of theoretical and applied mathematics. Numerous authors in the domains of topology and analysis have since

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extensively employed this concept. Kannan [10] expanded the Banach Contraction Principle in 1968 and derived several fixed point results. Following Kannan, many mathematicians referenced as [4,8,14] continued to investigate this area and made their own important contributions.
We start with the definition of fuzzy metric space in the sense of George-Veeramani [7].

Definition 1.1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is called a continuous t-norm if it satisfies the following conditions:*

(i) $*$ *is associative and commutative,*

(ii) $*$ *is continuous,*

(iii) $a * 1 = a$ *for all* $a \in [0,1]$*,*

(iv) $a * b \leq c * d$ *wherever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 1.2. *The triple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary non-empty set,* $*$ *is a continuous t-norm and* M *is a function from* $X \times X \times (0,\infty)$ *to* $[0,1]$ *such that for all* $x, y, z \in X$ *and* $t, s > 0$*:*

(F1) $M(x,y,t) > 0$,

(F2) $M(x,y,t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(F3) $M(x,y,t) = M(y,x,t)$,

(F4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$,

(F5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

We follow the paper by Grabiec [9] to define the G-Cauchy sequence and G-completeness.

Definition 1.3. *Let* $(X, M, *)$ *be a fuzzy metric space and* $\{x_n\}$ *be a sequence in* X.

1. *The sequence* $\{x_n\}$ *is said to be convergent if there exists* $x \in X$ *such that* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for* $t > 0$.

2. *The sequence* $\{x_n\}$ *is said to be a G-Cauchy sequence if* $\lim_{n\to\infty} M(x_n, x_{n+q}, t) = 1$ *for* $t > 0$ *and* $q \in \mathbb{N}$.

3. *A fuzzy metric space in which every G-Cauchy sequence is convergent is called a G-complete fuzzy metric space.*

Definition 1.4. *Let* A *be a non-empty subset of a fuzzy metric space* $(X, M, *)$ *and* $t > 0$. *The fuzzy distance* $\mathcal{M}$ *between an element* $\rho \in X$ *and the subset* $A \subset X$ *is given by* $\mathcal{M}(\rho, A, t) = \sup\{M(\rho, \mu, t) : \mu \in A\}$. *We also define that* $\mathcal{M}(\rho, A, t) = \mathcal{M}(A, \rho, t)$.

Definition 1.5. *Let* $(X, M, *)$ *be a fuzzy metric space. Define a function* $\Theta_{\mathcal{M}}$ *on* $\hat{C}_0(X) \times \hat{C}_0(X) \times (0,\infty)$ *by*

$$\Theta_{\mathcal{M}}(A, B, t) = \min\{\inf_{\rho\in A} \mathcal{M}(\rho, B, t), \inf_{\mu\in B} \mathcal{M}(A, \mu, t)\},$$

for all $A, B \in \hat{C}_0(X)$ and $t > 0$, where $\hat{C}_0(X)$ is the collection of nonempty compact subset of X. The triple $(\hat{C}_0(X), \Theta_{\mathcal{M}}, *)$ is called a Hausdorff fuzzy metric space.

Remark 1.6. *For each $x, y \in X$, $M(x, y, t)$ is a non-decreasing function on $(0, \infty)$.*

Remark 1.7. *From the continuity of M and Remark 1.6, for a given $x, y \in X$ if $M(x, y, t) > 1 - t$ for any $t > 0$, then $x = y$.*

Remark 1.8. *From Remark 1.6, for any $A \subset X$ and $\mu \in X$, $\mathcal{M}(A, \mu, t)$ is a non-decreasing function on $(0, \infty)$.*

Remark 1.9. *From Remark 1.8, for any $A, B \in \hat{C}_0(X)$, $\Theta_{\mathcal{M}}(A, B, t)$ is a non-decreasing function on $(0, \infty)$.*

Lemma 1.10. *If $A \in Cl(X)$, then $\rho \in A$ if and only if $\mathcal{M}(A, \rho, t) = 1$ for all $t > 0$, where $Cl(X)$ is the collection of nonempty closed subsets of X.*

Lemma 1.11. *Let $(X, M, *)$ be a fuzzy metric space such that $(\hat{C}_0(X), \Theta_{\mathcal{M}}, *)$ is a Hausdorff fuzzy metric space on $\hat{C}_0(X)$. Assume that for all $A, B \in \hat{C}_0(X)$, for each $\rho \in A$ and for $t > 0$, there exists $\mu_\rho \in B$ so that $\mathcal{M}(\rho, B, t) = M(\rho, \mu_\rho, t)$. Then $\Theta_{\mathcal{M}}(A, B, t) \le M(\rho, \mu_\rho, t)$, holds.*

