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Edge Graceful Irregularity Strength of Wheel Related Graphs

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Abstract: For a simple graph G, the edge graceful irregular s-labeling is a mapping $f : V \bigcup E \to \{1, 2, 3, \ldots, s\}$ such that if for any two distinct edges e and g, $wt(e) \neq wt(g)$, $wt(uv) = |f(u) + f(v) - f(uv)|$. The edge graceful irregularity strength of G, denoted by $egs(G)$ is the smallest k for which G has an edge graceful irregular s-labeling. In this paper we determine the exact value of an edge graceful irregularity strength of graphs, namely gear, helm, closed helm and flower graph.

MSC: 05C78.

Keywords: Graceful irregularity, edge graceful irregularity strength, gear, helm, closed helm, flower graph.

1. Introduction

In this paper, we consider only finite simple undirected graphs with order p and size q. For graph theoretic notation we follow [4, 6]. A labeling of a graph G is a mapping that carries a set of graph elements, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [3]. Motivated by irregular assignments, irregularity strength of graphs was introduced by Chartrand et al., [1] and papers [2, 5, 8]. Ahmad, Al-Mushayt and Baca [7] defined the notion of an edge irregular k-labeling of a graph G to be labeling of the vertices of G, $\phi : V(G) \to \{1, \ldots, k\}$such that, the edge weights $w_\phi(uv) = \phi(u) + \phi(v)$ are distinct for every edges. The minimum k for which the graph G has an edge irregular k-labeling is called an edge irregularity strength of G.

From an edge irregularity strength of G, we have found an edge graceful irregularity strength of graphs. For a simple graph G, the edge graceful irregular s-labeling is a mapping $f : V \bigcup E \to \{1, 2, 3, \ldots, s\}$ such that if for any two distinct edges e and g, $wt(e) \neq wt(g)$, $wt(uv) = |f(u) + f(v) - f(uv)|$. The edge graceful irregularity strength of G, denoted by $egs(G)$ is the smallest k for which G has an edge graceful irregular s-labeling.

In this paper, we study an edge graceful irregular labeling and determine a value of the edge graceful irregularity strength for classes of wheel related graph, such as gear,helm, closed helm and flower graph.

2. Main Results

Theorem 2.1. *For any graph G of size q, the lower bound for $egs(G) = \left\lceil \frac{q}{2} \right\rceil$.*

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Definition 2.2. *A Gear graph G_n is a wheel graph with a vertex added between each pair of adjacent vertices of the outer cycle.*

Theorem 2.3. *An edge graceful irregularity strength of gear graph G_n, $n \geq 3$ is $\lceil \frac{3n}{2} \rceil$.*

Proof. Let

$$\begin{aligned}
&V(G_n) = \{u\} \bigcup \{u_i | 1 \leq i \leq n\} \bigcup \{v_i | 1 \leq i \leq n\} \quad \text{and} \\
&E(G_n) = \{uu_i | 1 \leq i \leq n\} \bigcup \{u_i v_i | 1 \leq i \leq n\} \bigcup \{v_i u_{i+1} | 1 \leq i \leq n-1\} \bigcup \{v_n u_1\}.
\end{aligned}$$

Then from Theorem 2.1, $egs(G_n) \geq \lceil \frac{3n}{2} \rceil$.

To prove the upper bound.

We define a function $f : V(G_n) \bigcup E(G_n) \to \{1, 2, \ldots, \lceil \frac{3n}{2} \rceil\}$ as follows.

$$\begin{aligned}
f(u) &= 1 \\
f(u_i) &= \left\lceil \frac{n-1}{2} \right\rceil + i, \quad 1 \leq i \leq n \\
f(v_i) &= \left\lfloor \frac{3n+1}{2} \right\rfloor, \quad 1 \leq i \leq n \\
f(uu_i) &= \left\lfloor \frac{n+4}{2} \right\rfloor, \quad 1 \leq i \leq n \\
f(u_i v_i) &= n+1, \quad 1 \leq i \leq n \\
f(v_i u_{i+1}) &= 1, \quad 1 \leq i \leq n-1 \\
f(v_n u_1) &= 1
\end{aligned}$$

