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The Minimum Resolving Energy of a Graph

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Abstract: A subset W of vertrices in a connected graph $G = (V, E)$ is called a resolving set of G if all other vertices are uniquely determined by their distances in W. The metric dimension $dim(G)$ of a graph G is the minimum cardinality of a resolving set of G. In this paper, for a minimum resolving set R of a graph G, we define the minimum resolving energy $E_R(G)$ of G. We study this parameter for some standard graphs. Some properties of $E_R(G)$ and bounds were also obtained.

MSC: 05C12, 05C50, 05C69.

Keywords: Minimum resolving set, metric dimension, minimum resolving matrix, minimum resolving eigenvalue, minimum resolving energy of a graph.

1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected and connected graph with neither loops nor parallel edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology, we refer to Chartrand and Lesniak [4]. The concept of energy of a graph was introduced by I. Gutman [9] in the year 1978. Let $A = (a_{ij})$ be the adjacency matrix of a graph $G = (V, E)$. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. Since, A is real and symmetric, the eigenvalues of G are real and $\sum_{i=1}^{n} \lambda_i = 0$. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G. That is $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

For more details on the theory of graph energy we refer to ([6, 7, 10, 11]). The upper and lower bounds for energy of a graph G and its basic properties can be found in ([14, 15]). The topic graph energy is a part of Spectral Graph Theory which deals with the eigenvalues of various matrices associated with graphs namely adjacency matrix, Incidence matrix, Laplacian Matrix etc. Spectral graph theory has applications in chemistry in the study of molecular orbital theory of conjugated molecules ([8, 12]). For various matrices associated with graphs, we refer to [1].

Adiga et al. [3] and Rajesh Kanna et al. [16] have defined, the minimum covering energy $E_C(G)$ and the minimum dominating energy $E_D(G)$ respectively for a graph G and computed them for several families of graphs. Various upper and lower bounds are also established. In this paper, we initaite a similar study for the minimum resolving set of a graph.

The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest u-v path in G. By an ordered set of vertices we mean a set $W = \{w_1, w_2, \cdots, w_k\}$ on which the ordering $(w_1, w_2, \cdots, w_k)$ has been imposed. For an ordered subset $W = \{w_1, w_2, \cdots, w_k\}$ of V, we refer to the k-vector (ordered k-tuple) $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ as

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the (metric) representation of v with respect to W. The set W is called a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$ for all $u, v \in V(G)$. Hence, if W is a resolving set of cardinality k for a graph G of order n, then the set $\{r(v|W) : v \in V\}$ consists of n distinct k-vectors. A resolving set of minimum cardinality for a graph G is called a basis for G and the metric dimension of G is defined to be the cardinality of a basis of G and is denoted by $dim(G)$. A vertex $x \in V$ is said to resolve a pair of vertices $\{u, v\}$ in G if $d(u, x) \neq d(v, x)$. It is well known that for a connected graph G with n vertices, $1 \leq dim(G) \leq n - 1$.

The idea of resolving sets has appeared in the literature previously. In [17] and later in [18], Slater introduced the concept of a resolving set for a connected graph G under the term locating set. He referred to a minimum resolving set as a reference set for G. He called the cardinality of a minimum resolving set (reference set) the location number of G. Independently, Harary and Melter [13], discovered these concepts as well but used the term metric dimension. For a survey of results in metric dimension, we refer to Chartrand and Ping [5].

2. The Minimum Resolving Energy

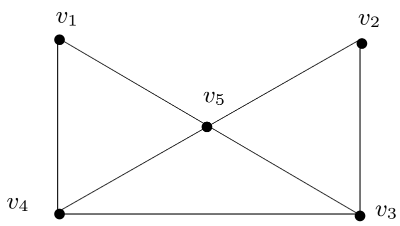
Definition 2.1. *Let $G = (V, E)$ be a connected graph of order n, with $V = \{v_1, v_2, \cdots, v_n\}$. Let $R \subset V$ be a minimum resolving set of G. Then $|R| = dim(G)$, the metric dimension of G. The minimum resolving matrix of G is the $n \times n$ matrix defined by $A_R(G) = (a_{ij})$, where*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in R \\ 0 & \text{otherwise.} \end{cases}$$

$A_R(G)$ is a real and symmetric matrix. Also note that trace of $A_R(G) = dim(G)$.

