An Extension of the Compression-Expansion Fixed Point Theorem of Functional Type

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Abstract

In this article we use an interval of functional type as the underlying set in our compression-expansion fixed point theorem argument which can be used to exploit properties of the operator to improve conditions that will guarantee the existence of a fixed point in applications. An example is provided to demonstrate how intervals of functional type can improve conditions in applications to boundary value problems. We also show how one can use suitable k-contractive conditions to prove that a fixed point in a functional-type interval is unique.

Keywords: Fixed-point theorem; k-contractive; expansion; compression.

2020 Mathematics Subject Classification: 47H10.

1. Introduction

The compression-expansion fixed point theorems of functional type, see for example [2,3], have relied on functional frustrums of a cone which are sets of the form

$$P(\beta, b, \alpha, a) = \{x \in P : a < \alpha(x) \text{ and } \beta(x) < b\},$$

and if compression-expansion conditions are met, then one concludes that there is a fixed point for an operator T in this set. In this paper we show that the underlying set can be generalized by using functional-type intervals which are subsets of $P(\beta, b, \alpha, a)$, that is,

$$A(\beta, b, \alpha, a) = \{x \in A : a < \alpha(x) \text{ and } \beta(x) < b\},$$

where A is an open subset of the cone P and most importantly

$$\partial A \cap \overline{A(\beta, b, \alpha, a)} = \varnothing.$$

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In the spirit of the Leggett-Williams fixed point theorem [9], which many of the compression-expansion fixed point theorems of functional type have generalized, we do not know that T is invariant on $A(\beta, b, \alpha, a)$. However, if suitable k-contractive conditions are met we can use similar arguments as presented in the Banach fixed point theorem (see [13] for a presentation of these concepts, and one can also see these techniques in the work of Petryshyn [11]), to prove that the fixed point of T in $A(\beta, b, \alpha, a)$ is unique. The uniqueness argument does not require iterates to converge nor does it require the operator to be invariant on the functional interval.

2.   Preliminaries

For completeness we provide the following definitions and theorems, which are nearly identical to the presentation in other compression-expansion fixed point papers, in particular [3].

Definition 2.1. *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a* cone *if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$ and if $x, -x \in P$ then $x = 0$.*

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Definition 2.3. *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and*

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

In Theorem 3.1 we will show how one can use functional-type intervals instead of functional frustrums of a cone (sets of the form $P(\beta, b, \alpha, a)$). The definition of a functional type interval appears next.

Definition 2.4. *Let A be a relatively open subset of a cone P, a and b be nonnegative numbers, α be a concave functional on P, and β be a convex functional on P. Then the set*

$$A(\beta, b, \alpha, a) = \{x \in A : a < \alpha(x) \text{ and } \beta(x) < b\}$$

is an interval of functional type.

Definition 2.5. *Let D be a subset of a real Banach space E. If $r : E \rightarrow D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a* retract *of E, and the map r is a* retraction.

Remark 2.6. *The* convex hull *of a subset D of a real Banach space X is given by*

$$\text{conv}(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D, \ \lambda_i \in [0,1], \ \sum_{i=1}^{n} \lambda_i = 1, \ and \ n \in \mathbb{N} \right\}.$$

The following theorem is due to Dugundji and its proof can be found in [4].

Theorem 2.7. *For Banach spaces X and Y, let $D \subset X$ be closed and let $F : D \rightarrow Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \rightarrow Y$ such that $\tilde{F}(X) \subset \overline{\text{conv}(F(D))}$.*

Corollary 2.8. *Every closed convex set of a Banach space is a retract of the Banach space.*

The following theorem is developed from topological degree theory and the abstract (axiomatic) form that appears below can be found in [4]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 2.9. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \rightarrow X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

(1) *Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in \overline{U}$;*

(2) *Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$;*

(3) *Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0,1]$ whenever $H : [0,1] \times \overline{U} \rightarrow X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0,1] \times \partial U$;*

(4) *Solution: If $i(A, U, X) \neq 0$, then A has at least one fixed point in U.*

Moreover, $i(A, U, X)$ is uniquely defined.