From the above labeling we get,

$$\begin{aligned}
wt(uu_i) &= |f(u) + f(u_i) - f(uu_i)| \\
&= \left| 1 + \left\lceil \frac{n-1}{2} \right\rceil + i - \left\lfloor \frac{n+4}{2} \right\rfloor \right| \\
&= i - 1, \ 1 \leq i \leq n \\
wt(u_i v_i) &= |f(u_i) + f(v_i) - f(u_i v_i)| \\
&= \left| \left\lceil \frac{n-1}{2} \right\rceil + i + \left\lfloor \frac{3n+1}{2} \right\rfloor - (n+1) \right| \\
&= n + i - 1, \ 1 \leq i \leq n \\
wt(v_i u_{i+1}) &= |f(v_i) + f(u_{i+1}) - f(v_i u_{i+1})| \\
&= \left| \left\lfloor \frac{3n+1}{2} \right\rfloor + \left\lceil \frac{n-1}{2} \right\rceil + i + 1 - 1 \right| \\
&= 2n + i, \ 1 \leq i \leq n-1 \\
wt(v_n u_1) &= |f(v_n) + f(u_1) - f(v_n u_1)| \\
&= \left| \left\lfloor \frac{3n+1}{2} \right\rfloor + \left\lceil \frac{n-1}{2} \right\rceil + 1 - 1 \right| \\
&= 2n
\end{aligned}$$

The above labeling shows that, $egs(G_n) \leq \lceil \frac{3n}{2} \rceil$. Combining this with the lower bound, we get, $egs(G_n) = \lceil \frac{3n}{2} \rceil$.

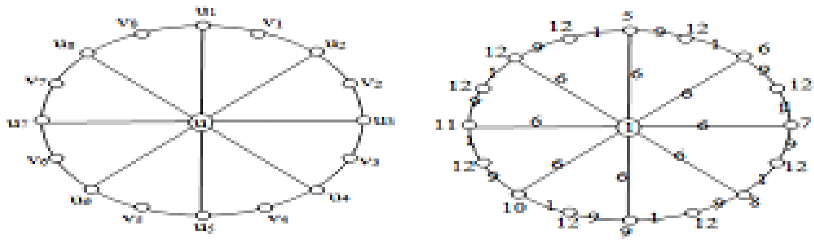

Figure 1. $egs\,(G_8) = 12$

□

Definition 2.4. *The helm graph H_n is the graph obtained from the wheel graph by adding a pendent edge at each vertex of the cycle.*

Theorem 2.5. *An edge graceful irregularity strength of Helm graph H_n, $n \geq 3$ is $\lceil \frac{3n}{2} \rceil$.*

Proof. Let

$$V(H_n) = \{u\} \bigcup \{u_i | 1 \leq i \leq n\} \bigcup \{v_i | 1 \leq i \leq n\} \quad and$$
$$E(H_n) = \{uu_i | 1 \leq i \leq n\} \bigcup \{u_i u_{i+1} | 1 \leq i \leq n-1\} \bigcup \{u_n u_1\} \bigcup \{u_i v_i | 1 \leq i \leq n\}.$$

Then from theorem 1, $egs\,(H_n) \geq \lceil \frac{3n}{2} \rceil$.

To prove the upper bound.

We define a function $f : V(H_n) \bigcup E(H_n) \to \{1, 2, \ldots, \lceil \frac{3n}{2} \rceil\}$ as follows.

$$\begin{aligned}
f(u) &= 1 \\
f(u_i) &= n + \left\lfloor \frac{i}{2} \right\rfloor, && 1 \leq i \leq n \\
f(v_i) &= n + \left\lceil \frac{i}{2} \right\rceil, && 1 \leq i \leq n \\
f(uu_i) &= n + 2 - \left\lceil \frac{i}{2} \right\rceil, && 1 \leq i \leq n \\
f(u_i u_{i+1}) &= n + 1, && 1 \leq i \leq n-1 \\
f(u_n u_1) &= \left\lceil \frac{n+1}{2} \right\rceil \\
f(u_i v_i) &= 1, && 1 \leq i \leq n
\end{aligned}$$

From the above labeling, we get,

$$\begin{aligned}
wt(uu_i) &= |f(u) + f(u_i) - f(uu_i)| \\
&= \left| 1 + n + \left\lfloor \frac{i}{2} \right\rfloor - \left(n + 2 - \left\lceil \frac{i}{2} \right\rceil \right) \right| \\
&= i - 1, \; 1 \leq i \leq n \\
wt(u_i u_{i+1}) &= |f(u_i) + f(u_{i+1}) - f(u_i u_{i+1})| \\
&= \left| n + \left\lfloor \frac{i}{2} \right\rfloor + n + \left\lfloor \frac{i+1}{2} \right\rfloor - (n+1) \right|
\end{aligned}$$

$$
\begin{aligned}
&= n+i-1,\ 1 \le i \le n-1 \\
wt\left(u_n u_1\right) &= \left|f\left(u_n\right)+f\left(u_1\right)-f\left(u_n u_1\right)\right| \\
&= \left|n+\left\lfloor \frac{n}{2} \right\rfloor + n + \left\lfloor \frac{1}{2} \right\rfloor - \left\lceil \frac{n+1}{2} \right\rceil\right| \\
&= 2n+i,\ 1 \le i \le n-1 \\
wt\left(u_i v_i\right) &= \left|f\left(u_i\right)+f\left(v_i\right)-f\left(u_i v_i\right)\right| \\
&= \left|n+\left\lfloor \frac{i}{2} \right\rfloor + n + \left\lceil \frac{i}{2} \right\rceil - 1\right| \\
&= 2n-1+i,\ 1 \le i \le n
\end{aligned}
$$

The above labeling shows that , $egs\left(H_n\right) \le \left\lceil \frac{3n}{2} \right\rceil$. Combining this with the lower bound, we get, $egs\left(H_n\right) = \left\lceil \frac{3n}{2} \right\rceil$.

