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Existence Result for Riemann-Liouville Fractional Differential Equation with Boundary Condition

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Abstract: Investigation of existence property of Riemann-Liouville Fractional Differential Equation with Boundary Condition is done in this paper

$$-D_{0+}^{p}x\left(t\right)=f\left(t,x\left(t\right)\right),\quad 0<t<1$$

$$x\left(0\right)=x'\left(0\right)=x''\left(0\right)=\cdots=x^{(n-2)}\left(0\right)=0,\quad x\left(1\right)=\lambda\int_{0}^{1}x\left(s\right)ds$$

the technique we have employed is coupled lower and upper solutions with fixed point theory on cone, where $2\leq n-1<p\leq n$, $p\in R$, is the order of Riemann-Liouville Fractional derivative and $0<\lambda<p$.

Keywords: Riemann-Liouville fractional differential equation, boundary condition, fixed point theory.

1. Introduction

History of differential and fractional calculus is of same age. It begins with pioneering work of Leibniz, Eular, Lagrange, Liouville in the 17^{th} century. For forty years theory related to non-integer order calculus achieves great importance. This is due to its applicability in the areas like Control theory, Physics, Mechanics, Electrochemistry, Engineering, Electrodynamics of complex medium, Aerodynamics, Polymer Science etc [1–3, 12].

Fractional differential equation with boundary conditions has several applications. Existence result for multipoint that is two, three point by using Leray-Schauder, coincidence degree theory, fixed point index theory, fixed point theorems in cones and so on. We refer to the reader with references [4–6, 9–11, 13, 16, 17, 19, 20].

Since accuracy of physical systems depends on proper choice of boundary conditions that gives different applications such as population control, models of blood flows in arteries. Hence, many point boundary value conditions has of much importance.. Also, note that many point boundary value integral condition contains so much sub cases, see papers [6–9, 17, 20] and references therein. In [7] Benchohra study for fractional differential equations with mixed boundary conditions of the form

$$D^{\alpha}x\left(t\right)=f\left(t,x\left(t\right)\right),\quad \text{for each } t=I=[0,1]$$

$$x\left(0\right)+\mu\int_{0}^{T}x\left(s\right)ds=x\left(T\right)$$

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where D^{α}, $0 < p \leq 1$ is the caputo fractional derivative $f : I \times E \to E$ is a given function satisfying some condition, E is a Banach space with norm $\|\bullet\|$ and $\mu \in R$. In [6], Wang discussed following integral boundary value problem

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad 0 < t < 1, 1 < \alpha \leq 2$$
$$u(0) = 0, \qquad u(1) = \int_0^1 u(s)\, ds$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and D_{0+}^{p} is the standard Riemann Liouville derivative. In paper [21], G.U.Pawar and J.N. Salunke study the existence and uniqueness of positive solutions of the following problem

$$D^{\alpha} v(t) = f(t, v(t)), \ \text{ foreach } t = I = [0,1]$$
$$v(0) + \lambda \int_0^1 v(s)\, ds = v(1)$$

where D_{0+}^{α} is the standard Riemann-Liouville derivative and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. The papers as above motivated us to the study of existence of solution for fractional differential equation with boundary condition of multipoint integral type

$$-D_{0+}^{p} x(t) = f(t, x(t)), \quad 0 < t < 1 \tag{1}$$
$$x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)\, ds$$

we employ coupled method of lower and upper solution with fixed point theorem on cone, where $2 \leq n-1 < p \leq n,\ p \in R$, is the order of Riemann-Liouville derivative and $0 < \lambda < p$. To our knowledge, less work appears on existence of solution for fractional differential equation with boundary condition of multipoint integral type. Our aim is to this for the problem (1). We organize the paper as follows: section 2, devoted for derivation of Green's function and its properties. Also discussion of preliminary facts are presented in this section. Section 3 deals with proof of existence result by using lower and upper solution for fractional differential equation with multipoint integral boundary conditions.

2. Preliminary Results

This section is about to the study of Green's function and its properties which are important to our further study.

Definition 2.1 ([2]). *The fractional order integral of a function $f : (0,\infty) \to R$ is given by*

$$I_{0+}^{\alpha} f(t) = \frac{1}{G(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

where I_{0+}^{α} is integral of order $\alpha > 0$.

Definition 2.2 ([2]). *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds$$

where $n-1 < \alpha \leq n$, the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3 ([1]). *Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$. Then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has unique solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in R, \quad i = 1, 2, \ldots, N, \quad N = [\alpha] + 1.$$

Lemma 2.4 ([1]). *Assume that* $u \in C(0,1) \cap (0,1)$ *with fractional derivative of order* $\alpha \geq 0$ *that belongs to* $C(0,1) \cap L(0,1)$ *. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in R, \quad i = 1, 2, \ldots, N.$$

We begin by deriving for linear differential equations of non-integer order with multipoint integral boundary value condition, the Green's function.