3.   Main Results

There are many functional fixed point theorems in the literature, for example see the following papers [1–3,6–12] for foundational arguments related to functional fixed point theorems. The following functional fixed point theorem extends the conclusions of the fixed point theorems of functional type in the literature by providing a uniqueness condition and a generalization of the underlying set in which the fixed point is known to exist. The underlying set in our arguments is called a functional type interval and provides a mechanism for properties of the operator to be introduced into applications of the theorem which is illustrated at the conclusion of this paper.

Theorem 3.1. *Suppose P is a cone in a real Banach space E, A is a relatively open subset of P, α and ψ are nonnegative continuous concave functionals on P, β and θ are nonnegative continuous convex functionals on P, and $T : P \to P$ is a completely continuous operator. If there exist nonnegative numbers a, b, c, and d such that*

(A1) $A(\beta, b, \alpha, a)$ is bounded, $A(\beta, b, \alpha, a) \cap A(\theta, c, \psi, d) \neq \varnothing$, and
$$\partial A \cap \overline{A(\beta, b, \alpha, a)} = \varnothing;$$

(A2) if $x \in \partial A(\beta, b, \alpha, a)$ with $\alpha(x) = a$ and either $\theta(x) \leq c$ or $\theta(Tx) > c$, then $\alpha(Tx) > a$;

(A3) if $x \in \partial A(\beta, b, \alpha, a)$ with $\beta(x) = b$ and either $\psi(Tx) < d$ or $\psi(x) \geq d$, then $\beta(Tx) < b$;

then T has a fixed point $x^* \in A(\beta, b, \alpha, a)$. Moreover, if for all $x \in A(\beta, b, \alpha, a)$ there exists a $k \in [0, 1)$ such that $\|Tx - x^*\| \leq k \|x - x^*\|$ then x^* is the unique fixed point of T in $A(\beta, b, \alpha, a)$.

Proof. By Corollary 2.8, P is a retract of the Banach space E since it is convex and closed.

Claim 1: $Tx \neq x$ for all $x \in \partial A(\beta, b, \alpha, a)$. The functional interval $A(\beta, b, \alpha, a) = A \cap P(\beta, b, \alpha, a)$, hence

$$
\begin{aligned}
\partial A(\beta, b, \alpha, a) &= \partial(A \cap P(\beta, b, \alpha, a)) \\
&= \overline{(A \cap P(\beta, b, \alpha, a))} \cap \overline{(P - (A \cap P(\beta, b, \alpha, a)))} \\
&= \overline{(A \cap P(\beta, b, \alpha, a))} \cap \overline{(P - A) \cup (P - P(\beta, b, \alpha, a))} \\
&= \overline{(A \cap P(\beta, b, \alpha, a))} \cap (\overline{(P - A)} \cup \overline{(P - P(\beta, b, \alpha, a))}) \\
&\subseteq (\overline{A} \cap \overline{P(\beta, b, \alpha, a)}) \cap (\overline{(P - A)} \cup \overline{(P - P(\beta, b, \alpha, a))}) \\
&= (\overline{A} \cap \overline{P(\beta, b, \alpha, a)} \cap \overline{(P - A)}) \cup (\overline{A} \cap \overline{P(\beta, b, \alpha, a)} \cap \overline{(P - P(\beta, b, \alpha, a))}) \\
&= (\partial A \cap \overline{P(\beta, b, \alpha, a)}) \cup (\overline{A} \cap \partial P(\beta, b, \alpha, a)) \\
&= \overline{A} \cap \partial P(\beta, b, \alpha, a)
\end{aligned}
$$

since $\partial A \cap \overline{P(\beta, b, \alpha, a)} = \partial A \cap \overline{A(\beta, b, \alpha, a)} = \varnothing$. Thus, if $z_0 \in \partial A(\beta, b, \alpha, a)$ then $z_0 \in \partial P(\beta, b, \alpha, a)$ so either $\beta(z_0) = b$ or $\alpha(z_0) = a$. We want to show that z_0 is not a fixed point of T; so, suppose to the contrary that $T(z_0) = z_0$.