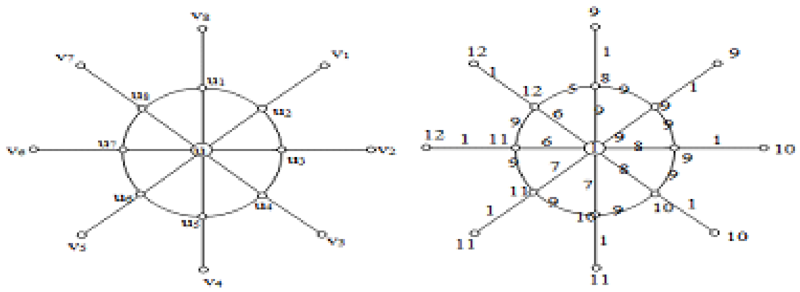

Figure 2. $egs\left(H_8\right) = 12$

□

Definition 2.6. *A closed helm CH_n is the graph obtained by taking a helm graph and adding edges between the pendent vertices.*

Theorem 2.7. *An edge graceful irregularity strength of Closed Helm graph CH_n, $n \ge 3$ is $2n$.*

Proof. Let

$$V\left(CH_n\right) = \{u\} \bigcup \{u_i | 1 \le i \le n\} \bigcup \{v_i | 1 \le i \le n\} \quad and$$
$$E\left(CH_n\right) = \{uu_i | 1 \le i \le n\} \bigcup \{u_i u_{i+1} | 1 \le i \le n-1\} \bigcup \{v_i v_{i+1} | 1 \le i \le n-1\} \bigcup \{u_n u_1\} \bigcup \{v_n v_1\} \bigcup \{u_i v_i | 1 \le i \le n\}$$

Then from Theorem 2.1, $egs\left(CH_n\right) \ge 2n$.

To prove the upper bound.

We define a function $f : V\left(CH_n\right) \bigcup E\left(CH_n\right) \to \{1, 2, \ldots, 2n\}$ as follows.

$$
\begin{aligned}
f(u) &= 2n \\
f\left(u_i\right) &= n+i, && 1 \le i \le n \\
f\left(v_i\right) &= i, && 1 \le i \le n \\
f\left(uu_i\right) &= 1, && 1 \le i \le n \\
f\left(u_i u_{i+1}\right) = f\left(v_i v_{i+1}\right) &= 2+i, && 1 \le i \le n-1 \\
f\left(u_n u_1\right) = f\left(v_n v_1\right) &= 2
\end{aligned}
$$

$$f(u_i v_i) = 1 + i, \qquad\qquad 1 \le i \le n$$

From the above labeling, we get,

$$\begin{aligned}
wt(uu_i) &= |f(u) + f(u_i) - f(uu_i)| \\
&= |2n + n + i - 1| \\
&= 3n + i - 1, \ 1 \le i \le n \\
wt(u_i u_{i+1}) &= |f(u_i) + f(u_{i+1}) - f(u_i u_{i+1})| \\
&= |n + i + n + i + 1 - (2 + i)| \\
&= 2n - 1 + i, \ 1 \le i \le n - 1 \\
wt(u_n u_1) &= |f(u_n) + f(u_1) - f(u_n u_1)| \\
&= |n + n + n + 1 - 2| \\
&= 3n - 1 \\
wt(v_i v_{i+1}) &= |f(v_i) + f(v_{i+1}) - f(v_i v_{i+1})| \\
&= |i + i + 1 - (2 + i)| \\
&= i - 1, \ 1 \le i \le n - 1 \\
wt(v_n v_1) &= |f(v_n) + f(v_1) - f(v_n v_1)| \\
&= |n + 1 - 2| \\
&= n - 1 \\
wt(u_i v_i) &= |f(u_i) + f(v_i) - f(u_i v_i)| \\
&= |n + i + i - (1 + i)| \\
&= n + i - 1, \ 1 \le i \le n.
\end{aligned}$$

The above labeling shows that, $egs(CH_n) \le 2n$. Combining this with the lower bound, we get, $egs(CH_n) = 2n$.