Theorem 2.5. *Let* $n-1 < p < n$ *and* $\lambda \neq p$*. Assume* $u(t) \in C[0,1]$*, then unique solution of following problem*

$$-D_{0+}^{p} x(t) = u(t), \quad 0 < t < 1 \tag{2}$$

$$x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \lambda \int_0^1 x(s)\, ds \tag{3}$$

is given by

$$x(t) = \int_0^1 G(t,s)\, u(s)\, ds$$

where Green's function $G(t,s)$ *is given by*

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{p-1}(p-\lambda+\lambda s) - (p-\lambda)(t-s)^{p-1}}{(p-\lambda)\Gamma(p)}, & 0 \leq s \leq t \leq 1 \\ \frac{[t(1-s)]^{p-1}(p-\lambda+\lambda s)}{(p-\lambda)\Gamma(p)}, & 0 \leq t \leq s \leq 1 \end{cases} \tag{4}$$

Proof. Applying I^p on both sides of (4) by Lemma equivalent integral equation of (4) is

$$\begin{aligned} x(t) &= -I_{0+}^{p} u(t) + c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n} \\ &= -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)}\, u(s)\, ds + c_1 t^{p-1} + c_2 t^{p-2} + \cdots + c_n t^{p-n} \end{aligned} \tag{5}$$

where $c_1, c_2, \ldots, c_n \in R$. From $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$ we have $c_2 = c_3 = \cdots = c_n = 0$. Then the general solution of (2) is

$$x(t) = -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s)\, ds + c_1 t^{p-1} \tag{6}$$

and it follows from

$$x(1) = \lambda \int_0^1 x(s)\, ds$$

that $x(1) = -\int\limits_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)}\, u(s)\, ds + c_1$.

$$c_1 = \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} u(s)\, ds + \lambda \int_0^1 x(s)\, ds$$

Put this value of c_1 in (6)

$$x(t) = -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s)\, ds + t^{p-1} \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} u(s)\, ds + \lambda t^{p-1} \int_0^1 x(s)\, ds \tag{7}$$

Let $\varepsilon = \int_0^1 x(s)\, ds$ by (7), we get

$$\begin{aligned} \int_0^1 x(t)\, dt &= -\int_0^1 \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)}\, u(s)\, ds dt + \int_0^1 t^{p-1} \int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} u(s)\, ds dt + \lambda\varepsilon \int_0^1 t^{p-1} dt \\ &= -\int_0^1 \frac{(1-s)^{p}}{p\Gamma(p)} + \int_0^1 \frac{(1-s)^{p-1}}{p\Gamma(p)} u(s)\, ds + \frac{\lambda\varepsilon}{p} \end{aligned}$$

$$= \int_0^1 \frac{s(1-s)^{p-1}}{p\Gamma(p)} u(s)\, ds + \frac{\lambda\varepsilon}{p}$$

so

$$\varepsilon = \int_0^1 \frac{s(1-s)^{p-1}}{p\Gamma(p)} u(s)\, ds + \frac{\lambda\varepsilon}{p}$$
$$\varepsilon = \int_0^1 \frac{s(1-s)^{p-1}}{(p-\lambda)\Gamma(p)} u(s)\, ds \tag{8}$$

Substitute value of ε from (8) into (7), we have

$$x(t) = -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s)\, ds + t^{p-1}\int_0^1 \frac{(1-s)^{p-1}}{\Gamma(p)} u(s)\, ds + \lambda t^{p-1}\int_0^1 \frac{s(1-s)^{p-1}}{(p-\lambda)\Gamma(p)} u(s)\, ds$$
$$x(t) = -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s)\, ds + \int_0^1 \frac{(1-s)^{p-1}t^{p-1}}{\Gamma(p)} u(s)\, ds + \int_0^1 \frac{\lambda s(1-s)^{p-1}t^{p-1}}{(p-\lambda)\Gamma(p)} u(s)\, ds$$
$$x(t) = -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s)\, ds + \int_0^1 \left[\frac{p-\lambda+\lambda s}{p-\lambda}\right] \frac{[(1-s)t]^{p-1}}{\Gamma(p)} u(s)\, ds$$
$$x(t) = -\int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} u(s)\, ds + \int_0^t \left[\frac{p-\lambda+\lambda s}{p-\lambda}\right] \frac{[(1-s)t]^{p-1}}{\Gamma(p)} u(s)\, ds + \int_t^1 \left[\frac{p-\lambda+\lambda s}{p-\lambda}\right] \frac{[(1-s)t]^{p-1}}{\Gamma(p)} u(s)\, ds$$
$$x(t) = \int_0^t \frac{(p-\lambda+\lambda s)[(1-s)t]^{p-1} - (p-\lambda)(t-s)^{p-1}}{(p-\lambda)\Gamma(p)} u(s)\, ds + \int_t^1 \frac{(p-\lambda+\lambda s)[(1-s)t]^{p-1}}{(p-\lambda)\Gamma(p)} u(s)\, ds$$
$$x(t) = \int_0^1 G(t,s)\, u(s)\, ds$$

