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Minimal and Maximal Solution for Nonlinear Volterra Type Random Integral Equations

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Abstract: In this research paper the existence of minimal and maximal solution for Nonlinear Volterra type random integral equations is proved under some contraction, continuity and monotonicity conditions.

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1. Introduction

The study of a natural or physical phenomenon with the help of random models or equations forms an important branch of the analysis. Nonlinear Volterra type random integral equation are more important in number of physical problems, many physical phenomena in life science, engineering, and technology, biological problems, neutron transportation theory, and specially to the application of theoretical physics Chandrasekhar [8]. And such type of Volterra integral equation occurs in the mathematical description of such phenomena as a concentration highly toxic drugs in a human body Padgett W. J. and Tsokos C. P. [9], some system theory and numbers of differential system with random parameters may be reduced to the Volterra type random integral equations in Toskos [4–6]. Now consider the nonlinear Volterra type perturbed random integral equation of the type

$$x\left(t,\omega\right)=h\left(t,x\left(t,\omega\right)\right)+\int_{0}^{t}k\left(t,\tau,\omega\right)f\left(\tau,x\left(\tau,\omega\right)\right)d\tau \tag{1}$$

for every $t\geq 0$, where the random perturbed term $h\left(t,x\left(t,\omega\right)\right)$ is a mapping from $\mathbb{R}_{+}\times\mathbb{R}\rightarrow\mathbb{R}$. ω is the point of Ω and $x\left(t,\omega\right)$ is an unknown random variable for each $t\geq 0$. The kernel $k\left(t,\tau,\omega\right)$ is a mapping from $\mathbb{R}_{+}\times\mathbb{R}_{+}\times\Omega\rightarrow\mathbb{R}$ and is defined for $0\leq\tau\leq t\leq\infty$ and $\omega\in\Omega$, the kernel $k\left(t,\tau,\omega\right)$ is an essentially bounded function with respect to measure μ for every t and τ such that $0\leq\tau\leq t\leq\infty$. The function $f(\tau,x(\tau,\omega))$ is a mapping from $\mathbb{R}_{+}\times\mathbb{R}\rightarrow\mathbb{R}$ for $t\geq 0$ i.e. the function f is a scalar valued function for $\tau\geq 0$.

2. Auxiliary Results

In this section, we present here some notations, definitions and preliminary facts that will be used in the proofs of our main results.

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Definition 2.1 ([1, 3]). *Let a closed subset $\mathbb{K}$ of a Banach Space $\mathbb{X}$, then the set $\mathbb{K}$ is called a cone, if*

(1). $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$,

(2). $\delta\mathbb{K} \subset \mathbb{K}$ *for all* $\delta \in \mathbb{R}_+$,

(3). $\{-\mathbb{K}\} \cap \mathbb{K} = \{0\}$,

where the element 0 *is an zero element of a Banach space* $\mathbb{X}$.

Now we introduce an ordered relation "$\leq$"in a Banach space $\mathbb{X}$ with help of a cone $\mathbb{K}$ as defined above in a Banach space $\mathbb{X}$ as follows.

Let for any elements $x, y \in \mathbb{X}$, then we define the relation $x \leq y$ if and only if $y - x \in \mathbb{X}$.

Definition 2.2. *A cone $\mathbb{K}$ in a Banach space $\mathbb{X}$ is known as normal, if the norm $\|.\|$ is defined on $\mathbb{X}$ is an semi-monotone on a cone $\mathbb{K}$ i.e. for every $x, y \in \mathbb{K}$ then $\|x + y\| \leq \|x\| + \|y\|$.*

Definition 2.3. *A cone $\mathbb{K}$ in a Banach space $\mathbb{X}$ is known as a regular, if every increasing, bounded sequence in $\mathbb{X}$ is convergent in norm, and also similarly the cone $\mathbb{K}$ is called a fully regular, if every increasing normed bounded sequence converges in that Banach space $\mathbb{X}$.*

The all details about the different types of cones and some interesting properties of a cone are appeared in Zeidler [10] and Ladde G. S. and V. Lakshmikantham [7]. The space $C(\mathbb{R}_+, \mathbb{R})$ be the space of all real valued continuous function defined on $\mathbb{R}_+$ and the space $C(\mathbb{R}_+, \mathbb{R})$ be the Banach space with the suprimum norm which is denoted as $\|x\|$ and defined as

$$\|x\| = \sup_{t \in \mathbb{R}_+} |x(t)|$$

and also this space $C(\mathbb{R}_+, \mathbb{R})$ is a separable Banach space.