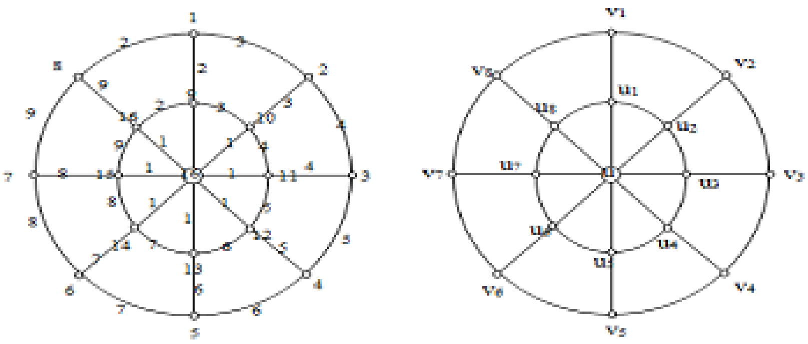

Figure 3. $egs(H_8) = 12$

□

Definition 2.8. *A flower graph Fl_n is the graph obtained from a helm graph by joining each pendent vertex to the central vertex of the helm graph.*

Theorem 2.9. *An edge graceful irregularity strength of Flower graph Fl_n, $n \geq 3$ is $2n$.*

Proof. Let

$$V(Fl_n) = \{u\} \bigcup \{u_i | 1 \leq i \leq n\} \bigcup \{v_i | 1 \leq i \leq n\} \quad and$$
$$E(Fl_n) = \{uu_i | 1 \leq i \leq n\} \bigcup \{u_i u_{i+1} | 1 \leq i \leq n-1\} \bigcup \{u_n u_1\} \bigcup \{uv_i | 1 \leq i \leq n\} \bigcup \{u_i v_i | 1 \leq i \leq n\}.$$

Then from Theorem 2.1, $egs(Fl_n) \geq 2n$.

To prove the upper bound.

We define a function $f : V(Fl_n) \bigcup E(Fl_n) \to \{1, 2, \ldots, 2n\}$ as follows.

$$\begin{aligned}
f(u) &= 2n \\
f(u_i) &= n + i, && 1 \leq i \leq n \\
f(v_i) &= 1, && 1 \leq i \leq n \\
f(uu_i) &= 1, && 1 \leq i \leq n \\
f(u_i u_{i+1}) &= 2 + i, && 1 \leq i \leq n-1 \\
f(u_n u_1) &= 2 \\
f(u_i v_i) &= n + 2, && 1 \leq i \leq n \\
f(uv_i) &= n + 2 - i, && 1 \leq i \leq n
\end{aligned}$$

From the above labeling, we get,

$$\begin{aligned}
wt(uu_i) &= |f(u) + f(u_i) - f(uu_i)| \\
&= |2n + n + i - 1| \\
&= 3n + i - 1, \ 1 \leq i \leq n \\
wt(u_i u_{i+1}) &= |f(u_i) + f(u_{i+1}) - f(u_i u_{i+1})| \\
&= |n + i + n + i + 1 - (2 + i)| \\
&= 2n - 1 + i, \ 1 \leq i \leq n - 1 \\
wt(u_n u_1) &= |f(u_n) + f(u_1) - f(u_n u_1)| \\
&= |n + n + n + 1 - 2| \\
&= 3n - 1 \\
wt(u_i v_i) &= |f(u_i) + f(v_i) - f(u_i v_i)| \\
&= |n + i + 1 - (n + 2)| \\
&= i - 1, \ 1 \leq i \leq n \\
wt(uv_i) &= |f(u) + f(v_i) - f(uv_i)| \\
&= |2n + 1 - (n + 2 - i)| \\
&= n + i - 1, \ 1 \leq i \leq n.
\end{aligned}$$

The above labeling shows that , $egs(Fl_n) \leq 2n$. Combining this with the lower bound, we get, $egs(Fl_n) = 2n$.

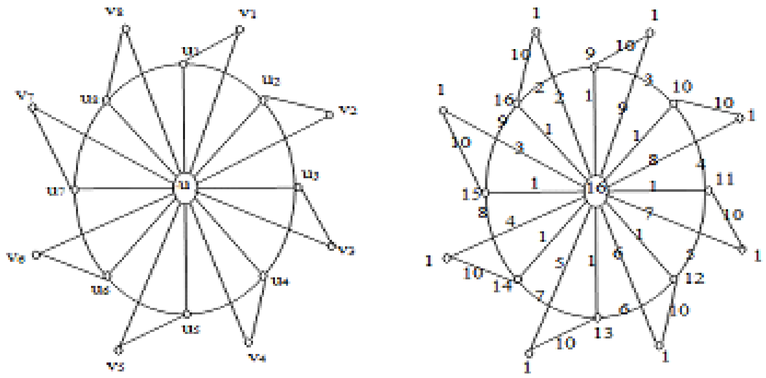

Figure 4. $egs\,(Fl_8) = 16$

$\square$

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