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Various Spectral Properties in the Banach Algebra $\mathcal{A} \times_c \mathcal{I}$ with the Convolution Product

H. V. Dedania[1,*] and H. J. Kanani[2,†]

1 Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, Gujarat, India.

2 Department of Mathematics, Bahauddin Science College, Junagadh, Gujarat, India.

Abstract: Let $\mathcal{I}$ be an ideal of an associative algebra $\mathcal{A}$ over the field $\mathbb{C}$. Then the vector space $\mathcal{A}\times\mathcal{I}$ with pointwise linear operations becomes an algebra with the product $(a,x)(b,y) = (ab+xy, ay+bx)$ $((a,x),(b,y) \in \mathcal{A}\times\mathcal{I})$. This product is known as the convolution product and this algebra is denoted by $\mathcal{A}\times_c\mathcal{I}$. In this paper, some well-known spectral properties of Banach algebras are studied for $\mathcal{A}\times_c\mathcal{I}$.

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1. Introduction

Throughout this paper, $\mathcal{A}$ is an algebra over the complex field $\mathbb{C}$ and $\mathcal{I}$ is an ideal in $\mathcal{A}$. Then the product $\mathcal{A}\times\mathcal{I}$ is an algebra with pointwise linear operations and the *convolution product* defined as

$$(a,x)(b,y) = (ab+xy, ay+bx) \qquad ((a,x),(b,y) \in \mathcal{A}\times\mathcal{I}).$$

This algebra will be denoted by $\mathcal{A}\times_c\mathcal{I}$. It should be noted that $\mathcal{A} \cong \mathcal{A}\times\{0\}$ is a closed subalgebra of $\mathcal{A}\times_c\mathcal{I}$, while $\mathcal{I}$ need not be even a subalgebra of $\mathcal{A}\times_c\mathcal{I}$. Thus the later algebra can be considered as an extension algebra of $\mathcal{A}$. Further, if $\mathcal{A}$ is a Banach algebra and the ideal $\mathcal{I}$ is closed in $\mathcal{A}$, then $\mathcal{A}\times_c\mathcal{I}$ is a Banach algebra with respect to the norm

$$\|(a,x)\|_1 = \|a\| + \|x\| \qquad ((a,x) \in \mathcal{A}\times_c\mathcal{I}).$$

Note that the linear norm $\|(a,x)\|_\infty = \max\{\|a\|,\|x\|\}$ $((a,x)\in\mathcal{A}\times_c\mathcal{I})$ is equivalent to the algebra norm $\|\cdot\|_1$, but it may not be submultiplicative. Thus $\|\cdot\|_\infty$ is a linear norm but it may not be an algebra norm on $\mathcal{A}\times_c\mathcal{I}$. For example, let $\mathcal{A} = C_0(\mathbb{R})$ and $\mathcal{I} = \{f\in\mathcal{A} : f(x) = 0\,(x\le 0)\}$. Then clearly $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ in $\|\cdot\|_\infty$. Define the function $f:\mathbb{R}\to\mathbb{R}$ as $f(x) = x(2-x)$ if $x\in[0,2]$ and 0 otherwise. Then $f\in\mathcal{I}$. Also, observe that $\|(f,f)^2\|_\infty = 2$ and $\|(f,f)\|_\infty^2 = 1$. i.e, $\|(f,f)^2\|_\infty > \|(f,f)\|_\infty^2$. Thus $\|\cdot\|_\infty$ is not submultiplicative on $\mathcal{A}\times_c\mathcal{I}$. In this article, we study the behavior of some

* E-mail: hvdedania@yahoo.com
† E-mail: hitenmaths69@gmail.com

spectral properties (also known as approximation properties) in the Banach algebra $\mathcal{A} \times_c \mathcal{I}$; namely, Topological divisor of zero, Quasi divisor of zero, Topological annihilator condition, Multiplicative Hahn-Banach property, Ditkin's condition, and Tauberian condition. The results proved here are part of the second author's Ph.D. thesis. We also note that the spectral properties studied in this paper are also studied in [3], but the Banach algebra therein is $\mathcal{A} \times_d \mathcal{I}$ with the direct-sum product.