Case 1.1: $\beta(z_0) = b$. If $\psi(Tz_0) < d$ or $\psi(z_0) = \psi(Tz_0) \geq d$, then $\beta(Tz_0) < b$ by condition (A3). Hence we have that $Tz_0 \neq z_0$.

Case 1.2: $\alpha(z_0) = a$. If $\theta(Tz_0) > c$ or $\theta(Tz_0) = \theta(z_0) \leq c$, then $\alpha(Tz_0) > a$ by condition (A2). Hence we have that $Tz_0 \neq z_0$.

Therefore, T does not have any fixed points on $\partial A(\beta, b, \alpha, a)$.

Let $w_0 \in A(\beta, b, \alpha, a) \cap A(\theta, c, \psi, d)$ (see condition (A1)), and define $H : [0, 1] \times \overline{A(\beta, b, \alpha, a)} \to P$ by

$$H(t, x) = (1 - t)Tx + tw_0.$$

Clearly, H is continuous and $H([0, 1] \times \overline{A(\beta, b, \alpha, a)})$ is precompact.

Claim 2: $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial A(\beta, b, \alpha, a)$. Suppose not; that is, suppose there exists $(t_0, x_0) \in [0,1] \times \partial A(\beta, b, \alpha, a)$ such that $H(t_0, x_0) = x_0$. Since $x_0 \in \partial A(\beta, b, \alpha, a)$, we have that $\beta(x_0) = b$ or $\alpha(x_0) = a$ since $\partial A \cap \overline{A(\beta, b, \alpha, a)} = \varnothing$. Also, since T has no fixed point on $\partial A(\beta, b, \alpha, a)$, we have that $t_0 \neq 0$.

Case 2.1: $\beta(x_0) = b$. Either $\psi(Tx_0) < d$ or $\psi(Tx_0) \geq d$.

Subcase 2.1.1: $\psi(Tx_0) < d$. By condition (A3) we have $\beta(Tx_0) < b$, thus it follows that

$$b = \beta(x_0) = \beta((1 - t_0)Tx_0 + t_0w_0) \leq (1 - t_0)\beta(Tx_0) + t_0\beta(w_0) < b,$$

which is a contradiction.

Subcase 2.1.2: $\psi(Tx_0) \geq d$. We have that $\psi(x_0) \geq d$ since

$$\psi(x_0) = \psi((1 - t_0)Tx_0 + t_0w_0) \geq (1 - t_0)\psi(Tx_0) + t_0\psi(w_0) \geq d,$$

and thus by condition (A3) we have $\beta(Tx_0) < b$, which is the same contradiction we arrived at in the previous subcase.

Case 2.2: $\alpha(x_0) = a$. Either $\theta(Tx_0) \leq c$ or $\theta(Tx_0) > c$.

Subcase 2.2.1: $\theta(Tx_0) > c$. By condition (A2) we have $\alpha(Tx_0) > a$, thus we have

$$a = \alpha(x_0) = \alpha((1 - t_0)Tx_0 + t_0w_0) \geq (1 - t_0)\alpha(Tx_0) + t_0\alpha(w_0) > a,$$

which is a contradiction.

Subase 2.2.2: $\theta(Tx_0) \leq c$. We have that $\theta(x_0) \leq c$ since

$$\theta(x_0) = \theta((1 - t_0)Tx_0 + t_0w_0) \leq (1 - t_0)\theta(Tx_0) + t_0\theta(w_0) \leq c,$$

and thus by condition (A2) we have $\alpha(Tx_0) > a$, which is the same contradiction we arrived at in the previous case.