Definition 2.4. *A random operator $A(\omega) : \Omega \times X \to X$ is called a increasing operator, if $\forall\ x, y \in X$ with $x \leq y$ then $A(\omega)x \leq A(\omega)y$ for every $\omega \in \Omega$.*

Here now we introduce an ordered relation "$\leq$"on $C(\mathbb{R}_+, \mathbb{R})$ with the help of a cone $\mathbb{K}$, which is defined as follows

$$\mathbb{K} = \{x \in C(\mathbb{R}_+, \mathbb{R})\, |x(t) \geq 0,\ \forall\, t \in \mathbb{R}_+\}$$

thus we have,

$$x \leq y \Rightarrow x(t) \leq y(t) \quad \forall \quad t \in \mathbb{R}_+$$

It is known as the cone $\mathbb{K}$ is a normal in $C(\mathbb{R}_+, \mathbb{R})$. For any functions $p, q : \Omega \to C(\mathbb{R}_+, \mathbb{R})$ we define a random interval $[p, q]$ in $C(\mathbb{R}_+, \mathbb{R})$ by

$$[p, q] = \{x \in C(\mathbb{R}_+, \mathbb{R}) \setminus p(\omega) \leq x \leq q(\omega)\,, \quad \forall \quad t \in \mathbb{R}_+$$

$$[p, q] = \bigcap_{\omega \in \Omega} [p(\omega), q(\omega)]$$

Definition 2.5. *A random solution $x_M(t, \omega)$ of the nonlinear Volterra type perturbed random integral equation (1) is called a maximal random solution, if for all random solutions to the nonlinear Volterra type perturbed random integral equation (1), one has $x(t, \omega) \leq x_M(t, \omega)$ for all $\omega \in \Omega$, and $\forall\ t \in \mathbb{R}_+$. Similarly a random solution $x_m(t, \omega)$ of the Volterra type perturbed nonlinear random integral equation (1) is called a minimal random solution, if for all random solutions to the nonlinear Volterra type perturbed random integral equation (1), one has $x(t, \omega) \geq x_m(t, \omega)$ for all $\omega \in \Omega$, and $\forall\ \ t \in \mathbb{R}_+$, on $\mathbb{R}_+$ is defined.*

We use the random fixed point theorem of Dhage B. C. [2] which is as follows.

Theorem 2.6. *Let $A(\omega), B(\omega) : \Omega \times X \to X$ be two increasing random operators such that for each $\omega \in \Omega$*

(1). $A(\omega)$ is a contraction operator.

(2). $B(\omega)$ is completely continuous operator.

(3). There exist two measurable functions $a, b : \Omega \to X$ such that $a \leq b$ on Ω, satisfies $a(\omega) \leq A(\omega)a + B(\omega)a$ and $b(\omega) \leq A(\omega)b + B(\omega)b$.

Further the cone $\mathbb{K}$ in a Banach space $\mathbb{X}$ is normal, then the random equation $A(\omega)x(t) + B(\omega)x(t) = x(t,\omega)$ has a minimal random solution $x_m(t,\omega)$ in $[a,b]$, moreover $x_m(\omega) = \lim_{n\to\infty} x_n(\omega)$ and $x_M(\omega) = \lim_{n\to\infty} y_n(\omega)$ for all $t \in \mathbb{R}_+$, and for all $\omega \in \Omega$, where the random sequence $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given as,

$$x_{n+1}(\omega) = A(\omega)x_n + B(\omega)x_n \quad for \quad n \geq 0 \quad and \quad x_0 = a$$

and,

$$y_{n+1}(\omega) = A(\omega)y_n + B(\omega)y_n \quad for \quad n \geq 0 \quad and \quad y_0 = b$$