Definition 2.2. *The characteristic polynomial of $A_R(G)$ is defined by $f_n(G, \lambda) = det(\lambda I - A_R(G))$, and we call it the minimum resolving polynomial of A corresponding to R. Eigenvalues of $A_R(G)$ are called minimum resolving eigenvalues w.r.t R. They are real numbers and we can arrange them in non-increasing order say $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$. The minimum resolving energy of G is defined as $E_R(G) = \sum_{i=1}^{n} |\lambda_i|$.*

The minimum resolving energy of a graph G may not be unique and it depends on the resolving set of G. Consider the following graph.



Graph G

Example 2.3. *The metric dimension of the above graph G is 2 and all resolving sets in G of cardinality 2 are $R_1 = \{v_1, v_2\}$, $R_2 = \{v_1, v_3\}$, $R_3 = \{v_1, v_4\}$, $R_4 = \{v_2, v_3\}$ and $R_5 = \{v_3, v_4\}$.*

(i). $R_1 = \{v_1, v_2\}$

$$A_{R_1}(G) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Its characteristic equation is $\lambda^5 - 2\lambda^4 - 6\lambda^3 + 4\lambda^2 + 8\lambda = 0$. *Minimum resolving eigenvalues w.r.t* R_1 *are* $\lambda_1 \approx 3.2361, \lambda_2 \approx -1.4142, \lambda_3 \approx 1.4142, \lambda_4 \approx -1.2361$ *and* $\lambda_5 = 0$. *So the minimum resolving energy* $E_{R_1}(G) = \sum_i^5 \lambda_i \approx 7.3006$.

(ii). $R_2 = \{v_1, v_3\}$

$$A_{R_2}(G) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Its characteristic equation is $\lambda^5 - 2\lambda^4 - 6\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0$. *Minimum resolving eigenvalues w.r.t* R_2 *are* $\lambda_1 = -1, \lambda_2 \approx -0.15644, \lambda_3 \approx 3.33297, \lambda_4 \approx 1.29940$ *and* $\lambda_5 = -1.47593$. *So the minimum resolving energy* $E_{R_2}(G) = \sum_i^5 \lambda_i \approx 7.26474$. *For* $R_4 = \{v_2, v_3\}$, *the charactersitic equation is same as that of* A_{R_2} *and so* $E_{R_4}(G) \approx 7.26474$.

(iii). For $R_3 = \{v_1, v_4\}$, *the characteristic equation of* $A_{R_3}(G)$ *is* $\lambda^5 - 2\lambda^4 - 6\lambda^3 + 3\lambda^2 + 6\lambda - 1 = 0$. *Minimum resolving eigenvalues w.r.t* R_3 *are* $\lambda_1 \approx -0.15829, \lambda_2 \approx -1.35190, \lambda_3 \approx 1.09405, \lambda_4 \approx -1.26823$ *and* $\lambda_5 = 3.36777$. *So the minimum resolving energy* $E_{R_3}(G) \approx 7.24024$.

(iv). For $R_5 = \{v_3, v_4\}$, *the characteristic equation of* $A_{R_5}(G)$ *is* $\lambda^5 - 2\lambda^4 - 6\lambda^3 + 2\lambda^2 + 5\lambda = 0$. *Minimum resolving eigenvalues w.r.t* R_5 *are* $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 \approx 3.4495$ *and* $\lambda_5 = -1.4495$. *So the minimum resolving energy* $E_{R_5}(G) \approx 6.899$.

Now we give the minimum resolving energy of some standard graphs.

Theorem 2.4. *For the path* $P_n, 1 \leq n \leq 9$, *the minimum resolving energy are given in the following table*

P_n	$E_R(P_n) \approx$
1	1
2	2.2361
3	3.4939
4	4.7588
5	6.0267
6	7.296
7	8.5668
8	9.8379
9	11.1095

Proof. Let the path P_n on n vertices be $P_n = (v_1, v_2, \cdots v_n)$. For a connected graph G, it is well known that the metric dimension $dim(G) = 1$ if and only if G is a path. Also the resolving sets of P_n with cardinality one are the end vertices $\{v_1\}$ and $\{v_n\}$. Let $R = \{v_1\}$. When $n = 1$, clearly $E_R(P_1) = 1$. When $n = 2$, the characteristic equation for $A_R(G)$ is $\lambda^2 - \lambda - 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx 1.61803$ and $\lambda_2 \approx 0.61803$. Thus the minimum resolving energy $E_R(P_2) \approx 2.2361$.