Next we have following famous fixed point theorems,

Theorem 1.12 ([10])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists a constant $c \in (0, \frac{1}{2})$ satisfying, for any $x, y \in X$, $d(Tx, Ty) \le c[d(x, Tx) + d(y, Ty)]$. Then, T has a unique fixed point.*

Theorem 1.13 ([3])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a Chatterjea type mapping such that there exists a constant a, $0 < a < \frac{1}{2}$ satisfying, for any $x, y \in X$, $d(T(x), T(y)) < a[d(x, T(y)) + d(y, T(x))]$. Then, T has a unique fixed point.*

Theorem 1.14 ([14])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that there exists a constant a, b, c, $0 < a + b + c < 1$ satisfying, for any $x, y \in X$, $d(T(x), T(y)) < ad(x, T(x)) + bd(y, T(y)) + cd(x, y)$. Then, T has a unique fixed point.*

Now we define the following generalized contraction of a multi-valued mapping based on Romaguera's definition as follows.

Definition 1.15. *Let $(X, M, *)$ be a fuzzy metric space. We say that a multi-valued mapping $T : X \to \hat{C}_0(X)$ is a (1)-generalized contraction on X if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,*

$$\min\{\mathcal{M}(x, Tx, t), \mathcal{M}(y, Ty, t)\} > 1 - t \Rightarrow \Theta_{\mathcal{M}}(Tx, Ty, ct) > 1 - ct. \tag{1}$$

Definition 1.16. *Let $(X, M, *)$ be a fuzzy metric space. We say that a multi-valued mapping $T : X \to \hat{C}_0(X)$ is a (1)-generalized contraction on X if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,*

$$\min\{\mathcal{M}(y, Tx, t), \mathcal{M}(x, Ty, t)\} > 1 - t \Rightarrow \Theta_{\mathcal{M}}(Tx, Ty, ct) > 1 - ct. \tag{2}$$

Definition 1.17. *Let $(X, M, *)$ be a fuzzy metric space. We say that a multi-valued mapping $T : X \to \hat{C}_0(X)$ is a (1)-generalized contraction on X if there is a constant $c \in (0, 1)$ such that for any $x, y \in X$ and $t > 0$,*

$$\begin{aligned} &\min\{\mathcal{M}(x, Tx, t), \mathcal{M}(y, Ty, t), \mathcal{M}(y, Tx, t), \mathcal{M}(x, Ty, t), \mathcal{M}(x, y, t)\} > 1 - t \\ &\Rightarrow \Theta_{\mathcal{M}}(Tx, Ty, ct) > 1 - ct. \end{aligned} \tag{3}$$

We call (s)-generalized contraction if the constant can be taken in the range $(0, s)$.

2. Main Result

In this section, we prove a generalized contraction-type fixed point theorem for multi-valued mappings on G-complete fuzzy metric spaces. Recall that, given a multi-valued mapping $T : X \to \hat{C}_0(X)$, a point z is said to be a fixed point of T if $z \in Tz$.

Theorem 2.1. *Let $(X, M, *)$ be a G-complete fuzzy metric space and $(\hat{C}_0(X), \Theta_{\mathcal{M}}, *)$ be a Hausdorff fuzzy metric space. Let $T : X \to \hat{C}_0(X)$ be a multi-valued generalized contraction mapping, then T has a fixed point.*

Proof. Take any $x_0 \in X$. Let $x_1 \in X$ such that $x_1 \in Tx_0$. By Lemma 1.11, we can choose $x_2 \in Tx_1$ such that for all $t > 0$,

$$M(x_1, x_2, t) \geq \Theta_{\mathcal{M}}(Tx_0, Tx_1, t).$$

Inductively, we have $x_{n+1} \in Tx_n$ satisfying

$$M(x_n, x_{n+1}, t) \geq \Theta_{\mathcal{M}}(Tx_{n-1}, Tx_n, t), \ \forall n \in \mathbb{N}.$$

Fix $t_0 > 1$. For any $x, y \in X$ we have

$$\mathcal{M}(x, Tx, t_0) > 1 - t_0, \quad \mathcal{M}(y, Ty, t_0) > 1 - t_0. \tag{4}$$

Then, from the assumption, we obtain

$$\Theta_{\mathcal{M}}(Tx, Ty, ct_0) > 1 - ct_0.$$

In particular, we have

$$\Theta_{\mathcal{M}}(Tx_0, Tx_1, ct_0) > 1 - ct_0, \quad \Theta_{\mathcal{M}}(Tx_1, Tx_2, ct_0) > 1 - ct_0.$$

