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Topological Structure of Quasi-Partial b-Metric Spaces

Poonam Rani[1,*], Arti Sexana[1] and Pragati Gautam[2]

1 Department of Mathematics, Manav Rachna International Institute of Research and Studies, Faridabad, Haryana, India.

2 Department of Mathematics, Nehru College, Delhi, India.

Abstract: In this paper we discuss the topological properties of quasi-partial b-metric spaces. The notion of quasi-partial b-metric space was introduced and fixed point theorem and coupled fixed point theorem on this space were studied. Here the concept of quasi-partial b-metric topology is discussed and notion of product of quasi-partial b-metric spaces is also introduced.

Keywords: Topological Structure, Metric Spaces, b-Metric Spaces, fixed point theorem.

1. Introduction

The study of ordinary metric spaces is fundamental in topology and functional analysis. In the late nineties metric spaces structure has gained much attention of the mathematicians because of development of fixed point theory in ordinary metric spaces. The concept of b-metric space was introduced by Czerwick as a generalization of metric space. Several authors have focused on fixed point theorems for a metric space, a partial metric space, quasi-partial metric spaxce and a partial b-metric space. The concept of a quasi-partial-metric space was introduced by Karapinar. He studied some fixed point theorems on these spaces. Motivated by this a modest attempt has been made to introduce the notion of quasi-partial b-metric space where we have discussed fixed point theorem on it. Further, we have proved coupled fixed point theorem on the same space. The aim of this paper is to study then topological properties of quasi-partial b-metric spaces. Here we also introduce product of quasi-partial b-metric spaces and some relevant results are discussed on it.

2. Preliminaries

Definition 2.1. *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:*

$(b1)$ $d(x, y) = 0$ *if and only if* $x = y$,

$(b2)$ $d(x, y) = d(y, x)$,

$(b3)$ $d(x, y) \leq s[d(x, y) + d(y, z)]$.

* *E-mail: poonammalhotra2692@gmail.com*

In this case, the pair (X, d) is called a b-metric space.

Definition 2.2 (Metric Space). *A metric space is a set together with a metric on the set. The metric is a function that defines a concept of distance between any two numbers of the set which are called points metric spaces are generalization of real line. A distance or metric on a metric space x is a function.*

$$d : X^2 \rightarrow R^+$$
$$(x, y) \rightarrow d(x, y)$$

With properties

(1). $d(x, y) = 0 \Leftrightarrow x = y$

(2). $d(y, x) = \partial(x, y)$

(3). $d(x, y) \leq \partial(x, z) + \partial(y, z)$

Definition 2.3 (Partial metric space). *A partial metric space is a pair $(X, p : X \times X \rightarrow R)$ such that*

$P1:$ $p(x, x) \leq p(x, y)$ *(non negitivety and small self distance)*

$P2:$ *If* $p(x, x) = p(x, y) = p(y, y)$ *then* $x = y$.

$P3:$ $P(x, y) = p(y, x)$ *symmetry*

$P4:$ $P(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ *(triangularity).*

Definition 2.4 (Partial b-metric space). *A partial b-metric space on non-empty set X is a function $b : X \times X \rightarrow R^+$ such that $x, y, z \in X$.*

$(Pb1)$ $x = y$, *iff* $b(x, x) = b(x, y) = b(y, y)$;

$(Pb2)$ $b(x, x) \leq b(x, y)$;

$(Pb3)$ $b(x, y) = b(y, x)$;

$(Pb4)$ *F a real number* $S \geq 1$ *such that* $b(x, y) \leq S[b(x, z) + b(z, y)] - b(z, z)$.

Definition 2.5 (Quasi-partial metric space). *A partial quasi-metric on a set X is a function $p : X \times X \rightarrow [0, \infty)$ such that*

(a). $p(x, x) \leq p(x, y)$ *when* $x, y \in X$.

(b). $p(x, x) \leq p(y, x)$ *when* $x, y \in X$.

(c). $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ *when* $x, y, z \in X$.

(d). $x = y$ *iff* $[p(x, x) = p(x, y)$ *and* $p(y, y) = p(y, x)$ *when* $x, y \in X]$.