When $n = 3$, the characteristic equation for $A_R(G)$ is $\lambda^3 - \lambda^2 - 2\lambda + 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx 0.44504, \lambda_2 \approx 1.24697$ and $\lambda_3 \approx 1.80193$. Thus the minimum resolving energy $E_R(P_3) \approx 3.4939$.

When $n = 4$, the characteristic equation for $A_R(G)$ is $\lambda^4 - \lambda^3 - 3\lambda^2 + 2\lambda + 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 = 1, \lambda_2 \approx -0.34729$, $\lambda_3 \approx -1.53208$ and $\lambda_4 \approx 1.87938$. Thus the minimum resolving energy $E_R(P_4) \approx 4.7588$.

When $n = 5$, the characteristic equation for $A_R(G)$ is $\lambda^5 - \lambda^4 - 4\lambda^3 + 3\lambda^2 + 3\lambda - 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx 0.28462, \lambda_2 \approx -1.68250$, $\lambda_3 \approx 1.91898$, $\lambda_4 \approx -0.83083$ and $\lambda_5 \approx 1.30972$. Thus the minimum resolving energy $E_R(P_5) \approx 6.0267$.

When $n = 6$, the characteristic equation for $A_R(G)$ is $\lambda^6 - \lambda^5 - 5\lambda^4 + 4\lambda^3 + 6\lambda^2 - 3\lambda - 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx -0.24107, \lambda_2 \approx 0.70920$, $\lambda_3 \approx -1.77091$, $\lambda_4 \approx 1.94188$, $\lambda_5 \approx -1.13612$ and $\lambda_6 \approx 1.49702$. Thus the minimum resolving energy $E_R(P_6) \approx 7.296$.

When $n = 7$, the characteristic equation for $A_R(G)$ is $\lambda^7 - \lambda^6 - 6\lambda^5 + 5\lambda^4 + 10\lambda^3 - 6\lambda^2 - 4\lambda + 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 = 1, \lambda_2 \approx 0.20905$, $\lambda_3 \approx 1.61803$, $\lambda_4 \approx -1.33826$, $\lambda_5 \approx -0.61803, \lambda_6 \approx -1.82709$ and $\lambda_7 \approx 1.95629$. Thus the minimum resolving energy $E_R(P_7) \approx 8.5668$.

When $n = 8$, the characteristic equation for $A_R(G)$ is $\lambda^8 - \lambda^7 - 7\lambda^6 + 6\lambda^5 + 15\lambda^4 - 10\lambda^3 - 10\lambda^2 + 4\lambda + 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx -0.18453, \lambda_2 \approx 1.20526$, $\lambda_3 \approx 1.96594$, $\lambda_4 \approx 1.70043$, $\lambda_5 \approx -1.86494, \lambda_6 \approx 0.54732, \lambda_7 \approx -0.89147$ and $\lambda_8 \approx -1.47801$. Thus the minimum resolving energy $E_R(P_8) \approx 9.8379$. When $n = 9$, the characteristic equation for $A_R(G)$ is $\lambda^9 - \lambda^8 - 8\lambda^7 + 7\lambda^6 + 21\lambda^5 - 15\lambda^4 - 20\lambda^3 + 10\lambda^2 + 5\lambda - 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx 0.16515, \lambda_2 \approx -1.09389$, $\lambda_3 \approx -1.89163$, $\lambda_4 \approx 1.97272$, $\lambda_5 \approx -1.57828, \lambda_6 \approx 1.75894, \lambda_7 \approx -0.49097, \lambda_8 \approx 0.80339$ and $\lambda_9 \approx 1.35456$. Thus the minimum resolving energy $E_R(P_9) \approx 11.1095$. $\square$

Theorem 2.5. *For the cycle $C_n, 3 \le n \le 9$, the minimum resolving energy are given in the following table*

C_n	$E_R(C_n) \approx$
3	3.4642
4	5.2361
5	7.1231
6	8.2527
7	8.9664
8	10.4834
9	12.0563

Proof. Let the cycle C_n on n vertices be $C_n = (v_1, v_2, \cdots v_n, v_1), n \ge 3$. The metric dimension $dim(C_n) = 2$ and any two adjacent vertices is a minimum resolving set of C_n. So take $R = \{v_1, v_2\}$.