Therefore, from

$$\begin{aligned} \mathcal{M}(x_1, Tx_1, ct_0) &\geq M(x_1, x_2, ct_0) \geq \Theta_{\mathcal{M}}(Tx_0, Tx_1, ct_0) > 1 - ct_0, \\ \mathcal{M}(x_2, Tx_2, ct_0) &\geq M(x_2, x_3, ct_0) \geq \Theta_{\mathcal{M}}(Tx_1, Tx_2, ct_0) > 1 - ct_0, \end{aligned}$$

and the assumption, we obtain

$$\Theta_{\mathcal{M}}(Tx_1, Tx_2, c^2 t_0) > 1 - c^2 t_0.$$

Similarly,

$$\Theta_{\mathcal{M}}(Tx_2, Tx_3, c^2 t_0) > 1 - c^2 t_0.$$

So, we have

$$\mathcal{M}(x_2, Tx_2, c^2 t_0) \geq M(x_2, x_3, c^2 t_0) \geq \Theta_{\mathcal{M}}(Tx_1, Tx_2, c^2 t_0) > 1 - c^2 t_0,$$
$$\mathcal{M}(x_3, Tx_3, c^2 t_0) \geq M(x_3, x_4, c^2 t_0) \geq \Theta_{\mathcal{M}}(Tx_2, Tx_3, c^2 t_0) > 1 - c^2 t_0.$$

By repeating n times we obtain

$$M(x_n, x_{n+1}, c^n t_0) > 1 - c^n t_0.$$

Here, given $t > 0$, there is $n(t) \in \mathbb{N}$ such that $c^n t_0 < t$ for all $n \geq n(t)$. Therefore, considering $\frac{t}{q}$ as t for $q \in \mathbb{N}$, we have

$$\begin{aligned} M(x_n, x_{n+q}, t) &\geq M\left(x_n, x_{n+1}, \frac{t}{q}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{q}\right) * \cdots * M\left(x_{n+q-1}, x_{n+q}, \frac{t}{q}\right) \\ &\geq M(x_n, x_{n+1}, c^n t_0) * M(x_{n+1}, x_{n+2}, c^{n+1} t_0) * \cdots * M(x_{n+q-1}, x_{n+q}, c^{n+q-1} t_0) \\ &> (1 - c^n t_0) * (1 - c^{n+1} t_0) * \cdots * (1 - c^{n+q-1} t_0), \end{aligned}$$

for any $n \geq n(\frac{t}{q})$. So, taking the limit as $n \to \infty$, $(x_n)_{n \in \mathbb{N}}$ is a G-Cauchy sequence in $(X, M, *)$. Then, there is $z \in X$ such that the sequence $(x_n)_{n \in \mathbb{N}}$ converges to z.

Next, we prove that z is a fixed point of T. Fix $r, s > 0$ such that $c < s < r < 1$.

First, we show

$$\mathcal{M}(z, Tz, r^k t_0) \geq 1 - r^k t_0, \tag{5}$$

for any $k \in \mathbb{N}$. We can assume $r^k t_0 \leq 1$ for all $k \in \mathbb{N}$ since the opposite case gives (5) obviously.

For each $k \in \mathbb{N}$ we define

$$A_{k,r,s} := \{\varepsilon \in (0,1) : \varepsilon + s r^{k-1} t_0 < r^k t_0\}.$$

To show (5) by induction, let $k = 1$. Then, we have

$$\mathcal{M}(z, Tz, t_0) > 1 - t_0, \quad \mathcal{M}(x_n, Tx_n, t_0) > 1 - t_0.$$

So, by the definition of $\Theta_{\mathcal{M}}$ and condition (3), we obtain

$$\mathcal{M}(Tz, x_{n+1}, s t_0) \geq \mathcal{M}(Tz, x_{n+1}, c t_0) \geq \inf_{\rho \in Tx_n} \mathcal{M}(Tz, \rho, c t_0) \geq \Theta_{\mathcal{M}}(Tz, Tx_n, c t_0)$$

$$> 1 - ct_0 > 1 - st_0,$$

for any $n \in \mathbb{N} \cup \{0\}$. In particular, since Tz is compact, there exists some $\rho \in X$ such that

$$M(\rho, x_n, st_0) = \mathcal{M}(Tz, x_n, st_0) > 1 - st_0.$$

Since $(x_n)_{n\in\mathbb{N}}$ converges to z, for any $\varepsilon \in A_{1,r,s}$, there exists $n_\varepsilon \in \mathbb{N}$ such that $M(z, x_n, \varepsilon) > 1 - \varepsilon$ for any $n \geq n_\varepsilon$. Therefore, we obtain

$$\begin{aligned}\mathcal{M}(z, Tz, rt_0) &\geq M(z, \rho, rt_0) \geq M(z, x_n, \varepsilon) * M(x_n, \rho, st_0)\\ &\geq (1-\varepsilon) * (1 - st_0) \geq (1-\varepsilon) * (1 - rt_0).\end{aligned}$$

