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Bessel Type Function J_β^θ, Bessel Type Operator Δ_β^θ and Fractional Fourier-Bessel Type Transform $\mathcal{F}_\beta^\theta$

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Abstract

Fractional Fourier-Bessel type transformation is defined. Then using these transformations the pseudo-differential Bessel type operators $\mathscr{B}_{\beta,\,a}^\theta$ is also defined. After that we introduce some class of symbols, Sobolev and Bessel type potentials spaces. Properties of these transformations and operators are investigated.

Keywords: Fractional Fourier transform; Bessel potential spaces; Sobolev type spaces; pseudo-differential operators; Poly-axially operators.

2020 Mathematics Subject Classification: 35S05, 46E35, 46F12.

1. Introduction and Motivation

The term "pseudo-differential operators" [1–4] has a fairly broad definition and covers such chapters as harmonic analysis, partial differential equation, computations, quantum mechanics. In mathematics, medicine, scientific computing, and engineering, natural sciences current trends and novel applications are highlighted. The emphasis is on contemporary developments in mathematics, engineering, medicine, scientific computers, and the natural sciences. In reality, Kohn-Nirenberg and Hörmander were the ones who first introduced the pseudo-differential calculus. Pseudo-differential operators on $\mathbb{R}_+$ are standard or conventional generalizations of partial differential operators or ordinary differential operators and singular integrals. Many faculties, scientists, Ph.D students and researchers of other field developed the theory of pseudo-differential operators with the help of following types of integral operators similar as Fourier transforms ([5,6]), Hankel transform ([7–9]), Fourier Bessel Transform on $\mathbb{R}_+$ ([10,11]), Weinstein transform ([12]), Laguerre hypergroups ([13]) and Jacobi differential operators ([14]). From 19th century Fourier analysis is a most frequently used tools in scientific studies/streams [15–18]. In mathematical literature, a generalized concept of the Fourier

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transform well known as the fractional Fourier transform was considered in 1980-1987, by Mcbride, Kerr and Namias [19,20]. The fractional Fourier transform (FrFT) [**?**,21,22] has been defined as follows:

$$(\mathcal{F}^\theta\varphi)(\xi) = \widehat{\varphi}^\theta(\xi) = \int_{\mathbb{R}} K^\theta(x,\xi)\varphi(x)dx \tag{1}$$

$$K^\theta(x,\xi) = \begin{cases} C^\theta e^{\frac{i(x^2+\xi^2)\cot\theta}{2} - ix\xi\csc\theta}, & \theta \neq n\pi, n\epsilon\mathbb{Z} \\ \frac{1}{\sqrt{2\pi}}e^{-ix\xi}, & \theta = \frac{\pi}{2} \\ \delta(x-\xi), & \theta = 2n\pi \\ \delta(x+\xi), & \theta = (2n+1)\pi, \end{cases}$$

Where $C^\theta = \sqrt{\frac{1-i\cot\theta}{2\pi}}$. In the present manuscript, we consider first order Bessel operator Δ_β with the theory of Bessel potentials which is mentioned in [24], defined for $x \in \mathbb{R}_+,\ \beta \in \mathbb{R}$ such that $\beta > -\frac{1}{2}$ by

$$\Delta_\beta = \frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}.$$

Now using the Kernel of (1) we define the Bessel type function J_β^θ as follows: for $\xi \in \mathbb{R}_+$, $\beta > -\frac{1}{2}$,

$$J_\beta^\theta(\xi) = \frac{\Gamma(\beta+1)}{\sqrt{\pi}\Gamma(\beta+\frac{1}{2})}\int_{-1}^{1}(1-x^2)^{\beta-\frac{1}{2}}K^\theta(x,\xi)dx. \tag{2}$$

In this manuscript, we introduce first order Bessel type operator Δ_β^θ, defined for $x \in \mathbb{R}_+$, $\beta > -\frac{1}{2}$ by

$$\Delta_\beta^\theta = \frac{d^2}{dx^2} + \Big(\frac{2\beta+1}{x} + 2ix\cot\theta\Big)\frac{d}{dx} + \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta. \tag{3}$$