When $n = 3$, the characteristic equation for $A_R(G)$ is $\lambda^3 - 2\lambda^2 - 2\lambda = 0$ and minimum resolving eigenvalues are $\lambda_1 = 0, \lambda_2 \approx 2.7321$ and $\lambda_3 \approx -0.7321$. Thus the minimum resolving energy $E_R(C_3) \approx 3.4642$.

When $n = 4$, the characteristic equation for $A_R(G)$ is $\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda - 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx 0.381966, \lambda_2 \approx 0.618033$, $\lambda_3 \approx 1.618033$ and $\lambda_4 \approx 2.6180339$. Thus the minimum resolving energy $E_R(C_4) \approx 5.2361$.

When $n = 5$, the characteristic equation for $A_R(G)$ is $\lambda^5 - 2\lambda^4 - 4\lambda^3 + 6\lambda^2 + 3\lambda - 4 = 0$ and minimum resolving eigenvalues are $\lambda_1 = -1, \lambda_2 = 1$, $\lambda_3 = 1$, $\lambda_4 \approx 2.5615528$ and $\lambda_5 \approx -1.5615528$. Thus the minimum resolving energy $E_R(C_5) \approx 7.1231$.

When $n = 6$, the characteristic equation for $A_R(G)$ is $\lambda^6 - 2\lambda^5 - 5\lambda^4 + 8\lambda^3 + 6\lambda^2 - 6\lambda - 3 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx -0.44504, \lambda_2 \approx 2.53208$, $\lambda_3 \approx -1.80193$, $\lambda_4 \approx 1.34729$, $\lambda_5 \approx -0.87938$ and $\lambda_6 \approx 1.24627$. Thus the minimum resolving energy $E_R(C_6) \approx 8.2527$.

When $n = 7$, the characteristic equation for $A_R(G)$ is $\lambda^7 - 2\lambda^6 - 6\lambda^5 + 10\lambda^4 + 10\lambda^3 - 12\lambda^2 - 4\lambda = 0$ and minimum resolving eigenvalues are $\lambda_1 = 0, \lambda_2 \approx -0.28733$, $\lambda_3 \approx 1.55240$, $\lambda_4 \approx -1.78165$, $\lambda_5 \approx 2.51657, \lambda_6 \approx 1.41421$ and $\lambda_7 \approx -1.41421$. Thus the minimum resolving energy $E_R(C_7) \approx 8.9664$.

When $n = 8$, the characteristic equation for $A_R(G)$ is $\lambda^8 - 2\lambda^7 - 7\lambda^6 + 12\lambda^5 + 15\lambda^4 - 20\lambda^3 - 10\lambda^2 + 8\lambda - 1 = 0$ and minimum resolving eigenvalues are $\lambda_1 = -1, \lambda_2 \approx 0.17421$, $\lambda_3 \approx 0.34729$, $\lambda_4 \approx 2.50848$, $\lambda_5 \approx -1.87938, \lambda_6 \approx 1.67964, \lambda_7 \approx -1.36233$ and $\lambda_8 \approx 1.53208$. Thus the minimum resolving energy $E_R(C_8) \approx 10.4834$.
When $n = 9$, the characteristic equation for $A_R(G)$ is $\lambda^9 - 2\lambda^8 - 8\lambda^7 + 14\lambda^6 + 21\lambda^5 - 30\lambda^4 - 20\lambda^3 + 20\lambda^2 + 5\lambda - 4 = 0$ and minimum resolving eigenvalues are $\lambda_1 \approx 0.61803, \lambda_2 \approx -1.86993$, $\lambda_3 \approx 2.50430$, $\lambda_4 \approx -1.61803$, $\lambda_5 \approx 1.76213, \lambda_6 \approx -0.92215, \lambda_7 \approx 0.52565, \lambda_8 \approx -0.61803$ and $\lambda_9 \approx 1.61803$. Thus the minimum resolving energy $E_R(C_9) \approx 12.0563$. □