If we take the limit as $\varepsilon \to 0$, then by continuity of $*$ we have

$$\mathcal{M}(z, Tz, rt_0) \geq 1 - rt_0.$$

So, we have proved when $k = 1$. Next, suppose that the inequality (5) holds for $k = j$. Then, we will show

$$\mathcal{M}(z, Tz, r^{j+1}t_0) \geq 1 - r^{j+1}t_0.$$

From the assumption of induction, we have

$$\mathcal{M}(z, Tz, r^j t_0) > 1 - r^j t_0.$$

Also, since $(x_n)_{n\in\mathbb{N}}$ is a G-Cauchy sequence, there exists $n_j \in \mathbb{N}$ such that

$$M(x_n, x_{n+1}, r^j t_0) > 1 - r^j t_0,$$

for all $n \geq n_j$. Thus, by the definition of $\mathcal{M}$, we have

$$\mathcal{M}(x_n, Tx_n, r^j t_0) \geq M(x_n, x_{n+1}, r^j t_0) > 1 - r^j t_0.$$

Therefore, from condition (3), we obtain

$$\Theta_{\mathcal{M}}(Tz, Tx_n, cr^j t_0) > 1 - cr^j t_0,$$

for any $n \geq n_j$. From $s > c$ and non-decreasing property, we have

$$\Theta_{\mathcal{M}}(Tz, Tx_n, sr^j t_0) > 1 - sr^j t_0,$$

for any $n \geq n_j$. Then, we get

$$\mathcal{M}(Tz, x_{n+1}, sr^j t_0) \geq \inf_{\rho \in Tx_n} \mathcal{M}(Tz, \rho, sr^j t_0) \geq \Theta_{\mathcal{M}}(Tz, Tx_n, sr^j t_0)$$
$$> 1 - sr^j t_0.$$

In particular, since Tz is compact, there exists some $\rho \in X$ such that

$$M(\rho, x_n, sr^j t_0) = \mathcal{M}(Tz, x_n, sr^j t_0) > 1 - sr^j t_0.$$

Now let $\varepsilon \in A_{j+1,r,s}$. Then $\varepsilon + sr^j t_0 < r^{j+1} t_0$, and there exists $n_\varepsilon > n_j$ such that $M(z, x_{n_\varepsilon}, \varepsilon) > 1 - \varepsilon$. Therefore,

$$\mathcal{M}(z, Tz, r^{j+1} t_0) \geq M(z, \rho, r^{j+1} t_0) \geq M(z, x_{n_\varepsilon}, \varepsilon) * M(x_{n_\varepsilon}, \rho, sr^j t_0)$$
$$\geq (1 - \varepsilon) * (1 - sr^j t_0) \geq (1 - \varepsilon) * (1 - r^{j+1} t_0).$$

If we take the limit as $\varepsilon \to 0$, then by continuity of $*$ we have

$$\mathcal{M}(z, Tz, r^{j+1} t_0) \geq 1 - r^{j+1} t_0.$$

So, the inequality (5) holds.
Now, given $t > 0$, since there exists $k \in \mathbb{N}$ such that $r^k t_0 < t$, we have

$$\mathcal{M}(z, Tz, t) \geq \mathcal{M}(z, Tz, r^k t_0) > 1 - r^k t_0 > 1 - t.$$

By Lemma 1.10, we obtain $z \in Tz$. This completes the proof. □

Following results follows by (2.1),

Corollary 2.2. *Let $(X, M, *)$ be a G-complete fuzzy metric space and $(\hat{C}_0(X), \Theta_{\mathcal{M}}, *)$ be a Hausdorff fuzzy metric space. Let $T : X \to \hat{C}_0(X)$ be a multi-valued Kannan type contraction mapping* (1), *then T has a fixed point.*

Corollary 2.3. *Let $(X, M, *)$ be a G-complete fuzzy metric space and $(\hat{C}_0(X), \Theta_{\mathcal{M}}, *)$ be a Hausdorff fuzzy metric space. Let $T : X \to \hat{C}_0(X)$ be a multi-valued Chatterjea type contraction mapping* (1), *then T has a fixed point.*

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