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Comparative Properties of Nano πgp-Regular and Nano πgp-Normal Spaces

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Abstract

This paper presents a comparative study of nano πgp-regular and nano πgp-normal spaces in nano topology. These separation axioms are defined using nano πgp-closed sets, extending classical notions of regularity and normality. Fundamental properties, equivalent characterizations, and relationships between the two classes are investigated. Preservation results under subspaces and various nano mappings, including nano πgp-irresolute and nano πgp-homeomorphic functions, are also established, thereby enriching the theory of generalized separation axioms in nano topological spaces.

Keywords: Nano πgp-closed sets; nano πgp-regular space; nano πgp-normal space.
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1. Introduction

Nano topology, introduced as an extension of classical topology through rough set theory, has gained significant attention due to its applicability in dealing with finite structures and approximation spaces [7, 8]. It provides a framework for studying generalized forms of open and closed sets, which has led to the development of various weakened versions of classical topological concepts. In this direction, several types of generalized closed sets have been introduced and studied, enriching the structure of nano topological spaces and offering new perspectives on continuity and separation axioms [1, 4, 14]. Separation axioms, particularly regularity and normality, play a crucial role in understanding the structural behavior of topological spaces. These concepts have been extensively explored in nano topology using different generalized closed sets and related notions [2, 3, 5]. Further contributions have been made by various authors in defining and analyzing different forms of nano regular and nano normal spaces, along with their properties and interrelationships [9, 10, 11, 12]. In particular, Rajasekaran [6] introduced the concept of nano πgp-closed sets, which generalizes several existing classes of closed sets and provides a useful tool for defining weaker separation axioms. Motivated

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by these developments, the present paper focuses on the study of nano πgp-regular and nano πgp-normal spaces. The aim is to investigate their fundamental properties, obtain characterizations, analyze their mutual relationships, and examine their behavior under subspaces and various nano mappings. This study further contributes to the ongoing development of generalized separation axioms in nano topological spaces and extends existing results in the literature [10, 11, 12, 13].

2. Preliminaries

In this section, we review the fundamental definitions and results of nano topological spaces Throughout this paper, U denotes a non-empty finite universe and R an equivalence relation on U. The pair (U, R) is called an approximation space.

Definition 2.1 ([5]). *Let $X \subseteq U$. The lower approximation, upper approximation, and boundary region of X with respect to R are defined as $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$, $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$ and $B_R(X) = \overline{R}(X) \backslash \underline{R}(X)$, where $[x]_R$ denotes the equivalence class of x under R.*

Definition 2.2 ([5]). *The collection $\tau_R(X) = \{\emptyset, U, \underline{R}(X), \overline{R}(X), B_R(X)\}$ forms a topology on U, called the nano topology induced by X. The triple $(U, \tau_R(X))$ is called a nano topological space. Elements of $\tau_R(X)$ are called nano open sets, and their complements are called nano closed sets.*

Definition 2.3 ([4]). *Let $A \subseteq U$. The nano closure of A, denoted by $Cl(A)$, is the intersection of all nano closed sets containing A. The nano interior of A, denoted by $Int(A)$, is the union of all nano open sets contained in A.*

Definition 2.4 ([5]). *A subset A of a nano topological space $U$is said to be:*

(i) nano semi-open if $A \subseteq NCl\,(NInt\,(A))$

(ii) nano pre-open if $A \subseteq N\mathrm{Int}(NCl(A))$,

The complements of these sets are called nano pre-closed and nano semi-closed sets, respectively.

Definition 2.5 ([6]). *Let $(U, \tau_R(X))$ be a nano topological space and let $A \subseteq U$. The set $A$is said to be nano πgp-closed if $NCl(A) \subseteq G$ whenever $A \subseteq G$ and G is a nano π-open set in U. The complement of a nano πgp-closed set is called a nano πgp-open set.*

Definition 2.6 ([6]). *For any subset $A \subseteq U$, the nano πgp-closure of A, denoted by $NCl_{\pi gp}(A)$, is the intersection of all nano πgp-closed sets containing A.*

Definition 2.7 ([3,4]). *Let $f : (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a function. Then f is said to be:*

1. *nano continuous if the inverse image of every nano open set in V is nano open in U,*
2. *nano open if the image of every nano open set in U is nano open in V.*
3. *nano πgp-irresolute if and only if the inverse image of every nano πgp-open set in V is nano πgp-open in U.*

4. *almost nano πgp-irresolute if and only if the inverse image of every nano regular πgp-open set in $V$is nano πgp-open in U.*

Definition 2.8. *Let $f : (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a function. A bijection $f : U \to V$is a nano πgp-homeomorphism if both f and f^{-1} preserve nano πgp-open sets under inverse images.*