Theorem 2.6. *For the complete graph $K_n, n \geq 2$, the minimum resolving energy $E_R(K_n) = \sqrt{n^2 + 2n - 3}$.*

Proof. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. The metric dimension $dim(K_n) = n - 1$. So any subset of $(n-1)$ vertices is a minimum resolving set of K_n. Take $R = \{v_1, v_2, \cdots, v_{n-1}\}$. The resolving matrix of A for R is (a_{ij}), where $a_{ij} = 1 \forall i, j, i \neq n, j \neq n, a_{nn} = 0$. That is,

$$A_R(K_n) = \begin{pmatrix} 1 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & 1 & 1 & . & . & . & 1 & 1 \\ 1 & 1 & 1 & . & . & . & 1 & 1 \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ 1 & 1 & 1 & . & . & . & 1 & 0 \end{pmatrix}_{n \times n}$$

The characteristic polynomial is

$$\begin{vmatrix} \lambda - 1 & -1 & -1 & . & . & . & -1 & -1 \\ -1 & \lambda - 1 & -1 & . & . & . & -1 & -1 \\ -1 & -1 & \lambda - 1 & . & . & . & -1 & -1 \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ -1 & -1 & -1 & . & . & . & -1 & \lambda \end{vmatrix}$$

The characteristic equation is $\lambda^{n-2}(\lambda^2 - (n-1)\lambda - (n-1)) = 0$. The minimum resolving eigenvalues are $\lambda = 0[(n-2)$ times], $\lambda = \frac{(n-1) \pm \sqrt{n^2+2n-3}}{2}$ [one time each]. The minimum resolving energy is

$$E_R(K_n) = |0|(n-2) + \left| \frac{(n-1) + \sqrt{n^2 + 2n - 3}}{2} \right| + \left| \frac{(n-1) - \sqrt{n^2 + 2n - 3}}{2} \right| = \sqrt{n^2 + 2n - 3}.$$

□

Theorem 2.7. *For the complete bipartite graph $K_{n,n}, n \geq 2$, the minimum resolving energy $E_R(K_{n,n})$ is equal to $3(n-1) + \sqrt{n^2 - 2n + 5}$.*

Proof. Let the bipartition of $K_{n,n}$ be $V = (X, Y)$, where $X = \{v_1, v_2, \cdots, v_n\}$ and $Y = \{u_1, u_2, \cdots, u_n\}$. We know that the metric dimension of a complete bipartite graph G with n vertices is $(n-2)$, and so $dim(K_{n,n}) = n + n - 2 = 2n - 2$. Clearly the union of any $(n-1)$ vertices from each bipartite sets is a ninimum resolving set of $K_{n,n}$. Let $R = \{v_1, v_2, \cdots, v_{n-1}, u_1, u_2, \cdots, u_{n-1}\}$. Then R is a minimum resolving set of $K_{n,n}$. Now the minimum resolving matrix is

$$A_R(K_{n,n}) = \begin{bmatrix} 1 & 0 & .. & 0 & 0 & 1 & 1 & .. & 1 & 1 \\ 0 & 1 & .. & 0 & 0 & 1 & 1 & .. & 1 & 1 \\ . & . & .. & . & . & . & . & .. & . & . \\ 0 & 0 & .. & 1 & 0 & 1 & 1 & .. & 1 & 1 \\ 0 & 0 & .. & 0 & 0 & 1 & 1 & .. & 1 & 1 \\ 1 & 1 & .. & 1 & 1 & 1 & 0 & .. & 0 & 0 \\ 1 & 1 & .. & 1 & 1 & 0 & 1 & .. & 0 & 0 \\ . & . & .. & . & . & . & . & .. & . & . \\ 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 1 & 0 \\ 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 0 & 0 \end{bmatrix}_{2n\times 2n}$$