If $\theta = \frac{\pi}{2}$, we get $\Delta_\beta^\theta = \Delta_\beta$. Firstly, we define fractional Fourier-Bessel type transformation $\mathcal{F}_\beta^\theta$ with the help of (2) as follows: for any $\varphi \in \mathscr{S}_e'(\mathbb{R})$

$$(\mathcal{F}_\beta^\theta\varphi)(\xi) = \int_{\mathbb{R}_+}\varphi(x)J_\beta^\theta(\xi)d\mu_\beta(x),\ \ \forall\ \ \xi \in \mathbb{R}_+, \tag{4}$$

$d\mu_\beta(x) = x^{2\beta+1}dx$. Now we also introduce the definition of the Sobolev type spaces ${}^\theta\mathbb{H}_\beta^s(\mathbb{R}_+)$ as the set of all $\varphi \in \mathscr{S}_e'(\mathbb{R})$ such that

$$\left(\int_{\mathbb{R}_+}(1+|\xi|)^{2s}\left|(\mathcal{F}_\beta^\theta\varphi)(\xi)\right|^2 d\mu_\beta(\xi)\right)^{\frac{1}{2}} < \infty. \tag{5}$$

In this article we consider a set of symbols, denoted by Λ [25] which will be used in the upcoming sections. Next, we define pseudo-differential Bessel type operators $\mathscr{B}_{\beta,\ a}^\theta$ on $\mathscr{S}_e(\mathbb{R})$ associated with the set Λ as follows:

$$(\mathscr{B}_{\beta,a}^\theta\varphi)(\xi) = \int_{\mathbb{R}_+} J_\beta^\theta(\xi)a(x,\xi)(\mathcal{F}_\beta^\theta\varphi)(\xi)d\mu_\beta(x), \tag{6}$$

where $a(x,\xi)$ is the corresponding to the operator $\mathscr{B}_{\beta,a}^\theta$.

The current manuscript is was primarily inspired/motivated by the works of [10,11,26,27]

2. Definitions and Notes for Preliminary Terms

We start out by making some notations on the useful spaces that we will require for this article.

- $\mathbb{R}_+ = \{z \in \mathbb{R},\ z > 0\}$.
- $\mathscr{D}_e(\mathbb{R})$ is the space of all even C^∞-functions with compact support.
- $\mathscr{C}_e^\infty(\mathbb{R})$ is the space of all even $C^\infty-$functions on $\mathbb{R}$ and $\mathscr{C}_{e,0}(\mathbb{R}) = \Big\{h : h$ is an even continuousfunction defined on $\mathbb{R}$ such that $h(x) \to 0$ *as* $||x|| \to +\infty$ and $||h||_{\mathscr{C}_{e,0}} = \sup\{h(x) : x \in \mathbb{R}\} < +\infty\Big\}$.
- The Schwartz space $\mathscr{S}_e(R)$ consists of all even C^∞-functions on $\mathbb{R}$.
- $\mathscr{S}'_e(R)$ is the space of even tempered distributions on $\mathbb{R}$.
- $L_\beta^p(\mathbb{R}_+) = \Big\{h : \mathbb{R} \to \mathbb{C}\ \ such\ that\ \ ||h||_{L_\beta^p}^p = \int_{\mathbb{R}_+} |h(x)|^p d\mu_\beta(x) < +\infty,\ \ d\mu_\beta(x) = x^{2\beta+1}dx,\ \ \beta > -\frac{1}{2}\ and\ 1 \le p < \infty\Big\}$.
- $L_\beta^\infty(\mathbb{R}) = \Big\{h : \mathbb{R} \to \mathbb{C}\ \ such\ that\ \ ||h||_{L_\beta^\infty} = ess\ sup h(x)_{x\in\mathbb{R}} < +\infty\Big\}$.