Definition 2.6. *A Quasi-partial b-metric on a non-empty set X is a mapping $qp_b : X \times X \rightarrow R^+$ such that for some real numbers $S \geq 1$ and all $x, y, z \in X$*

(1). $qp_b(x, x) = qp_b(x, y)$.

(2). $qp_b(x, x) \leq qp_b(x, y)$.

(3). $qp_b(x, x) \leq qp_b(y, x)$.

(4). $qp_b(x, x) \leq S[qp_b(x, z) + qp_b(y, z)] - qp_b(z, z)$.

2.1. Topological Properties of Quasi-partial b-metric space

Theorem 2.7. *A quasi-partial b-metric space (X, qp_b) is a T_0-space.*

Proof. Let $x_0, y_0 \in (X, qp_b)$ such that $x_0 \neq y_0$. Consider the open ball $B_{qpb}(x_0, \varepsilon)$ in X where $qp_b(x_0, y_0) > \varepsilon$. Then by the Definition 2.5 it is seen that $y_0 \notin B_{qpb}(x_0, \varepsilon)$. For if $y_0 B_{qpb}(x_0, \varepsilon)$ then $qp_b(y_0, x_0) < \varepsilon$ and $qp_b(x_0, y_0) > \varepsilon$ which is a contradiction to the choice of ε. Hence (X, qp_b) is a T_0-space. □

Example 2.8. *Consider the usual metric $qp_b(x_0, y_0) = |x_0 - y_0|$ on $[0, 1]$. Let $x_0, y_0 \in [0, 1]$ be such that $x_0 \neq y_0$. Choose $\epsilon < \min\{|x_0 - y_0|, |x_0|, |x_0 - 1|\}$. Then $x_0 \in B_{qpb}(x_0, \varepsilon)$ but $y_0 \notin B_{qpb}(x_0, \varepsilon)$. For if $y_0 \in B_{qpb}(x_0, \varepsilon)$ then $qp_b(y_0, x_0) < \epsilon \Rightarrow |x_0 - y_0| < \epsilon$. But by the choice of ε, $\varepsilon < |x_0 - y_0|$ which is a contradiction. So $y_0 \notin B_{qpb}(x_0, \varepsilon)$. Hence (X, qp_b) is a T_0-space.*

2.2. Product of Quasi-Partial b-metric Spaces

Theorem 2.9. *For $I = 1, 2, 3, \ldots, n$ let (X_i, qp_{bi}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and let $X_M = \sum_{i=1}^{n} X_i$ then for qp_b defined by $qp_b(x, y) = \sum_{i=1}^{n} qp_{bi}(x_i, y_i)$ is symmetric quasi-partial b-metric space with coefficient s = max $\{s_i\}$, $1 \leq i \leq n$.*

Proof. We need to prove properties $QP_{b1} - QP_{b4}$ for (X_M, qp_b).

(QP_{b1})**:** Let $qp_b(x, y) = qp_b(x, x) \Rightarrow \sum_{i=1}^{n} qp_{bi}(x_i, y_i) = \sum_{i=1}^{n} qp_{bi}(y_i, x_i) = \sum_{i=1}^{n} qp_{bi}(x_i, x_i) \Rightarrow \sum_{i=1}^{n} [qp_{bi}(x_i, y_i) - qp_{bi}(x_i, x_i)] = 0$ and $\sum_{i=1}^{n} [qp_{bi}(x_i, y_i) - qp_{bi}(x_i, x_i)] = 0$. By (QP_{b2}) and (QP_{b3})

$$\sum_{i=1}^{n} [qp_{bi}(x_i, y_i) - qp_{bi}(x_i, x_i)] \geq 0 \quad \forall \quad i = 1, 2, 3, \ldots, n$$
$$\sum_{i=1}^{n} [qp_{bi}(y_i, x_i) - qp_{bi}(x_i, x_i)] \geq 0 \quad \forall \quad i = 1, 2, 3, \ldots, n$$

Hence

$$qp_{bi}(x_i, y_i) = qp_{bi}(x_i, x_i)] \geq 0 \quad \forall \quad i = 1, 2, 3, \ldots, n$$
$$\Rightarrow x_i = y_i \quad \forall \quad i = 1, 2, 3, \ldots, n$$
$$\Rightarrow x = y.$$

(QP_{b2})**:**
$$qp_b(x, x) = \sum_{i=1}^{n} qp_{bi}(x_i, x_i)$$
$$\leq \sum_{i=1}^{n} qp_{bi}(x_i, x_i) \quad \text{[by } (QP_{b2}) \text{ of } (X_i, qp_{bi})]$$
$$= qp_b(x, y).$$

(QP_{b3})**:** Similarly , as for (QP_{b2}).