Definition 2.7. *A measurable random function $a : \Omega \to C(\mathbb{R}_+, \mathbb{R})$ is called a lower random solution for the nonlinear Volterra type perturbed random integral equation* (**??**) *if*

$$a(t,\omega) \leq h(t, a(t,\omega)) + \int_0^t k(t,\tau,\omega) f(\tau, a(\tau,\omega)) d\tau$$

for every $\omega \in \Omega$. Similarly a measurable random function $b : \Omega \to C(\mathbb{R}_+, \mathbb{R})$ is called a upper random solution for the nonlinear Volterra type perturbed random integral equation (1) *if*

$$b(t,\omega) \geq h(t, b(t,\omega)) + \int_0^t k(t,\tau,\omega) f(\tau, b(\tau,\omega)) d\tau$$

for every $\omega \in \Omega$.

Definition 2.8 ([3])**.** *A Caratheodory function $\beta : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be random caratheodory function, if*

(1). The map $t \to \beta(t,x)$ is jointly measurable for all $x \in \mathbb{R}$.

(2). $x \to \beta(t,x)$ is continuous for almost every $t \in \mathbb{R}_+$.

Definition 2.9 ([3])**.** *A Caratheodory function $\beta : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is said to be random L^1 caratheodory function if*

(3). For each real number $r > 0$, there is measurable and bounded function $q_r \in L^1(\mathbb{R}_+, \mathbb{R})$ such that $|\beta(t, x(t,\omega))| \leq q_r(t,\omega)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$ with $|x| \leq r$.

Definition 2.10 ([3])**.** *An operator A on a Banach space X into itself is called Compact, if for any bounded subset S of X, $A(S)$ is a relatively compact subset of X. If A is continuous and compact, then it is called completely continuous on X.*

Theorem 2.11 (Arzela-Ascoli theorem)**.** *If every uniformly bounded and equi-continuous sequence $\{f_n\}$ of functions in $C(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.*

Remark 2.12. *A measurable random function $\xi : \Omega \to C(\mathbb{R}_+, \mathbb{R})$ is a random solution to the nonlinear Volterra type perturbed random integral equation* (1)*, if it is a lower as well as upper random solution of a corresponding nonlinear Volterra type perturbed random integral equation* (1) *on $\mathbb{R}_+$.*

3. Existence Result

Let $\mathbb{R}$ be the real line and Let $B(\mathbb{R}_+, \mathbb{R})$ and $M(\mathbb{R}_+, \mathbb{R})$ denotes the space of all bounded and measurable real valued functions on $\mathbb{R}_+$ respectively. Now we suppose the following hypothesis,

(B_0) $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

(B_1) For each $\omega \in \Omega$, the function $x \to h(t, \omega)$ in increasing almost every for each $t \in \mathbb{R}_+$,

(B_2) For each $\omega \in \Omega$, the function $x \to f(t, x, \omega)$ in increasing every for each $t \in \mathbb{R}_+$, and $x \in \mathbb{R}$,

(B_3) The Volterra type nonlinear random integral equation (1) has a lower random solution a and upper random solution b with $a \leq b$ on $\mathbb{R}_+$,

(B_4) The function $q : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is defined by

$$q(t, \omega) = |f(t, a(t, \omega), \omega)| + |f(t, b(t, \omega), \omega)|$$

is measurable and Lebesgue integrable on $\mathbb{R}_+$ and for all $\omega \in \Omega$. Furthermore, for every $\omega \in \Omega$,

$$|f(t, x(t, \omega), \omega)| \leq q(t, \omega)$$

for all $t \in \mathbb{R}_+$, $x \in [a, b]$ and also the mapping $\omega \to q(t, \omega)$ is a measurable on Ω.

(B_5) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is an random L^1-caratheodory function.