The characteristic polynomial is

$$\begin{vmatrix} \lambda-1 & 0 & .. & 0 & 0 & -1 & -1 & .. & -1 & -1 \\ 0 & \lambda-1 & .. & 0 & 0 & -1 & -1 & .. & -1 & -1 \\ . & . & .. & . & . & . & . & .. & . & . \\ 0 & 0 & .. & \lambda-1 & 0 & -1 & -1 & .. & -1 & -1 \\ 0 & 0 & .. & 0 & \lambda & -1 & -1 & .. & -1 & -1 \\ -1 & -1 & .. & -1 & -1 & \lambda-1 & 0 & .. & 0 & 0 \\ -1 & -1 & .. & -1 & -1 & 0 & \lambda-1 & .. & 0 & 0 \\ . & . & .. & . & . & . & . & .. & . & . \\ -1 & -1 & .. & -1 & -1 & 0 & 0 & .. & \lambda-1 & 0 \\ -1 & -1 & .. & -1 & -1 & 0 & 0 & .. & 0 & \lambda \end{vmatrix}$$

The characteristic equation is $(\lambda-1)^{2n-4}(\lambda^2-(n+1)\lambda+1)(\lambda^2+(n-1)\lambda-1)=0$. The minimum resolving eigenvalues are $\lambda = 1$ $[(2n-4)$ times], $\lambda = \frac{(n+1)\pm\sqrt{n^2+2n-3}}{2}$ [one time each], $\lambda = \frac{-(n-1)\pm\sqrt{n^2-2n+5}}{2}$ [one time each]. The minimum resolving energy is

$$\begin{aligned} E_R(K_{n,n}) &= |1|(2n-4) + \left|\frac{-(n-1)+\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{-(n-1)-\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{(n+1)+\sqrt{n^2+2n-3}}{2}\right| \\ &\quad + \left|\frac{(n+1)-\sqrt{n^2+2n-3}}{2}\right| \\ &= (2n-4)+\sqrt{n^2-2n+5}+(n+1) = 3(n-1)+\sqrt{n^2-2n+5}. \end{aligned}$$

□

In the next theorem, we compute the sum of the minimum resolving eigenvalues of a graph G and also sum of its squares.

Theorem 2.8. *Let $G=(V,E)$ be a graph and R be any minimum resolving set of G. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the matrix $A_R(G)$, then $\sum_i^n \lambda_i = dim(G)$ and $\sum_i^n \lambda_i^2 = dim(G)+2m(G)$.*

Proof. We have $|R| = dim(G)$. Consider the minimum resolving matrix $A_R(G)$. We know that the trace of a square matrix equals the sum of the eigenvalues counted with multiplicities. Thus $\sum_i^n \lambda_i = \sum_i^n a_{ii} = |R|$ and so $\sum_i^n \lambda_i = dim(G)$. Also we know that the sum of the squares of the eigenvalues of a square matrix A equals the trace of the matrix A^2. Thus $\sum_i^n \lambda_i^2 = \sum_i^n\sum_j^n a_{ij}a_{ji} = \sum_i^n (a_{ii})^2 + \sum_{i\neq j} a_{ij}a_{ji} = \sum_i^n (a_{ii})^2 + 2\sum_{i<j}(a_{ij})^2 = |R|+2|E|$. That is, $\sum_i^n \lambda_i^2 = dim(G)+2m(G)$. □

In the next theorem, we show that largest minimum resolving eigenvalue of a graph G is always greater than or equal to $\frac{2m+dim(G)}{n}$.

Theorem 2.9. *If $\lambda_1(G)$ is the largest minimum resolving eigenvalue of G, then $\lambda_1(G) \geq \frac{2m+dim(G)}{n}$.*

Proof. For any square matrix A, by [1] we have $\lambda_1(A) = \max_{X\neq 0}\{\frac{X'AX}{X'X} : X \text{ is a vector}\}$. Let J be the vector with all entries are 1. Then clearly, $\lambda_1(A) \geq \frac{J'AJ}{J'J} = \frac{2m+dim(G)}{n}$. □

Rajesh Kanna et al. [16], has obtained bounds for the minimum dominating energy $E_D(G)$ of a graph. Similar bounds can be obtained for $E_R(G)$. We state the following two theorems without proof.