2. Main Results

Throughout we assume that the Banach algebra $\mathcal{A}$ is always commutative and the ideal $\mathcal{I}$ is closed in $\mathcal{A}$. So that $\mathcal{A} \times_c \mathcal{I}$ is a commutative Banach algebra with the norm $\|\cdot\|_1$ which is defined above. Note that some concepts could be extendable to non-commutative Banach algebras also. However, we strictly restrict ourselves to the commutative case only.

Definition 2.1 ([5, Definition 1.6.1])**.** *A non-zero element $a \in \mathcal{A}$ is a* topological divisor of zero *if there exists a sequence (a_n) in $\mathcal{A}$ such that $\|a_n\| = 1$ $(n \in \mathbb{N})$ and $a_n a \longrightarrow 0$ as $n \longrightarrow \infty$. The Banach algebra $\mathcal{A}$ has* topological divisor of zero property *if every element of $\mathcal{A}$ is a topological divisor of zero.*

We should note that Bhatt and Dedania [1] have exhibited three classes of Banach algebras which have topological divisor of zero property. It follows from these results that if G is a non-discrete, LCA group, then $L^1(G)$ has topological divisor of zero property. Every non-unital, commutative C^*-algebra has this property. The next result guarantees that a Banach algebra with topological divisor of zero property can be extended to a larger Banach algebra with topological divisor of zero property keeping intact.

Theorem 2.2. *If $\mathcal{A}$ has topological divisor of zero property, then $\mathcal{A} \times_c \mathcal{I}$ has topological divisor of zero property.*

Proof. Let $(a, x) \in \mathcal{A} \times_c \mathcal{I}$ be a non-zero element. Then either $a \neq 0$ or $x \neq 0$. First, suppose that $a \neq 0$. Then, by the assumption, there exists a sequence (a_n) in $\mathcal{A}$ such that $\|a_n\| = 1$ $(n \in \mathbb{N})$ and $a_n a \longrightarrow 0$ as $n \longrightarrow \infty$. Then it is clear that $((a_n, 0))$ is a sequence in $\mathcal{A} \times_c \mathcal{I}$ such that $\|(a_n, 0)\|_1 = 1$ $(n \in \mathbb{N})$ and $(a_n, 0)(a, x) = (a_n a, a_n x) \longrightarrow (0, 0)$ as $n \longrightarrow \infty$. Next suppose that $a = 0$. Then we must have $x \neq 0$. Again by the assumption, there exists a sequence (x_n) in $\mathcal{A}$ such that $\|x_n\| = 1$ $(n \in \mathbb{N})$ and $x_n x \longrightarrow 0$ as $n \longrightarrow \infty$. Then $((x_n, 0))$ is a sequence in $\mathcal{A} \times_c \mathcal{I}$ such that $\|(x_n, 0)\|_1 = 1$ $(n \in \mathbb{N})$ and $(x_n, 0)(0, x) = (0, x_n x) \longrightarrow (0, 0)$ as $n \longrightarrow \infty$. Thus, in either case, (a, x) is a topological divisor of zero in $\mathcal{A} \times_c \mathcal{I}$. $\square$