3. Nano πgp-Regular Spaces

Definition 3.1. *A nano topological space (U, τ_R) is said to be nano πgp-regular if for every point $x \in U$ and for every nano πgp-closed set $F \subseteq U$ with $x \notin F$, there exist nano p-open sets G and H in U such that $x \in G, F \subseteq H$, and $G \cap H = \emptyset$.*

Theorem 3.2. *Every nano regular space is nano πgp-regular space.*

Proof. Let (U, τ_R) be a nano regular space. Take any point $x \in U$ and any nano πgp-closed set $F \subseteq U$ such that $x \notin F$. Since every nano closed set is, in particular, a nano πgp-closed set, the complement $U \backslash F$ is nano open and contains x. By nano regularity, there exist disjoint nano open sets G and H such that $x \in G$ and $F \subseteq H$. Because every nano open set is also nano p-open, G and H are nano p-open sets satisfying $x \in G, F \subseteq H$, and $G \cap H = \emptyset$. Hence, (U, τ_R) is a nano πgp-regular space. □

Remark 3.3. *The reverse implication of the above Theorem need not be true can be seen from the following example.*

For example, let $U = \{a, b, c\}$ and consider the nano topology $\tau_R = \{\emptyset, U, \{a\}, \{a, b\}\}$. In this space, nano regularity does not hold since the point c and the nano closed set $\{a\}$ cannot be separated by disjoint nano open sets. However, the family of nano πgp-closed sets is sufficiently large to ensure that, for any point and any disjoint nano πgp-closed set, there exist disjoint nano p-open sets separating them. Hence, the space is nano πgp-regular but not nano regular, showing that the converse fails.

Theorem 3.4. *Let (U, τ_R) be a nano topological space. The following conditions are equivalent:*

(i) *U is a nano πgp-regular space.*

(ii) *for each point $x \in U$ and each nano πgp-open set Acontaining x, there exists a nano p-open set G such that $x \in G \subseteq Cl(G) \subseteq A$.*

Proof. (i) ⇒ (ii): Assume that U is nano πgp-regular. Let $x \in U$ and let A be a nano πgp-open set containing x. Then $F = U \backslash A$ is a nano πgp-closed set with $x \notin F$. By nano πgp-regularity, there exist nano p-open sets G and H such that $x \in G, F \subseteq H, G \cap H = \emptyset$. Thus $G \subseteq U \backslash H \subseteq A$. Hence, $x \in G \subseteq Cl(G) \subseteq A$.

(ii) ⇒ (i): Let $x \in U$ and let F be a nano πgp-closed set such that $x \notin F$. Then $A = U \backslash F$ is nano πgp-open and contains x. By hypothesis, there exists a nano p-open set G such that $x \in G \subseteq Cl(G) \subseteq A$. Let $H = U \backslash Cl(G)$. Then H is nano p-open, $F \subseteq H$, and $G \cap H = \emptyset$. Hence, U is nano πgp-regular. □

Theorem 3.5. *Every nano p-closed subspace of a nano πgp-regular space is nano πgp-regular.*

Proof. Let (U, τ_R) be a nano πgp-regular space and let $V \subseteq U$ be nano p-closed. Let $x \in V$ and let $F \subseteq V$ be nano πgp-closed in V such that $x \notin F$. Then there exists a nano πgp-closed set $F_1 \subseteq U$ such that $F = F_1 \cap V$. Since $x \notin F$, we have $x \notin F_1$. By nano πgp-regularity of U, there exist nano p-open sets G and H in U such that $x \in G, F_1 \subseteq H, G \cap H = \emptyset$. Then $G \cap V$ and $H \cap V$ are nano p-open in V, satisfying $x \in G \cap V, F \subseteq H \cap V, (G \cap V) \cap (H \cap V) = \emptyset$. Hence, V is nano pgp-regular. □

Theorem 3.6. *Let $f : (U, t_R(X)) \longrightarrow (V, t_R(Y))$ be a bijective function. If f is nano pgp-irresolute and nano open, and if $(U, t_R(X))$ is a nano pgp-regular space, then $(V, t_R(Y))$ is also a nano pgp-regular space.*

Proof. Let $y \in V$ and let $F \subseteq V$ be a nano pgp-closed set such that $y \notin F$. Since f is bijective, there exists $x \in U$ with $f(x) = y$. As f is nano pgp-irresolute, the inverse image $f^{-1}(F)$ is a nano pgp-closed set in U, and clearly $x \notin f^{-1}(F)$. Because $(U, t_R(X))$ is nano pgp-regular, there exist nano open sets G and H in U such that $x \in G$, $f^{-1}(F) \subseteq H$, and $G \cap H = \emptyset$. Since f is nano open, $f(G)$ and $f(H)$ are nano open sets in V. Moreover, $y \in f(G)$, $F \subseteq f(H)$, and the injectivity of f implies $f(G) \cap f(H) = \emptyset$. Hence, there exist nano open sets in V separating y and F, and therefore $(V, t_R(Y))$ is a nano pgp-regular space. □