Therefore, we have shown that $H(t,x) \neq x$ for all $(t,x) \in [0,1] \times \partial A(\beta, b, \alpha, a)$, and thus by the homotopy invariance property of the fixed point index

$$i(T, A(\beta, b, \alpha, a), P) = i(w_0, A(\beta, b, \alpha, a), P),$$

and by the normality property of the fixed point index

$$i(T, A(\beta, b, \alpha, a), P) = i(w_0, A(\beta, b, \alpha, a), P) = 1.$$

Consequently by the solution property of the fixed point index, T has a fixed point $x^* \in A(\beta, b, \alpha, a)$.

Furthermore, if for all $x \in A(\beta, b, \alpha, a)$ there exists a $k \in [0, 1)$ such that $\|Tx - x^*\| \leq k\|x - x^*\|$, then for any fixed point $z^* \in A(\beta, b, \alpha, a)$ we have that $\|z^* - x^*\| = \|Tz^* - x^*\| \leq k\|z^* - x^*\|.$. Therefore $\|z^* - x^*\| = 0$ as $k < 1$, and we have verified that under this condition T has a unique fixed point in $A(\beta, b, \alpha, a)$. □

4. Application

In this section, using a non-standard functional technique (using an evaluation of the derivative as one of the functionals) and an open subset A of the cone P (the set A is not bounded nor is it a cone), we will illustrate the key techniques for verifying the existence and uniqueness of a positive solution for a right focal boundary value problem in our interval of functional type using our main result. Note that the resulting conditions for a fixed point to exist in our functional-type interval (conditions (a) and (b) of Theorem 4.1 below) do not force the operator T to be invariant on our functional-type interval. We consider the classical right focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \tag{1}$$

$$x(0) = 0 = x'(1), \tag{2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. It is well known that if x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t, s) f(x(s)) ds,$$

where

$$G(t, s) = \min\{t, s\}, \quad (t, s) \in [0, 1] \times [0, 1],$$

then x is a solution of the boundary value problem (1), (2).

Define the cone $P \subset E = C^1[0, 1]$ by

$$P := \{x \in E : x \text{ is nonnegative, nondecreasing, concave, and } x(0) = 0\}.$$

For $x \in P$, define the convex functional β on P by

$$\beta(x) := \max_{t \in [0,1]} x(t) = x(1),$$

the (concave) functional ψ on P by

$$\psi(x) := x'\left(\frac{1}{4}\right),$$

and the concave functional α on P by

$$\alpha(x) := \min_{t \in [\frac{1}{4}, 1]} x(t) = x\left(\frac{1}{4}\right).$$

We are now ready to prove the existence of a unique positive solution to (1), (2) in our functional-type interval, if the conditions in the following theorem are satisfied.

Theorem 4.1. *If $b > 0$, and $f : [0, 20b] \to [0, \infty)$ is a continuous differentiable function such that*

(a) $\frac{256b}{11} < f(w) < 80b$ *for $w \in [0, 8b]$,*

(b) $\frac{256b}{11} < f(w) < \frac{368b}{11}$ *for $w \in [8b, 20b]$,*

(c) $|f'(w)| < 16$ *for $w \in [0, 5b]$, and*

(d) $|f'(w)| < 1$ *for $w \in [5b, 20b]$,*

then the right focal problem (1), (2) has a unique positive solution $x^* \in A(\beta, 20b, \alpha, 5b)$.