Notations 2.3. *Let $\Delta(\mathcal{A})$ and $\partial\mathcal{A}$ denote the Gel'fand space and the Šilov boundary of $\mathcal{A}$, respectively. For $a \in \mathcal{A}$, the Gel'fand representation $\widehat{a}$ of the element a is a map $\widehat{a} : \Delta(\mathcal{A}) \longrightarrow \mathbb{C}$ defined as $\widehat{a}(\varphi) := \varphi(a)$. Let $\varphi \in \Delta(\mathcal{I})$ and $u \in \mathcal{I}$ such that $\varphi(u) = 1$. Then $\varphi^+, \varphi^- : \mathcal{A} \times_c \mathcal{I} \longrightarrow \mathbb{C}$ are defined as $\varphi^+((a, x)) = \varphi(au) + \varphi(x)$ and $\varphi^-((a, x)) = \varphi(au) - \varphi(x)$. It is easy to see that both φ^+ and φ^- are complex homomorphisms on $\mathcal{A} \times_c \mathcal{I}$, i.e., $\varphi^+, \varphi^- \in \Delta(\mathcal{A} \times_c \mathcal{I})$. Note that φ^+ and φ^- do not depend on the choice of u. Moreover, if $\mathcal{I} = \mathcal{A}$, then we do not need to choose u. For $F \subset \Delta(\mathcal{A})$, we define $F^+ = \{\varphi^+ : \varphi \in F\}$ and $F^- = \{\varphi^- : \varphi \in F\}$. Then $F^+, F^- \subset \Delta(\mathcal{A} \times_c \mathcal{I})$. If $F = \Delta(\mathcal{A})$ or $\partial\mathcal{A}$, then we shall write $\Delta^+(\mathcal{A})$ and $\partial^+\mathcal{A}$ for F^+. Similarly, we shall write $\Delta^-(\mathcal{A})$ and $\partial^-\mathcal{A}$ instead of $\Delta(\mathcal{A})^-$ and $\partial\mathcal{A}^-$.*

Definition 2.4 ([7, Definition 4, p-71])**.** *A Banach algebra $\mathcal{A}$ has* Quasi divisor of zero property *if there exists an open subset G of the Gel'fand space $\Delta(\mathcal{A})$ such that*

1. *The Šilov boundary $\partial\mathcal{A}$ is contained in the closure $\overline{G}$ of G.*

2. *For every open subset U of G, there exist an element $a \in \mathcal{A}$ and a non-empty open subset V of U such that*

$$\widehat{a}(\varphi) := \varphi(a) = \begin{cases} 0 & (\text{If } \varphi \in U^c) \\ 1 & (\text{If } \varphi \in V). \end{cases}$$

This property was first defined and studied by M. J. Meyer in his thesis [7].

Theorem 2.5. *The Banach algebra $\mathcal{A} \times_c \mathcal{I}$ has Quasi divisor of zero property iff $\mathcal{A}$ has Quasi divisior of zero property.*

Proof. Assume that $\mathcal{A} \times_c \mathcal{I}$ has Quasi divisor of zero property. Then there exists an open set $\widetilde{G} \subset \Delta(\mathcal{A} \times_c \mathcal{I})$ which satisfies the following properties.

1. $\partial(\mathcal{A} \times_c \mathcal{I}) \subset \overline{\widetilde{G}}$.
2. For every open subset $\widetilde{U}$ of $\widetilde{G}$, there exist $(a, x) \in \mathcal{A} \times_c \mathcal{I}$ and a non-empty open subset $\widetilde{V}$ of $\widetilde{U}$ such that

$$(a, x)^\wedge(\widetilde{\varphi}) = \begin{cases} 0 & (\text{If } \widetilde{\varphi} \in \widetilde{U}^c) \\ 1 & (\text{If } \widetilde{\varphi} \in \widetilde{V}). \end{cases}$$

Let $G_{\mathcal{A}} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{G}\}$, where φ^+ is defined above. Then $G_{\mathcal{A}}$ will be an open subset of $\Delta(\mathcal{A})$. Also, from (1) above, $\partial^+\mathcal{A} \subset \overline{G^+_{\mathcal{A}}}$ implies $\partial\mathcal{A} \subset \overline{G_{\mathcal{A}}}$. Now let $U \subset G_{\mathcal{A}}$ be open. Then U^+ is open in $\widetilde{G}$. Hence, by the hypothesis, there exist an element $(a, x) \in \mathcal{A} \times_c \mathcal{I}$ and a non-empty open set $V^+ \subset U^+$ such that $(a, x)^\wedge = 0$ on $(U^+)^c$ and $(a, x)^\wedge = 1$ on V^+. Now if $\varphi \in U^c$, then $\varphi^+ \in (U^+)^c$. Therefore, $(a + x)^\wedge(\varphi) = (a, x)^\wedge(\varphi^+) = 0$. Thus $(a + x)^\wedge|_{U^c} = 0$. On the other hand, if $\varphi \in V$, then $\varphi^+ \in V^+$. Therefore $(a + x)^\wedge(\varphi) = (a, x)^\wedge(\varphi^+) = 1$. Thus $(a + x)^\wedge|_V = 1$. Hence $\mathcal{A}$ has Quasi divisor of zero property.