Remark 3.1. *The hypothesis (B_3) is in general and used in several research papers on random differential equation and random integral equations (see Dhage [1, 2]), in particular, if there exists, measurable functions $a, b : \Omega \to C(\mathbb{R}_+, \mathbb{R})$ such that foe every $\omega \in \Omega$,*

$$a(t, \omega) \leq f(t, x(t, \omega), \omega) \leq b(t, \omega)$$

for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}$. The details about the lower and upper random solutions for different types of random differential and random integral equations may be found in Ladde and Lakshmikantham [7].

Theorem 3.2. *Assume that the hypothesis (B_0)-(B_5) holds, then the nonlinear Volterra type perturbed random integral equation (1) has a minimal random solution $x_*(t, \omega)$ and a maximal random solution $y^*(t, \omega)$ defined on $\mathbb{R}_+$, moreover $x_*(t, \omega) = \lim_{n\to\infty} x_n(t, \omega)$ and $y^*(t, \omega) = \lim_{n\to\infty} y_n(t, \omega)$ for all $t \in \mathbb{R}_+$, and for all $\omega \in \Omega$, where the random sequence $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given as,*

$$x_{n+1}(t, \omega) = h(t, x_n(t, \omega)) + \int_0^t k(t, \tau, \omega) f(\tau, x_n(\tau, \omega)) d\tau \quad a.e \text{ for } t \in \mathbb{R}_+, \text{ with } x_0 = a \tag{2}$$

$$y_{n+1}(t, \omega) = h(t, y_n(t, \omega)) + \int_0^t k(t, \tau, \omega) f(\tau, y_n(\tau, \omega)) d\tau \quad a.e \text{ for } t \in \mathbb{R}_+, \text{ with } y_0 = b \tag{3}$$

Proof. Let $X = BC(\mathbb{R}_+, \mathbb{R})$ be a separable Banach space, now we define the two operators A and B on X, by

$$A(\omega)x(t) = h(t, x(t, \omega)) \tag{4}$$

$$B(\omega)x(t) = \int_0^t k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \tag{5}$$

then the nonlinear Volterra type perturbed random integral equation (1) is equivalent to the random operator equation

$$A(\omega)\,x(t) + B(\omega)\,x(t) = x(t,\omega) \quad \forall \quad t \in \mathbb{R}_+. \tag{6}$$

We show that the random operators $A(\omega)$ and $B(\omega)$ satisfies the Theorem ??. Note that by the hypothesis (B_0), $A, B : [a,b] \to \mathbb{K}$, also note that (B_3) ensures that $a \leq Aa + Ba$ and $Ab + Bb \leq b$. Since the cone $\mathbb{K}$ in a banach space $\mathbb{X}$ is normal. And the interval $[a,b]$ is a norm bounded in $\mathbb{X}$.

Step I: Show that the random operator $A(\omega)$ is a contraction on X.

Let $x, y \in X$ be any arbitrary elements, then by the hypothesis (H_3), we get,

$$\begin{aligned}
|A(\omega)\,x(t) - A(\omega)\,y(t)| &= |h(t, x(t,\omega)) - h(t, y(t,\omega))| \\
&\leq \alpha(t)\,|x(t,\omega) - y(t,\omega)|
\end{aligned}$$

Taking maximum all over t, we get,

$$\begin{aligned}
\|A(\omega)\,x(t) - A(\omega)\,y(t)\| &\leq \|\alpha(t)\|\,\|x(t,\omega) - y(t,\omega)\| \\
&\leq \|\alpha(t)\|\,\|x - y\| \\
\|A(\omega)\,x(t) - A(\omega)\,y(t)\| &\leq \|\alpha(t)\|\,\|x - y\|
\end{aligned} \tag{7}$$

provided that $\|\alpha(t)\| < 1$. Thus the random operator $A(\omega)$ is a contraction on X.