Proof. Let

$$A = \left\{ x \in P : x\left(\frac{3}{8}\right) - x\left(\frac{1}{4}\right) > 2b \right\},$$

$$A(\beta, 20b, \alpha, 5b) = \left\{ x \in A : 5b < x(t) < 20b \text{ for all } \frac{1}{4} \leq t \leq 1 \right\}.$$

Thus A is an open subset of P and $A(\beta, 20b, \alpha, 5b)$ is an open, nonempty, bounded, subset of P. By properties of Green's function we have

$$(Tx)''(t) = -f(x(t)) \text{ and } Tx(0) = 0 = (Tx)'(1);$$

that is, fixed points of the operator T are solutions of the boundary value problem (1), (2). Applying the Arzela-Ascoli Theorem we have $T : \overline{A(\beta, 20b, \alpha, 5b)} \to P$ is a completely continuous operator, and applying Dugunji's Theorem, there is a continuous extension, which we will again denote by T, such that $T : P \to P$ (the extension is necessary to extend the domain of T from $\overline{A(\beta, 20b, \alpha, 5b)}$ to P since f is only defined on $[0, 20b]$).

Letting $x \in \overline{A(\beta, 20b, \alpha, 5b)}$ we see that $\alpha(x) = x(\frac{1}{4}) \geq 5b$; therefore

$$(Tx)\left(\frac{3}{8}\right) - (Tx)\left(\frac{1}{4}\right) = \int_{1/4}^{3/8} \left(s - \frac{1}{4}\right) f(x(s)) \, ds + \left(\frac{1}{8}\right) \int_{3/8}^{1} f(x(s)) \, ds$$

$$> \int_{1/4}^{3/8} \left(s - \frac{1}{4}\right) \frac{256b}{11} \, ds + \left(\frac{1}{8}\right) \int_{3/8}^{1} \frac{256b}{11} \, ds = 2b.$$

Hence, $x \notin \partial A$ thus $\partial A \cap \overline{A(\beta, 20b, \alpha, 5b)} = \emptyset$. Let

$$w_0(t) = \int_0^1 G(t, s) 36b \, ds.$$

thus

$$\beta(w_0) = \theta(w_0) = w_0(1) = \int_0^1 G(1,s)36b\,ds = 18b < 20b,$$

$$\alpha(w_0) = w_0\left(\frac{1}{4}\right) = \int_0^1 G\left(\frac{1}{4},s\right)36b\,ds = \frac{63b}{8} > 5b,$$

$$\psi(w_0) = w_0'\left(\frac{1}{4}\right) = \int_{1/4}^1 35b\,ds = \frac{105b}{4} > 24b$$

hence $w_0 \in A(\beta, 20b, \alpha, 5b) \cap A(\beta, 20b, \psi, 24b)$.

Claim 1: If $x \in \partial A(\beta, 20b, \alpha, 5b)$ with $\beta(x) = 20b$ and $\psi(x) \geq 24b$, then $\beta(Tx) < 20b$.

Since $\psi(x) \geq 24b$ and x is increasing, $x\left(\frac{1}{4}\right) \geq 6b$ since

$$6b \leq \frac{\psi(x)}{4} = \int_0^{1/4} x'\left(\frac{1}{4}\right)ds \leq \int_0^{1/4} x'(s)\,ds = x\left(\frac{1}{4}\right).$$

Also, since $x \in A$, $x\left(\frac{3}{8}\right) \geq x\left(\frac{1}{4}\right) + 2b \geq 8b$. Therefore,

$$\beta(Tx) = \int_0^1 G(1,s)f(x(s))\,ds = \int_0^1 sf(x(s))\,ds$$

$$= \int_0^{3/8} sf(x(s))\,ds + \int_{3/8}^1 sf(x(s))\,ds$$

$$< \int_0^{3/8} 80bs\,ds + \int_{3/8}^1 \frac{368bs}{11}\,ds$$

$$= 80b\left(\frac{9}{128}\right) + \left(\frac{368b}{11}\right)\left(\frac{55}{128}\right) = 20b.$$

Claim 2: If $x \in \partial A(\beta, 20b, \alpha, 5b)$ with $\beta(x) = 20b$ and $\psi(Tx) < 24b$, then $\beta(Tx) < 20b$.