□

Lemma 2.6. *The Green function given by (4) satisfies the following properties, for all $p \in (n-1, n]$ and $\lambda \geq 0$.*

(i). $G(t,s) \geq 0$, $t, s \in (0,1)$.

(ii). $(p-\lambda)\, G(1,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $p \neq \lambda$.

(iii). $G(t,s) \leq \frac{L}{(p-\lambda)\Gamma(p-1)}$ *for all* $t, s \in [0,1]$ *and* $\lambda \in [0,p)$.

(iv). $n-1 < p \leq n$ *and* $0 < \lambda < p$

$$t^{p-1}G(1,s) \leq G(t,s) \leq \frac{p}{\lambda} G(1,s) \quad \forall \ t, s \in (0,1).$$

Lemma 2.7. *Let D be a subset of the cone K of semi-order Banach space E, $T : D \to E$ be nondecreasing. If there exists $x_0, y_0 \in D$ such that $x_0 \leq y_0$, $\langle x_0, y_0\rangle \subset D$ and x_0, y_0 are the lower and upper solutions of equation $x - T(\mathrm{x}) = 0$, then the equation $x - T(x) = 0$ has maximum solution and minimum solution x^*, y^* in $\langle x_0, y_0\rangle$ such that $x^* \leq y^*$, when one of the following conditions hold*

(1). T is compact continuous and K is normal

(2). T is continuous and K is regular

(3). T is continuous or weak continuous and E is reflexive, K is normal.

3. Main Results

We establish, in this section, the existence result for positive solution of problem (1).

Let $E = C[0,1]$. Then E is a Banach space endowed with norm

$$\|x\| = \sup_{t\in[0,1]} |x(t)|.$$

The Cone $K \subset E$ is defined by

$$K = \{x \in E \,|x(t) \geq 0,\ 0 \leq t \leq 1\}$$

Assume that $u(t) = f(t, x(t))$, then it follows from the Theorem 2.1 that the equation (1) has a solution if and only if the operator T defined by

$$Tx(t) = \int_0^1 G(t,s)\ f(s, x(s))\, ds \tag{9}$$

has a fixed point. First of all, we give the definition of lower and upper solutions for the operator T

Definition 3.1. *The function $v(t) \in E$ is called a lower solution of operator T if*

$$v(t) \leq Tv(t) \quad (D^p v(t) \leq f(t, v(t)))\,,\ 0 < t < 1$$

and the function $u(t) \in E$ is called a upper solution of operator T if

$$w(t) \geq Tw(t) \qquad (D^p w(t) \geq\ f(t, w(t)))\ ,\ 0 < t < 1$$

We consider the following set of assumptions

(A_1): $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $f(t,\bullet)$ is nondecreasing for each $t \in [0,1]$, and there exists a positive constant K such that $f(t,\bullet)$ is strictly increasing on $[0,K]$ for each $t \in [0,1]$.

(A_2): $0 < \lim\limits_{x\to+\infty} f(t, x(t)) < +\infty$ for each $t \in [0,1]$.

(A_3): v_0 and w_0 are a lower and upper solutions of problem (1), satisfying $v_0 \leq w_0$, $0 \leq t \leq 1$.

Theorem 3.2 (Existence Theorem)**.** *Assume that $(A_1) - (A_3)$ holds. Then the problem (1) has a positive solution.*

Proof. We will give proof in the following four steps

Step 1: First of all we have to prove that the operator $T : K \to K$ is completely continuous. The operator $T : K \to K$ is continuous in view of continuity and non-negativeness of $G(t,s)$ and $f(t,x)$. Let $M \subset K$ be bounded i.e. $\exists$ a positive constant $l > 0$ such that $\|x\| \leq l$; $\forall\ x \in M$.

