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A Generalized Method to Find the Square Root of Matrix Whose Characteristic Equation is Quadratic

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Abstract: In this paper, we generalized the method to calculating the square root of matrix whose characteristic is quadratic and how to Cayley-Hamilton theorem may be used to determine the formula for all square root of matrix whose order is 2×2.

Keywords: Eigen values, Matrix equation, Square root of matrix.

1. Introduction

Let $M_n(C)$ be the set of all complex matrices whose order is $n \times n$. Matrix Q is said to be a square root of matrix P, if the matrix product $Q.Q = P$. Now, what is the square root of matrix such as $\begin{bmatrix} p & q \\ r & s \end{bmatrix}$. It is not, in general $\begin{bmatrix} \sqrt{p} & \sqrt{q} \\ \sqrt{r} & \sqrt{s} \end{bmatrix}$. It is easy to see that the upper left entry of its square is $p + \sqrt{q}$ and not p. In recent years, several article have been written about the root of a matrix, and one can refer to [4–6]. A number of method have been proposed to computing the square root of matrix and these are usually based on Newton's method, either directly or the sign function (see e.g., [1–3]).

2. Generalized Method

The set of all matrices which their square is P, denoted by $\sqrt{P}$, i.e.,

$$\sqrt{P} = \{Y : Y \in M_n(C),\ Y^2 = P\}$$

This set can be very large .For example, we will see that $\sqrt{I}$ has infinite members. We can define the n$^{\text{th}}$ root of a matrix P as follows.

$$\sqrt[n]{P} = \{Y : Y \in M_n(C),\ Y^n = P\}$$

It is well known to all, if $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, then characteristic equation is

$$\lambda^2 - (Trace\ P)\,\lambda + \det\ P = 0 \tag{1}$$

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Apply Cayley - Hamilton theorem, putting $\lambda = P$, then equation (1) is

$$P^2 - (Trace\ P)\, P + (\det\ P)\, I = 0$$

Thus, we have

$$P^2 = (Trace\ P)\, P - (\det\ P)\, I \tag{2}$$

Putting, $P^2 = Q$, then equation (2) is

$$Q = (Trace\ P)\, P - (\det\ P)\, I$$
$$Q + (\det\ P)\, I = (Trace\ P)\, P$$
$$\frac{1}{(Trace\ P)}\left[Q + (\det\ P)\, I\right] = P \tag{3}$$

Lemma 2.1. *Let P be a 2×2 matrix. Then trace $P^2 = (trace\ P)^2 - 2\det\ P$.*

Proof. Suppose λ_1 and λ_2 are the two Eigen values of the matrix P. Then we can easy to see that λ_1^2 and λ_2^2 are the Eigen values of P^2. We know that, $trace\ P = \lambda_1 + \lambda_2$ and $\det\ P = \lambda_1\lambda_2$. Then,

$$\begin{aligned} trace\ P^2 &= \lambda_1^2 + \lambda_2^2 \\ &= (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 \\ &= (trace\ P)^2 - 2\det\ P \end{aligned}$$

Second Proof. In other words, let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Then,

$$P^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$
$$P^2 = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}.$$

Therefore,

$$Trace\ P^2 = \left(p^2 + rq\right) + \left(s^2 + rq\right)$$
$$Trace\ P^2 = p^2 + s^2 + 2rq$$
$$Trace\ P^2 = p^2 + s^2 + 2ps - 2ps + 2rq$$
$$Trace\ P^2 = (p + s)^2 - 2(ps - rq) \tag{4}$$

But, $trace\ P = p + s$ and $\det\ P = ps - qr$, then equation (4), $Trace\ P$. Let $P, Q \in M_n^2(C) = (trace\ P)^2 - 2\det\ P$. □

Remark 2.2. *Let $P, Q \in M_2(C)$ and $P^2 = Q$. Then the following statements are holds:*

(1). $\det\ P = \sqrt{\det\ Q}$.

(2). $tracet\ P = \sqrt{trace\ Q + 2\sqrt{\det\ Q}}$.

Example 2.3. *Let* $Q = \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix}$. *So* $\det\ Q = 64 - 15 = 49$, *and trace* $Q = 8 + 8 = 16$, *therefore if* $P^2 = Q$, *then,* $\det\ P = \sqrt{\det\ Q} = \sqrt{49} = \pm 7$, *and trace* $P = \sqrt{trace\ Q + 2\sqrt{\det\ Q}} = \sqrt{16 + 2\sqrt{49}} = \sqrt{16 \pm 14}$, *taking positive and negative sign then, trace* $P = \pm\sqrt{30}$ *or trace* $P = \pm\sqrt{2}$, *thus, from equation* (3),

$$P = \frac{1}{(trace\ P)}\left[Q + (detP)\, I\right],$$
$$P = \frac{1}{\pm\sqrt{30}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (\pm 7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \quad or$$
$$P = \frac{1}{\pm\sqrt{2}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (\pm 7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Therefore,

$$P = \frac{1}{\pm\sqrt{30}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \quad or \quad P = \frac{1}{\pm\sqrt{30}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \quad and$$
$$P = \frac{1}{\pm\sqrt{2}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \quad or \quad P = \frac{1}{\pm\sqrt{2}}\left\{\begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (-7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\},$$

on calculating then we have,

$$P = \pm\frac{1}{\sqrt{30}}\begin{bmatrix} 15 & 5 \\ 3 & 15 \end{bmatrix} \quad or \quad P = \pm\frac{1}{\sqrt{30}}\begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}, \quad and$$
$$P = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 15 & 5 \\ 3 & 15 \end{bmatrix} \quad or \quad P = \pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}$$