(QP_{b4})**:** Here

$$qp_b(x, z) = \sum_{i=1}^{n} qp_{bi}(x_i, z_i) \leq \sum_{i=1}^{n} \{s_i[qp_{bi}(x_i, y_i) + qp_{bi}(y_i, z_i)] - qp_{bi}(z_i, z_i)\} \quad (\text{by } (QP_{b4}) \text{ of } (X_i, qp_{bi})).$$

By definition, $s = \max_{1 \leq i \leq n} \{s_i\} \Rightarrow s \geq s_i$ for all $i = 1, 2, \ldots, n$. Also $s \geq 1$ since $s_i \geq 1$ for all $i = 1, 2, \ldots, n$. Hence all the four properties of a quasi-partial b-metric space are satisfied by (X_M, qp_b) with, $s = \max_{1 \leq i \leq n} \{s_i\}$. Hence a quasi-partial b-metric space. It remains to show that it is symmetric. Let $x, y \in X$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $x_i, y_i \in X_i$ where $i = 1, 2, \ldots, n$. Since each (X_i, qp_{bi}) is qp_b-symmetric, therefore

$$qp_{bi}(x_i, y_i) = qp_{bi}(y_i, y_i) \Rightarrow \sum_{i=1}^{n} qp_{bi}(x_i, y_i) = \sum_{i=1}^{n} qp_{bi}(y_i, x_i) \Rightarrow qp_b(x, y) = qp_b(y, x)$$

Hence it is qp_b-symmetric. □

3. Literature Review

Partial metric space is first introduced by Matthews. A partial metric space is an attempt to find metric space by replacing $d(x, x) = 0$ with condition $d(x, x) \leq d(x, y)$ for all x and y. Some properties of convergence of sequence were discussed by Matthews. According to Matthews, any mapping T of complete partial metric space X into that satisfies for some $0 < k < 1$, the inequality $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ has a unique point. In the paper "Fixed point theorem on quasi partial metric space" by Erdal Karpinar, I.M.Erhan in mathematical and computer modelling. Quasi partial metric space introduced and discussed the space introduced and discussed the exixtence of fixed point of self mapping T on quasi-partial metric spaces. The concept of Quasi partial b metric space introduced to generalize the concept of quasi partial metric space. Some fixed points results are proved in paper quasi partial b metric space and some related fixed point theorm.Some of topological properties of quasi partial b metric space used in paper Anuradha Gupta and Pragati Gautam in International Journal of Pure Mathematical Sciences.

4. Literature Survey

S.No	Author	Title of the Paper	Name of Journal	Year	Pages
1.	"Dejan Ilic, Vladimir Pavlovic, Vladimir Rakocevic"	"Some new extensions of Banach's contraction principle to partial metric space"	"Applied Mathematics Letter 24"	2011	1326-1330
2.	"Erdal Karapinar, I.M. Erhan, Ali Ozturk"	"Fixed point theorems on quasi-partial metric spaces"	"Mathematical and Computer Modelling 57"	2013	2442-2448
3.	"Anuradha Gupta, Pragati Gautam"	"Quasi-partial b-metric spaces and some related fixed point theorems"	"Gupta and Gautam Fixed Point Theory and applications"	2015	2015-18
4.	"Pooja Dhawan, Jatinderdeep Kaur"	"Fixed Point Theorems for (ξ,α)-Expansive Mappings in Artially Ordered Sets"	"International Journal of Computer & Mathematical Sciences"	2017	2347-8527
5.	"Sandra Oltra and Oscar Valero"	"Banach's Fixed Point Theorems For Partial Metric Spaces"	"Rend. Istit. Mat. Univ. Trieste"	2004	17-26

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