Let $L = \max\limits_{0\leq t\leq 1, 0\leq x\leq l} |f(t, x(t))| + 1$. Then, for all $x \in M$, we have

$$|Tx(t)| \leq \int_0^1 G(t,s)\, f(s, x(s))\, ds \leq \frac{Lp}{(p-\lambda)\,\Gamma(p)},\quad \forall\ \ t \in [0,1]$$

Hence $T(M)$ is bounded. Take $x \in M$, then $\forall\ \ t \in [0,1]$, the following inequalities are satisfied by using Lemmas 2.2.

$$Tx(t) = \int_0^1 G(t,s)\ f(s, x(s))\, ds$$

$$
\begin{aligned}
&\geq t^{p-1}\int_0^1 G(1,s)\, f(s,x(s))\,ds\\
&\geq \frac{t^{p-1}\lambda}{p}\int_0^1 \max_{t\in[0,1]}\{G(t,s)\}\, f(s,x(s))\,ds\\
&\geq \frac{t^{p-1}\lambda}{p}\max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\, f(s,x(s))\,ds\right\}\\
&= \frac{t^{p-1}\lambda}{p}\|Tx\|
\end{aligned}
$$

For each $x\in M$, we have

$$
\begin{aligned}
\left|(Tx)'(t)\right| &= \left|-\int_0^t \frac{(t-s)^{p-2}}{\Gamma(p-1)} f(s,x(s))\,ds + \int_0^1 \frac{(p-\lambda+\lambda s)(1-s)^{p-1}t^{p-2}}{(p-\lambda)\Gamma(p-1)} f(s,x(s))\,ds\right|\\
&\leq \frac{L}{\Gamma(p-1)}\int_0^t (t-s)^{p-2}\left|f(s,x(s))\right|ds + \frac{pL}{(p-\lambda)\Gamma(p-1)}\int_0^1 (1-s)^{p-1}ds\\
&\leq \frac{L}{\Gamma(p-1)} + \frac{pL}{(p-\lambda)\Gamma(p-1)}\\
&:= N
\end{aligned}
$$

Consequently $\forall\ \ t_1,t_2\in[0,1]$, $t_1<t_2$, we have

$$
\|Tx(t_2)-Tx(t_1)\| \leq \int_{t_1}^{t_2}\left|(Tx)'(s)\right|ds \leq N(t_2-t_1)
$$

$T(M)$is equicontinuous. The Arzele-Ascoli Theorem implies that $\overline{T(\mathrm{M})}$ is compact. Therefore $T: K\to K$ is completely continuous.

Step 2: The operator T is an increasing. $x_0,y_0\in K$ and $x_0\leq y_0$ then by (A_1), we have that

$$
\begin{aligned}
Tx_0(t) &= \int_0^1 G(t,s)\, f(s,x_0(s))\,ds\\
&\leq \int_0^1 G(t,s)\, f(s,y_0(s))\,ds\\
&\leq Ty_0(t)
\end{aligned}
$$

Therefore T is an increasing operator. Applying definition of upper and lower solution, we have $Tx_0\geq x_0$ and $Ty_0\leq y_0$. Hence $T:\langle x_0,y_0\rangle\to\langle x_0,y_0\rangle$ is a compact continuous operator.

Step 3: By (A_3), there exists positive constants M_1 and N such that $x\geq N$ it holds $f(t,x(t))\leq M_1$: On the other hand, by (A_1), $f:[0,1]\times[0,N]$ is continuous, $\exists\ M_2>0$ such that, it holds $f(t,x(t))\leq M_2$. Let $M=\max\{M_1,M_2\}$, then we have $f(t,x(t))\leq M$, $\forall\ \ x\geq 0$. Now, we consider the following equation

$$
\begin{gathered}
D^p w(t) + M = 0,\quad 0<t<1,\quad n-1<p\leq n\\
w(0)=w'(0)=\cdots=w^{(n-2)}(0)=0,\quad w(1)=\lambda\int_0^1 w(s)\,ds
\end{gathered}
\tag{10}
$$

From the Theorem 2.1, we have the solution (10) is

$$
\begin{aligned}
w(t) &= \int_0^1 G(t,s)\ \ M ds\\
&\geq \int_0^1 G(t,s)\, f(s,w(s))\,ds\\
&= Tw(t)
\end{aligned}
$$

this implies $w(t)$ is an upper solution of the operator T.

Obviously, $w(t) \equiv 0$ is a upper solution of the operator T. On the other hand, it is obvious that $v(t) \equiv 0$ is a lower solution of the operator T, we have

$$v(t) \leq w(t).$$

Step 4: As K is a normal cone, Theorem 3.1 implies that T has a fixed point $x(t) \in \langle 0, w(t) \rangle$. Therefore, the equation (1) has a positive solution. Which completes the proof. □

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