Introducing the mono-axially operator Δ_β for $\beta > -\frac{1}{2}$ as follows

$$\Delta_\beta = \frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}.$$

3. The Fractional Fourier-Bessel Type Transform

Definition 3.1. *The fractional Fourier-Bessel type transform of $\varphi \in L_\beta([0,\ \infty))$ is the function $\mathcal{F}_\beta^\theta$ with the help of (2), defined as follows:*

$$(\mathcal{F}_\beta^\theta \varphi)(\xi) = \int_{\mathbb{R}_+} \varphi(x) J_\beta^\theta(\xi) d\mu_\beta(x), \qquad \forall \xi \in \mathbb{R}_+, \tag{7}$$

$d\mu_\beta(x) = x^{2\beta+1}dx.$

Example 3.2. *The mapping $\varphi(x) = e^{-x^2}$ belongs to $L_\beta([0,\ \infty))$ and we have*

$$(\mathcal{F}_\beta^\theta \varphi)(\xi) = \frac{1}{2}\Gamma(\beta+1) J_\beta^\theta(\xi),\ \ \forall\ \ \xi \geqq 0.$$

Example 3.3. *The function $\varphi(x) = e^{-x} \in L_\beta([0,\ \infty))$ and we get*

$$(\mathcal{F}_\beta^\theta \varphi)(\xi) = \Gamma(2\beta+2) J_\beta^\theta(\xi),\ \ \forall \xi \geqq 0.$$

Theorem 3.4. *For $\beta > -\frac{1}{2}$ and $\xi > 0$. The Kernel $K^\theta(x,\xi)$ satisfies the following equation*

$$\Delta_\beta^\theta K^\theta(x,\xi) = -\xi^2 \csc^2\theta K^\theta(x,\xi), \tag{8}$$

where

$$\Delta_\beta^\theta = \frac{d^2}{dx^2} + \Big(\frac{2\beta+1}{x} + 2ix\cot\theta\Big)\frac{d}{dx} + \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta.$$

Proof. We have

$$\begin{aligned} \frac{d(K^\theta(x,\xi))}{dx} &= \frac{d}{dx}\left(C^\theta e^{\frac{i(x^2+\xi^2)\cot\theta}{2} - ix\xi\csc\theta}\right) \\ &= K^\theta(x,\xi)i(x\cot\theta - \xi\csc\alpha). \end{aligned} \tag{9}$$

Similarly, we get

$$\begin{aligned} \frac{d^2(K^\theta(x,\xi))}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}K^\theta(x,\xi)\right) \\ &= -(x\cot\theta - \xi\csc\theta)^2 K^\theta(x,\xi) + i\cot\theta K^\theta(x,\xi). \end{aligned} \tag{10}$$

Obtain

$$\begin{aligned} \Delta_\beta^\theta K^\theta(x,\xi) &= \frac{d^2(K^\theta(x,\xi))}{dx^2} + \left(\frac{2\beta+1}{x} + 2ix\cot\theta\right)\frac{d(K^\theta(x,\xi))}{dx} \\ &+ \left(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\right)\cot\theta K^\theta(x,\xi). \end{aligned} \tag{11}$$

Using (9) and (10) in (11), we obtain

$$\Delta_\beta^\theta K^\theta(x,\xi) = -\xi^2\csc^2\theta K^\theta(x,\xi).$$

□

Remark 3.5. *By the Principle of Mathematical Induction, we get*

$$(\Delta_\beta^\theta)^l K^\theta(x,\xi) = (-1)^l(\xi\csc\theta)^{2l} K^\theta(x,\xi), \quad \forall \ l \in \mathbb{N}.$$

The transform $\mathcal{F}_\beta^\theta : S_{e,\ \beta}^2(\mathbb{R}_+) \to S_{e,\ \beta}(\mathbb{R}_+)$ is continuous and linear.