Step II: Now we show that the operator $B(\omega)$ is continuous operator on X. Let Y be a bounded subset of X, then there is real number $r > 0$ such that $\|x\| \leq r$ for all $y \in Y$. Let $\{y_n\}$ be a convergent sequence of points in Y converging to the point $y \in Y$. Then it is enough to prove that $\lim_{n\to\infty} B(\omega)\,y_n(t) = B(\omega)\,y(t)$, $t \in \mathbb{R}_+$. By the Lebesgue dominated converging theorem, we obtain,

$$\begin{aligned}
\lim_{n\to\infty} B(\omega)\,y_n(t) &= \lim_{n\to\infty} \int_0^t k(t,\tau,\omega)\, f(\tau, y_n(\tau,\omega))\, d\tau \\
&\leq \int_0^t k(t,\tau,\omega) \lim_{n\to\infty} f(\tau, y_n(\tau,\omega))\, d\tau \\
&\leq \int_0^t k(t,\tau,\omega)\, f(\tau, y(\tau,\omega))\, d\tau \\
&= B(\omega)\,y(t)
\end{aligned} \tag{8}$$

for every $t \in \mathbb{R}_+$, $\omega \in \Omega$. This proves that the random operator $B(\omega)$ is continuous operator on X.

Step III: The operator $B(\omega)$ is compact operator on X. Firstly, we show that $\{B(\omega)\,Y\}$ is uniformly bounded and equicontinuous set in X for each $\omega \in \Omega$. Let x be arbitrary element in Y. By hypothesis (B_5) the function $f(t, x(t,\omega))$ is L^1-caratheodory function, we have

$$\begin{aligned}
|B(\omega)\,x(t)| &= \left| \int_0^t k(t,\tau,\omega)\, f(\tau, x(\tau,\omega))\, d\tau \right| \\
&\leq \int_0^t |k(t,\tau,\omega)\, f(\tau, x(\tau,\omega))|\, d\tau \\
&\leq \int_0^t |k(t,\tau,\omega)|\, |f(\tau, x(\tau,\omega))| d\tau \\
&\leq K(\omega) \int_0^t |f(\tau, x(\tau,\omega))| d\tau
\end{aligned}$$

$$\leq K(\omega) \int_0^t |q_r(\tau)| d\tau$$

$$|B(\omega) x(t)| \leq K(\omega) v(t, \omega)$$

Taking suprimum all over t, we obtain

$$\|B(\omega) x(t)\| \leq K_1$$

for all $x(t) \in Y$, this shows that the $\{B(\omega) Y\}$ is a uniformly bounded subset in X for each $\omega \in \Omega$. Secondly, we show that $B(\omega) Y$ is an equicontinuous set in X. Let $x \in Y$ be arbitrary, then for any $s, t \in \mathbb{R}_+$ with $0 < s < t < \infty$ and for some $\omega \in \Omega$, (5) implies

$$\begin{aligned}
|B(\omega) x(t) - B(\omega) x(s)| &= \left| \int_0^t k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau - \int_0^s k(s, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \right| \\
&\leq \left| \int_0^s k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau + \int_s^t k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau - \int_0^s k(s, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \right| \\
&\leq \left| \int_0^s k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau - \int_0^s k(s, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \right| + \left| \int_s^t k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \right| \\
&\quad + \left| \int_0^s (k(t, \tau, \omega) - k(s, \tau, \omega)) f(\tau, x(\tau, \omega)) d\tau + \int_s^t k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \right| \\
&\leq \int_0^s |k(t, \tau, \omega) - k(s, \tau, \omega)| \, |f(\tau, x(\tau, \omega))| d\tau + \int_s^t |k(t, \tau, \omega)| \, |f(\tau, x(\tau, \omega))| d\tau \\
&\leq \int_0^s |k(t, \tau, \omega) - k(s, \tau, \omega)| \, |f(\tau, x(\tau, \omega))| d\tau + K(\omega) \int_s^t |f(\tau, x(\tau, \omega))| d\tau \\
&\leq \int_0^s |k(t, \tau, \omega) - k(s, \tau, \omega)| \, |q_r(\tau, \omega)| d\tau + K(\omega) \int_s^t |q_r(\tau, \omega)| d\tau \\
&\leq \int_0^s |k(t, \tau, \omega) - k(s, \tau, \omega)| \, |q_r(\tau, \omega)| d\tau + K(\omega) |v(t, \omega) - v(s, \omega)| \qquad (9)
\end{aligned}$$