Theorem 2.10. *For the graph $G = (V, E)$ with a minimum resolving set R, we have*

$$\sqrt{(2m + dim(G)) + n(n-1)P^{\frac{2}{n}}} \leq E_R(G) \leq \sqrt{n(2m + dim(G))}.$$

Theorem 2.11. *Let $G = (V, E)$ be a graph. If $(2m + dim(G)) \geq n$, then*

$$E_R(G) \leq \frac{2m + dim(G)}{n} + \sqrt{(n-1)[(2m + dim(G)) - (\frac{2m + dim(G)}{n})^2]}.$$

Bapat and Pati [2] proved that if the graph energy is a rational number then it is an even integer. In the following theorem, we give a similar result for minimum resolving energy of a graph G.

Theorem 2.12. *Let $G = (V, E)$ be a graph and R be any minimum resolving set of G. If the minimum resolving energy $E_R(G)$ is a rational number, then $E_R(G) \equiv dim(G) \pmod 2$.*

Proof. Let $\lambda_1, \lambda_2 \cdots, \lambda_n$ be minimum resolving eigenvalues of G. Assume that $\lambda_1, \lambda_2 \cdots, \lambda_r, r < n$ are positive and the rest are non-positive. Then $\sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{r} \lambda_i - \sum_{i=r+1}^{n} \lambda_i = 2\sum_{i=1}^{r} \lambda_i - \sum_{i=1}^{n} \lambda_i$. That is, $E_R(G) = 2\sum_{i=1}^{r} \lambda_i - |R|$. Thus $E_R(G) \equiv dim(G) \pmod 2$. □

Results on $E_R(G)$ for various other graphs G, graph products and other bounds are obtained and will be communicated shortly.

References

[1] R. B. Bapat, *Graphs and Matrices*, Hindustan Book Agency, (2011).

[2] R.B. Bapat and S. Pati, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc., 1(2011), 129-132.

[3] C. Adiga, A. Bayad, I. Gutman and S. A. Srinivas, *The minimum covering energy of a graph*, Kragujevac J. Sci., 34(2012), 39-56.

[4] G. Chartrand and L. Lesniak, *Graphs & Digraphs*, Fourth Edition, Chapman & Hall/CRC, (2005).

[5] G. Chartrand and Ping Zhang, *The theory and applications of resolvability in graphs: A Survey*, Congressus Numerantium, 160(2003), 47-68.

[6] D. Cvetkovic and I. Gutman, *Applications of Graph Spectra*, Mathematical Institution, Belgrade, (2009).

[7] D. Cvetkovic and I. Gutman, *Selected Topics on Applications of Graph Spectra*, Mathematical Institute Belgrade, (2011).

[8] A. Graovac, I. Gutman and N. Trinajstic, *Topological Approach to the Chemistry of Conjugated Molecules*, Springer, Berlin, (1977).

[9] I. Gutman, *The energy of a graph*, Ber. Math-Statist. Sekt. Forschungsz. Graz, 103(1978), 1-22.

[10] I. Gutman, X. Li, J. Zhang, *Graph Energy* (Ed-s: M. Dehmer, F. Emmert), Streib., *Analysis of Complex Networks*, From Biology to Linguistics, Wiley-VCH, Weinheim, (2009), 145-174.

[11] I. Gutman, *The energy of a graph: Old and new results* (Ed-s: A. Betten, A. Kohnert, R. Laue, A. Wassermann), *Algebraic Combinatorics and Applications*, Springer, Berlin, (2001), 196-211.

[12] I. Gutman and O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, (1986).

[13] F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin., 2(1976), 191-195.

[14] Huiqing Liu, Mei Lu and Feng Tian, *Some upper bounds for the energy of graphs*, Journal of Mathematical Chemistry, 41(1)(2007).

[15] B. J. McClelland, *Properties of the latent roots of a matrix: The estimation of p-electron energies*, J. Chem. Phys., 54(1971), 640-643.

[16] M. R. Rajesh Kanna, B. N. Dharmendra and G. Sridhara, *The Minimum Dominating Energy of a Graph*, International Journal of Pure and Applied Mathematics, 85(4)(2013), 707-718.

[17] P. J. Slater, *Leaves of trees*, Congressus Numerantium, 14(1975), 549-559.

[18] P. J. Slater, *Domination and location in acyclic graphs*, Networks, 17(1987), 55-64.