Proof. We assume $f \in S^2_{e,\,\beta}(\mathbb{R}_+)$. We have

$$\begin{aligned}\Delta^\theta_\beta &= \frac{d^2}{dx^2} + \Big(\frac{2\beta+1}{x} + 2ix\cot\theta\Big)\frac{d}{dx} + \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta \\ &= \frac{d^2}{dx^2} + c(x)\frac{d}{dx} + d(x),\end{aligned}$$

where, $c(x) = \frac{2\beta+1}{x} + 2ix\cot\theta$ and $d(x) = \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta$. Applying Leibniz's Theorem of successive differentiation of pointwise multiplication of functions, the differential operator $(\Delta^\theta_\beta)^p$ for any $p = 0,\ 1,\ 2,\ 3,\ 4, \ldots$ we obtain

$$(\Delta^\theta_\beta)^p(f) = \sum_{k=0}^{2p} e^\theta_k(x)\frac{d^k}{dx^k}(f),$$

where, $e^\theta_{2p}(x) = 1$ and remaining $e^\theta_k(x)$ are functions of $\frac{2\beta+1}{x} + 2ix\cot\theta$ or $\Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta$ and its derivatives, x and $\cot\theta$, $\csc\theta$, $\sec\theta$. Therefore, the result is got by using the fact that for all $p \geq 0$, there exist $h_p > 0$ and $j_p > 0$, R_p such that

$$\left|\frac{d^p}{dx^p}\left(\frac{2\beta+1}{x} + 2ix\cot\theta\right)\right| \leq h_p, \quad x \geq R_p,$$

and

$$\left|\frac{d^p}{dx^p}\left(\Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta\right)\right| \leq j_p, \quad x \geq R_p$$

and $\mathcal{F}^\theta_\beta$ is an isomorphism from $S^2_{e,\,\beta}(\mathbb{R}_+)$ onto $S_{e,\,\beta}(\mathbb{R}_+)$. Proof of the Theorem ? is completed. □

Theorem 3.6. *For $\xi > 0$, the function $h(x) = \Delta^\theta_\beta K^\theta(x,\xi)$ is a regular generalized function.*

Proof. Let $f \in S^2_e(\mathbb{R}_+)$. We get

$$\begin{aligned}\langle h, f\rangle = \langle \Delta^\theta_\beta K^\theta(x,\xi), f\rangle &= \int_{\mathbb{R}} \Delta^\theta_\beta K^\theta(x,\xi) f(x) x^{2\beta+1} dx \\ &= \int_{\mathbb{R}} -\xi^2\csc^2\theta K^\theta(x,\xi) f(x) x^{2\beta+1} dx\end{aligned}$$

Thus,

$$\begin{aligned}|\langle h, f\rangle| = |\langle \Delta^\theta_\beta K^\theta(x,\xi), f\rangle| &\leq |\xi^2\csc^2\theta| Q_{m,0}(f) \int_{\mathbb{R}} (1+x^2)^{-m} |K^\theta(x,\xi)||x^{2\beta+1}| dx \\ &= |\xi^2\csc^2\theta| Q_{m,0}(f) ||(1+x^2)^{-m} K^\theta(x,\xi) x^{2\beta+1}||^1_{L^1_\beta}.\end{aligned}$$

Proof of the Theorem 3.6 is completed. □

Theorem 3.7. *Let $f \in S^2_e(\mathbb{R}_+)$. We have*

$$\langle \Delta^\theta_\beta K^\theta(x,\xi), f\rangle = \langle K^\theta(x,\xi), (\Delta^\theta_\beta)^* f\rangle, \qquad \text{for all } \xi > 0 \tag{12}$$

where

$$(\Delta^\theta_\beta)^* = \frac{d^2}{dx^2} + \Big(\frac{2\beta+1}{x} - 2ix\cot\theta\Big)\frac{d}{dx} + \Big(3x^2\cot\theta - 6(\beta+1)i + \frac{2\beta+1}{x}i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta. \tag{13}$$