Let $x \in \partial A(\beta, 20b, \alpha, 5b)$ with $\beta(x) = 20b$ and $\psi(Tx) < 24b$. Then

$$24b > \psi(Tx) = (Tx)'\left(\frac{1}{4}\right)$$

$$= \int_{1/4}^1 f(x(s))\,ds$$

$$= \int_{1/4}^1 sf(x(s))\,ds + \int_{1/4}^1 (1-s)f(x(s))\,ds,$$

and since we have that

$$\int_{1/4}^1 (1-s)f(x(s))\,ds \geq \int_{1/4}^1 \frac{256b(1-s)}{11}\,ds > \frac{13b}{2},$$

we also have that

$$\int_{1/4}^1 sf(x(s))\,ds < \frac{35b}{2}.$$

Thus,

$$
\begin{aligned}
\beta(Tx) &= \int_0^1 G(1,s)f(x(s))\,ds \\
&= \int_0^{1/4} sf(x(s))\,ds + \int_{1/4}^1 sf(x(s))\,ds \\
&< \int_0^{1/4} 80bs\,ds + \frac{35b}{2} \\
&= \frac{5b}{2} + \frac{35b}{2} = 20b.
\end{aligned}
$$

Hence, the previous two claims have verified condition (A3) of Theorem 3.1 is satisfied.

Claim 3: If $x \in \partial A(\beta, 20b, \alpha, 5b)$ with $\alpha(x) = 5b$, then $\alpha(Tx) > 5b$.

Let $x \in \partial A(\beta, 20b, \alpha, 5b)$ with $\alpha(x) = 5b$. Then

$$
\begin{aligned}
\alpha(Tx) &= Tx\left(\frac{1}{4}\right) = \int_0^1 G\left(\frac{1}{4}, s\right) f(x(s))\,ds \\
&= \int_0^{1/4} sf(x(s))\,ds + \int_{1/4}^1 \frac{f(x(s))}{4}\,ds \\
&> \int_0^{1/4} s\frac{256b}{11}\,ds + \int_{1/4}^1 \frac{256b}{44}\,ds > 5b.
\end{aligned}
$$

Therefore, condition (A2) of Theorem 3.1 is satisfied since we have shown that

$$
\text{if } x \in \partial A(\beta, 20b, \alpha, 5b) \text{ with } \alpha(x) = 5b \text{ implies } \alpha(Tx) > 5b
$$

which guarantees that

$$
\text{if } x \in \partial A(\beta, b, \alpha, a) \text{ with } \alpha(x) = a \text{ and either } \theta(x) \le c \text{ or } \theta(Tx) > c \text{ implies } \alpha(Tx) > a.
$$

Therefore, by Theorem 3.1, the operator T has a fixed point $x^* \in A(\beta, 20b, \alpha, 5b)$ which is a desired solution of (1), (2). Furthermore, if $x \in A(\beta, 20b, \alpha, 5b)$, then

$$
\begin{aligned}
\|Tx - Tx^*\| &= \max_{t \in [0,1]} \left| \int_0^1 G(t,s)(f(x(s)) - f(x^*(s)))\,ds \right| \\
&\le \max_{t \in [0,1]} \int_0^1 G(t,s)|f(x(s)) - f(x^*(s))|\,ds \\
&= \int_0^1 s|f(x(s)) - f(x^*(s))|\,ds \\
&= \int_0^{1/4} s|f(x(s)) - f(x^*(s))|\,ds + \int_{1/4}^1 s|f(x(s)) - f(x^*(s))|\,ds \\
&\le \int_0^{1/4} 16s\|x - x^*\|\,ds + \int_{1/4}^1 s\|x - x^*\|\,ds \\
&= \left(\frac{31}{32}\right) \|x - x^*\|.
\end{aligned}
$$

Thus

$$
\|Tx - x^*\| = \|Tx - Tx^*\| \le \left(\frac{31}{32}\right) \|x - x^*\|,
$$

hence x^* is the unique fixed point of T in $A(\beta, 20b, \alpha, 5b)$.   □

Acknowledgements

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