Lemma 2.4. *Let* $P \in M_2(C)$. *If trace* $P = 0$, *then* $P^2 \in \langle I \rangle$.

Proof. We will prove this lemma in two ways. In general, we have

$$P^2 - (trace\ P)\, P + (detP)\, I = 0 \tag{5}$$

Therefore, if *trace* $P = 0$, then from (5) we obtain,

$$P^2 + (\det\ P)\, I = 0$$
$$P^2 = -(\det\ P)\, I \quad and \quad P^2 \in \langle I \rangle$$

Second Proof. let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, and $p + s = 0$. Then,

$$P^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$P^2 = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

Putting $p = -s$, then

$$P^2 = \begin{bmatrix} p^2 + rq & 0 \\ 0 & s^2 + rq \end{bmatrix}$$

Hence, when $p^2 = s^2$ then $P^2 = (p^2 + rq)$. $\square$

Example 2.5. *Let* $Q = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. *Then* $\det\ Q = 2-6 = -4$, *and trace* $Q = 1+2 = 3$. *If* $P^2 = Q$ *then* $\det\ P = \sqrt{\det\ Q} = \sqrt{-4} = 2i$, *and*

$$\begin{aligned} trace\ P &= \sqrt{trace\ Q + 2\sqrt{\det\ Q}} \\ &= \sqrt{3 + 2\sqrt{-4}} \\ &= \sqrt{3+4i}. \end{aligned}$$

Now,

$$P = \frac{1}{(trace\ P)}\left[Q + (detP)\, I\right],$$

$$P = \frac{1}{\pm\sqrt{3+4i}}\left\{\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} + 2i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$P = \frac{1}{\pm\sqrt{3+4i}}\left\{\begin{bmatrix} 1+2i & 3 \\ 2 & 2+2i \end{bmatrix}\right\}$$

Lemma 2.6. *For each* $\beta \in C$ *and any matrix* P, $\sqrt{\beta P} = \sqrt{\beta}\sqrt{P}$.

Proof. Suppose that $\beta \neq 0$ and $Y \in \sqrt{BP}$. So $Y^2 \in \beta P$, hence $\frac{Y}{\sqrt{\beta}} \in \sqrt{P}$, which implies that $Y \in \sqrt{\beta}\sqrt{P}$. Conversely, if $Y \in \sqrt{BP}$, then $\frac{Y^2}{\beta} = P$. Hence $Y^2 = \beta P$ and $Y \in \sqrt{\beta P}$. Now, we try to compute $\sqrt{I}$. Suppose that $P \in M_2(C)$ and $P^2 = I$. Let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Then,

$$P^2 = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix},$$

but $P^2 = I$, then

$$I = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$$

Hence we have,

$$p^2 + rq = 1 \tag{6}$$

$$pq + qs = 0 \tag{7}$$

$$pr + rs = 0 \tag{8}$$

$$s^2 + rq = 1 \tag{9}$$

From (7) and (8), $q = 0$ or $p + s = 0$ and $r = 0$ or $p + s = 0$. We consider two cases:

(1). If $p+s = 0$, then equation (7) and (8) hold. We have $p^2 + rq = 1$ or $p = \sqrt{1 - rq}$ and since $a + d = 0$ and since $p + s = 0$ we have $p = -s = -\sqrt{1 - rq}$. Therefore

$$P = \left\{ \begin{bmatrix} \sqrt{1-rq} & 0 \\ 0 & -\sqrt{1-rq} \end{bmatrix} : b, c \in C \right\}.$$

(2). If $p + s \neq 0$ we must have $q = 0$ and $r = 0$. Hence $p = \pm 1$ and $s = \pm 1$. Therefore there are two solutions $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Hence we can write

$$\sqrt{I} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cup \begin{bmatrix} \sqrt{1-rq} & 0 \\ 0 & -\sqrt{1-rq} \end{bmatrix} \ b, c \ \in C \right\}.$$

$\square$

Example 2.7. *Let* $Q = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$. *Therefore* $Q = 16 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 16I$. *Then* $\sqrt{Q} = 4\sqrt{I}$, *hence we have*

$$\sqrt{I} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \cup \begin{bmatrix} 4\sqrt{1-rq} & 2q \\ 2r & -4\sqrt{1-rq} \end{bmatrix} : \ b, c \ \in C \right\}$$

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