Proof. We get firstly

$$\begin{aligned}
\langle \Delta^\theta_\beta K^\theta(x,\xi), f\rangle &= \int_{\mathbb{R}_+} \Delta^\theta_\beta K^\theta(x,\xi) f(x) x^{2\beta+1} dx \\
&= \int_{\mathbb{R}_+} \Big[\frac{d^2}{dx^2} + \Big(\frac{2\beta+1}{x} + 2ix\cot\theta\Big)\frac{d}{dx} \\
&+ \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x} i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta\Big] \\
&\times K^\theta(x,\xi) f(x) x^{2\beta+1} dx \\
&= \int_{\mathbb{R}_+} \Big(\frac{d^2}{dx^2}K^\theta(x,\xi) + \frac{2\beta+1}{x}\frac{d}{dx}K^\theta(x,\xi)\Big) f(x) x^{2\beta+1} dx \\
&+ \int_{\mathbb{R}_+} 2ix\cot\theta \frac{d}{dx} K^\theta(x,\xi) f(x) x^{2\beta+1} dx \\
&+ \int_{\mathbb{R}_+} \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x} i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta K^\theta(x,\xi) f(x) x^{2\beta+1} dx \\
&= I^\theta_1 + I^\theta_2 + I^\theta_3 \qquad (say),
\end{aligned}$$

where

$$\begin{aligned}
I^\theta_1 &= \int_{\mathbb{R}_+} \Big(\frac{d^2}{dx^2}K^\theta(x,\xi) + \frac{2\beta+1}{x}\frac{d}{dx}K^\theta(x,\xi)\Big) f(x) x^{2\beta+1} dx \\
&= \int_{\mathbb{R}_+} \frac{1}{x^{2\beta+1}}\frac{d}{dx}\Big\{x^{2\beta+1}\frac{d}{dx}K^\theta(x,\xi)\Big\} f(x) x^{2\beta+1} dx, \\
I^\theta_2 &= \int_{\mathbb{R}_+} 2ix\cot\theta \frac{d}{dx}\big(K^\theta(x,\xi)\big) f(x) x^{2\beta+1} dx
\end{aligned}$$

and

$$I^\theta_3 = \int_{\mathbb{R}_+} \Big(3x^2\cot\theta - 2(\beta+1)i + \frac{2\beta+1}{x} i\xi\sec\theta - 4x\xi\csc\theta\Big)\cot\theta K^\theta(x,\xi) f(x) x^{2\beta+1} dx.$$

From the fact that $x^{2\beta+1} \simeq e^{2\beta x}, \quad x \to \infty$. Applying integration by parts, we obtain

$$I^\theta_1 = -\int_{\mathbb{R}_+} \frac{d}{dx}\big(K^\theta(x,\xi)\big) x^{2\beta+1} \frac{df(x)}{dx} dx$$

Again applying integration by parts, we obtain

$$\begin{aligned} I_1^{\theta} &= \int_{\mathbb{R}_+}\left((2\beta+1)x^{\beta}\frac{df(x)}{dx}+x^{2\beta+1}\frac{d^2f(x)}{dx^2}\right)K^{\theta}(x,\xi)dx \\ &= \int_{\mathbb{R}_+}K^{\theta}(x,\xi)\frac{d^2f(x)}{dx^2}x^{2\beta+1}dx+\int_{\mathbb{R}_+}K^{\theta}(x,\xi)(2\beta+1)\frac{x^{2\beta}}{x^{2\beta+1}}\left(\frac{df(x)}{dx}\right)x^{2\beta+1}dx. \end{aligned}$$

By the same process, we obtain

$$I_2^{\theta}=-4i(\beta+1)\cot\theta\int_{\mathbb{R}_+}x^{2\beta+1}f(x)K^{\theta}(x,\xi)dx-2i\cot\theta\int_{\mathbb{R}_+}x^{2\beta+1}\frac{df(x)}{dx}K^{\theta}(x,\xi)dx.$$