Theorem 3.7. *Let $f : (U, t_R(X)) \longrightarrow (V, t_R(Y))$ be a nano pgp-homeomorphism. If $(U, t_R(X))$ is a nano pgp-regular space, then $(V, t_R(Y))$ is also a nano pgp-regular space.*

Proof. Let $y \in V$ and let $F \subseteq V$ be a nano pgp-closed set such that $y \notin F$. Since f is a nano pgp-homeomorphism, it is bijective, nano pgp-irresolute, and nano open; hence $f^{-1}(F)$ is a nano pgp-closed set in U. Let $x \in U$ be such that $f(x) = y$. Then $x \notin f^{-1}(F)$. As $(U, t_R(X))$ is nano pgp-regular, there exist nano open sets G and H in U such that $x \in G$, $f^{-1}(F) \subseteq H$, and $G \cap H = \emptyset$. Since f is nano open, $f(G)$ and $f(H)$ are nano open in V. Further, $y \in f(G)$, $F \subseteq f(H)$, and the injectivity of f implies $f(G) \cap f(H) = \emptyset$. Hence y and F can be separated by nano open sets in V, and therefore $(V, t_R(Y))$ is a nano pgp-regular space. □

4. Nano πgp-Normal Spaces

Definition 4.1. *A nano topological space (U, τ_R) is said to be nano pgp-normal if for any two disjoint nano pgp-closed subsets A and B of U, there exist disjoint nano p-open sets G and H in U such that $A \subseteq G$ and $B \subseteq H$.*

Theorem 4.2. *Every nano space is a nano pgp-normal space.*

Proof. Every nano topological space (U, τ_R) is a nano pgp-normal space because, under the definition of nano pgp-normality, there do not exist two nonempty disjoint nano pgp-closed sets in U, so the condition is satisfied trivially, and hence every nano space is nano pgp-normal. □

Remark 4.3. *The reverse implication of the above Theorem need not be true can be seen from the following example.*

For example, In the nano topological space $U = \{a, b, c\}$ with $t_N = \{\emptyset, U, \{a\}, \{b\}, \{a, b\}\}$, any two disjoint nano pgp-closed sets can be separated by disjoint nano p-open sets, so the space is nano pgp-normal, but the topology is not trivial, showing that a nano pgp-normal space need not be a trivial nano space.

Theorem 4.4. *Let (U, τ_R) be a nano topological space. Then the following statements are equivalent:*

(i) U is a nano pgp-normal space.

(ii) For every pair of nano pgp-open subsets V and W of U such that $V \cup W = U$, there exist nano p-closed subsets A and B of U such that $A \subseteq V, B \subseteq W$, and $A \cup B = U$.

(iii) For every nano pgp-closed set A and every nano pgp-open set B with $A \subseteq B$, there exists a nano p-open set W such that $A \subseteq W \subseteq pCl(W) \subseteq B$.

(iv) For every pair of disjoint nano pgp-closed subsets A and B of U, there exists a nano p-open set W such that $A \subseteq W$ and $pCl(W) \cap B = \emptyset$.

(v) For every pair of disjoint nano pgp-closed subsets A and B of U, there exist nano p-open subsets V and W such that $A \subseteq V, B \subseteq W$, and $pCl(V) \cap pCl(W) = \emptyset$.

Proof. (i) ⇒ (ii): Let (U, τ_R) be a nano topological space. Suppose first that U is nano pgp-normal. Let V and W be nano pgp-open sets such that $V \cup W = U$. Then $U \backslash V$ and $U \backslash W$ are disjoint nano pgp-closed sets. By nano pgp-normality, there exist disjoint nano p-open sets G and H such that $U \backslash V \subseteq G$ and $U \backslash W \subseteq H$. Let $A = U \backslash G$ and $B = U \backslash H$; then A and B are nano p-closed sets with $A \subseteq V$, $B \subseteq W$, and $A \cup B = U$.

(ii) ⇒ (iii): Next, assume (ii) holds and let A be a nano pgp-closed set and B a nano pgp-open set with $A \subseteq B$. Then B and $U \backslash A$ are nano pgp-open sets whose union is U; hence there exist nano p-closed sets $C \subseteq B$ and $D \subseteq U \backslash A$ such that $C \cup D = U$. Setting $W = U \backslash D$, we obtain a nano p-open set satisfying $A \subseteq W \subseteq pCl(W) \subseteq B$.