for all $\omega \in \Omega$. Since v is uniformly continuous function, it follows that $|B(\omega) x(t) - B(\omega) x(s)| \to 0$ as $t \to s$ for all $x \in Y$ and $\omega \in \Omega$. Therefore $\{B(\omega) Y\}$ is an equicontinuous set in X for all $s, t \in \mathbb{R}_+$. As $\{B(\omega) Y\}$ is uniformly bounded and equicontinuous subset in X, it is compact in X by Arzela-Ascoli theorem for each $\omega \in \Omega$. As a consequence, $B(\omega)$ is compact and continuous operator on X. Thus B is completely continuous on X.

Step IV: Hypothesis (B_1) and (B_2) implies that the operators A and B are the increasing on $[a, b]$. to see this let $x, y \in [a, b]$ be such that $x \leq y$ then by (B_1)

$$\begin{aligned}
A(\omega) x(t) &= h(t, x(t, \omega)) \\
&\leq h(t, y(t, \omega)) \\
&= A(\omega) y(t)
\end{aligned}$$

for all $t \in \mathbb{R}_+$, similarly by the hypothesis (B_2), we have

$$\begin{aligned}
B(\omega) x(t) &= \int_0^t k(t, \tau, \omega) f(\tau, x(\tau, \omega)) d\tau \\
&\leq \int_0^t k(t, \tau, \omega) f(\tau, y(\tau, \omega)) d\tau \\
&= B(\omega) y(t)
\end{aligned}$$

for all $t \in \mathbb{R}_+$. Thus the random operators $A(\omega)$ and $B(\omega)$ are increasing operators on $[a, b]$ and by the hypothesis (B_3),

$$a(t, \omega) \leq h(t, a(t, \omega)) + \int_0^t k(t, \tau, \omega) f(\tau, a(\tau, \omega)) d\tau$$

$$
\begin{aligned}
&\le h\left(t, x\left(t, \omega\right)\right) + \int_0^t k\left(t, \tau, \omega\right) f\left(\tau, x\left(\tau, \omega\right)\right) d\tau \\
&\le h\left(t, b\left(t, \omega\right)\right) + \int_0^t k\left(t, \tau, \omega\right) f\left(\tau, b\left(\tau, \omega\right)\right) d\tau \\
&\le b\left(t, \omega\right)
\end{aligned}
$$

thus we get $a(t,\omega) \le b(t,\omega)$ for all $t \in \mathbb{R}_+$ and $x \in [a,b]$ as a result $a(t,\omega) \le A(\omega)x(t) + B(\omega)x(t) \le b(t,\omega)$ for all $t \in \mathbb{R}_+$ and $x \in [a,b]$. Hence $A(\omega)x(t) + B(\omega)x(t) \in [a,b]$ for all $x \in [a,b]$. Again the ordered cone $\mathbb{K}$ is normal in banach space $\mathbb{X}$, therefore an application of Theorem 3.1 yields that the random operator equation (6) and consequently the nonlinear Volterra type perturbed random integral equation (1) has a minimal random solution x_* and a maximal solution y^* on $\mathbb{R}_+$. Moreover, $x_*(\omega) = \lim_{n\to\infty} x_n(t,\omega)$ and $y^*(\omega) = \lim_{n\to\infty} y_n(t,\omega)$, where random sequence $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are defined by equation

$$
\begin{aligned}
x_{n+1}(t,\omega) &= h\left(t, x_n(t,\omega)\right) + \int_0^t k(t,\tau,\omega) f\left(\tau, x_n(\tau,\omega)\right) d\tau \quad \text{a.e for } t \in \mathbb{R}_+, \text{ with } x_0 = a \text{ and} \\
y_{n+1}(t,\omega) &= h\left(t, y_n(t,\omega)\right) + \int_0^t k(t,\tau,\omega) f\left(\tau, y_n(\tau,\omega)\right) d\tau \quad \text{a.e for } t \in \mathbb{R}_+, \text{ with } y_0 = b
\end{aligned}
$$

respectively. □

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