Therefore,

$$\begin{aligned} I_1^{\theta}+I_2^{\theta}+I_3^{\theta} &= \int_{\mathbb{R}_+}\left((2\beta+1)x^{\beta}\frac{df(x)}{dx}+x^{2\beta+1}\frac{d^2f(x)}{dx^2}\right)K^{\theta}(x,\xi)dx \\ &+ \int_{\mathbb{R}_+}K^{\theta}(x,\xi)\frac{d^2f(x)}{dx^2}x^{2\beta+1}dx+\int_{\mathbb{R}_+}K^{\theta}(x,\xi)(2\beta+1)\frac{x^{2\beta}}{x^{2\beta+1}}\left(\frac{df(x)}{dx}\right)x^{2\beta+1}dx \\ &- 4i(\beta+1)\cot\theta\int_{\mathbb{R}_+}x^{2\beta+1}f(x)K^{\theta}(x,\xi)dx-2i\cot\theta\int_{\mathbb{R}_+}x^{2\beta+1}\frac{df(x)}{dx}K^{\theta}(x,\xi)dx \\ &+ \int_{\mathbb{R}_+}\left(3x^2\cot\theta-2(\beta+1)i+\frac{2\beta+1}{x}i\xi\sec\theta-4x\xi\csc\theta\right)\cot\theta K^{\theta}(x,\xi)f(x)x^{2\beta+1}dx \\ &= \int_{\mathbb{R}_+}K^{\theta}(x,\xi)\Bigg[\Bigg\{\frac{d^2}{dx^2}+\left(\frac{2\beta+1}{x}-2ix\cot\theta\right)\frac{d}{dx} \\ &+ \left(3x^2\cot\theta-6(\beta+1)i+\frac{2\beta+1}{x}i\xi\sec\theta-4x\xi\csc\theta\right)\cot\theta\Bigg\}f(x)\Bigg]x^{2\beta+1}dx \\ &= \int_{\mathbb{R}_+}K^{\theta}(x,\xi)(\Delta_{\beta}^{\theta})^{*}f(x)x^{2\beta+1}dx \\ &= \langle K^{\theta}(x,\xi),(\Delta_{\beta}^{\theta})^{*}f\rangle. \end{aligned}$$

This completes the proof. ◻

Theorem 3.8. *For $f\in S_e^2(\mathbb{R}_+)$. We get*

$$\mathcal{F}_{\beta}^{\theta}\big((\Delta_{\beta}^{\theta})^{*}f\big)(\xi)=-\xi^2\csc^2\theta\mathcal{F}_{\beta}^{\theta}(f)(\xi),\qquad \forall\xi>0.$$

Proof. Applying the Theorem 3.4 and Theorem 3.7, We have for $\xi>0$

$$\begin{aligned} \mathcal{F}_{\beta}^{\theta}\big((\Delta_{\beta}^{\theta})^{*}f\big)(\xi) &= \int_{\mathbb{R}_+}K^{\theta}(x,\xi)J_{\beta}^{\theta}(\xi)\big((\Delta_{\beta}^{\theta})^{*}f(x)\big)x^{2\beta+1}dx \\ &= -\xi^2\csc^2\theta\mathcal{F}_{\beta}^{\theta}(f)(\xi). \end{aligned}$$

Therefore,

$$\mathcal{F}_{\beta}^{\theta}\big((\Delta_{\beta}^{\theta})^{*}f\big)(\xi)=-\xi^2\csc^2\theta\mathcal{F}_{\beta}^{\theta}(f)(\xi),\qquad \forall\xi>0$$

which achieves the result. □

4. Conclusion and Discussions for Further Research

In the manuscript, we have defined the Bessel type function J_β^θ, first order Bessel type operator Δ_β^θ, fractional Fourier-Bessel type transformation $\mathcal{F}_\beta^\theta$. We have also introduced the definition of the Sobolev type spaces ${}^\theta\mathbb{H}_\beta^s(\mathbb{R}_+)$ and pseudo-differential Bessel type operators $\mathscr{B}_{\beta,\ a}^\theta$. Some properties of these transformations and operators have been investigated in this article. The future extension of domain of fractional Fourier-Bessel type transform will be $\mathscr{S}_*(\mathbb{R})$ and $\mathscr{D}_*(\mathbb{R})$, $L^2(\ [0,\ +\infty[\ )$, $L^p(\ [0,\ +\infty[\ ),\ 1 \leq p \leq 2$, $M^b(\ [0,\ +\infty[\ )$. Wavelet type transform associated with fractional Fourier-Bessel type transform will be introduced with some mathematical properties.

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