(iii) ⇒ (iv): Now, assume (iii) and let A and B be disjoint nano pgp-closed sets. Since $A \subseteq U \backslash B$ and $U \backslash B$ is nano pgp-open, there exists a nano p-open set W such that $A \subseteq W \subseteq pCl(W) \subseteq U \backslash B$, which yields $pCl(W) \cap B = \emptyset$.

(iv) ⇒ (v): Statement (iv) implies (v) by applying the same argument symmetrically to A and B.

(v) ⇒ (i): (v) directly implies nano pgp-normality, completing the proof of equivalence of all conditions.

□

Theorem 4.5. *Every nano pgp-regular space is nano pgp-normal.*

Proof. Let U be a nano pgp-regular space. Let F_1 and F_2 be two disjoint nano pgp-closed sets in U. For each $x \in F_1$, since $x \notin F_2$, there exist nano p-open sets G_x and H_x such that $x \in G_x, F_2 \subseteq H_x, G_x \cap H_x = \emptyset$. Let $G = \bigcup_{x \in F_1} G_x, H = \bigcap_{x \in F_1} H_x$. Then G and H are nano p-open sets, $F_1 \subseteq G$, $F_2 \subseteq H$, and $G \cap H = \emptyset$. Hence, U is nano pgp-normal. □

Theorem 4.6. *A nano pgp-closed and nano p-open subspace of a nano pgp-normal space is nano pgp-normal.*

Proof. Let (U, τ_R) be a nano pgp-normal space and let $M \subseteq U$ be a nano pgp-closed and nano p-open subspace of U. Let A and B be two disjoint nano pgp-closed subsets of M. Then there exist nano pgp-closed subsets A_1 and B_1 of U such that $A = A_1 \cap M$ and $B = B_1 \cap M$. Since M is nano pgp-closed in U, the sets A_1 and B_1 are disjoint nano pgp-closed subsets of U. By the nano pgp-normality of U, there exist disjoint nano p-open subsets G and H of U such that $A_1 \subseteq G$ and $B_1 \subseteq H$. Since M is nano p-open, the intersections $G \cap M$ and $H \cap M$ are nano p-open subsets of M. Moreover, $A \subseteq G \cap M$, $B \subseteq H \cap M$, and $(G \cap M) \cap (H \cap M) = \emptyset$. Hence, the disjoint nano pgp-closed subsets A and B of M can be separated by disjoint nano p-open subsets of M. Therefore, M is a nano pgp-normal space. □

Theorem 4.7. *Let $f : (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a bijective, nano pgp-irresolute, and nano M-pre-open function. If $(U, \tau_R(X))$ is a nano pgp-normal space, then $(V, \tau_R(Y))$ is also a nano pgp-normal space.*

Proof. Let A and B be two disjoint nano pgp-closed subsets of V. Since f is nano pgp-irresolute, the inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nano pgp-closed subsets of U. As $(U, \tau_R(X))$ is nano pgp-normal, there exist disjoint nano p-open sets G and H in U such that $f^{-1}(A) \subseteq G$ and $f^{-1}(B) \subseteq H$. Since f is nano M-pre-open, the images $f(G)$ and $f(H)$ are nano M-pre-open sets in V. Moreover, because f is bijective, we have $A \subseteq f(G), B \subseteq f(H)$, and $f(G) \cap f(H) = \emptyset$. Thus, the disjoint nano pgp-closed sets A and B in V can be separated by disjoint nano M-pre-open sets. Hence, $(V, t_R(Y))$ is a nano pgp-normal space. □

Theorem 4.8. *Let $f : (U, \tau_R(X)) \to (V, \tau_R(Y))$ be a surjective, nano M-pre-open, nano pgp-irresolute, and almost nano pgp-irresolute open function. If $(U, \tau_R(X))$ is a nano pgp-normal space, then $(V, \tau_R(Y))$ is also a nano pgp-normal space.*

Proof. Let A be a nano pgp-closed subset of V and let B be a nano pgp-open subset of V such that $A \subseteq B$. Since f is nano pgp-irresolute, $f^{-1}(A)$ is a nano pgp-closed subset of U and $f^{-1}(B)$ is a nano pgp-open subset of U with $f^{-1}(A) \subseteq f^{-1}(B)$. As $(U, t_R(X))$ is nano pgp-normal, there exists a nano p-open set W in U such that $f^{-1}(A) \subseteq W \subseteq pCl(W) \subseteq f^{-1}(B)$. Since f is nano M-pre-open and surjective, $f(W)$ is a nano M-pre-open subset of V satisfying $A \subseteq f(W) \subseteq pCl(f(W)) \subseteq B$. Hence, for every nano pgp-closed set A and nano pgp-open set B with $A \subseteq B$, there exists a nano M-pre-open set separating them. Therefore, $(V, \tau_R(Y))$ is a nano pgp-normal space. □

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