Conversely, assume that $\mathcal{A}$ has Quasi divisor of zero property. Then there exists open subset $G \subset \Delta(\mathcal{A})$ satisfying the axioms of the definition of Quasi divisor of zero property. Let $\widetilde{G} = G^+ \cup G^-$. Then

$$\partial(\mathcal{A} \times_c \mathcal{I}) = \partial^+(\mathcal{A}) \cup \partial^-(\mathcal{I}) \subset \overline{G^+} \cup \overline{G^-} = \overline{G^+ \cup G^-} = \overline{\widetilde{G}}.$$

Let $\widetilde{U} \subset \widetilde{G}$ be open. Set $U_{\mathcal{A}} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{U} \text{ or } \varphi^- \in \widetilde{U}\}$. Then $U_{\mathcal{A}}$ is an open subset of G such that $U^+_{\mathcal{A}} \cup U^-_{\mathcal{A}} = \widetilde{U}$. Hence there exists $a \in \mathcal{A}$ such that $\widehat{a} = 0$ on $U^c_{\mathcal{A}}$ and $\widehat{a} = 1$ on some non-empty open subset $V_{\mathcal{A}} \subset U_{\mathcal{A}}$. Now, let $\widetilde{\eta} \in \widetilde{U}^c$. Then $\widetilde{\eta} = \varphi^+$ or $\widetilde{\eta} = \varphi^-$ for some $\varphi \in U^c_{\mathcal{A}}$. If $\widetilde{\eta} = \varphi^+$, then $(a, 0)^\wedge(\widetilde{\eta}) = (a, 0)^\wedge(\varphi^+) = \widehat{a}(\varphi) = 0$. If $\widetilde{\eta} = \varphi^-$, then also $(a, 0)^\wedge(\widetilde{\eta}) = 0$. Thus $(a, 0)^\wedge = 0$ on $\widetilde{U}^c$. Similarly, $(a, 0)^\wedge = 1$ on $\widetilde{V}$. Hence $\mathcal{A} \times_c \mathcal{I}$ has Quasi divisor of zero property. □

Definition 2.6 ([7])**.** *Let $\mathcal{I}$ be an ideal in a Banach algebra $\mathcal{A}$. A* separating net *for $\mathcal{I}$ is a net $(q_\lambda)_{\lambda\in\Lambda}$ of quasi divisors of zeros in $\mathcal{A}$ such that*

1. $\sup\{r_{\mathcal{A}}(q_\lambda) : \lambda \in \Lambda\} < \infty$;
2. $\lim_{\lambda\to\infty} r_{\mathcal{A}}(aq_\lambda) = 0 \ \ (a \in \mathcal{I})$;
3. *There exists an element $b \in \mathcal{A}$ such that $q_\lambda b = q_\lambda$ $(\lambda \in \Lambda)$.*

Definition 2.7 ([7])**.** *A Banach algebra $\mathcal{A}$ satisfies* topological annihilator condition *if there exists a dense set $D \subset \partial\mathcal{A}$ such that, for every $\varphi \in D$, the maximal ideal $\ker\varphi$ contains a separating net.*

The concepts of "separating net" and "topological annihilator" were also defined and studied by M. J. Meyer [7]. The next result enables us to construct a larger Banach algebra satisfying topological annihilator condition.

Theorem 2.8. *The Banach algebra $\mathcal{A} \times_c \mathcal{I}$ satisfies the topological annihilator condition if and only if $\mathcal{A}$ satisfies the topological annihilator condition.*

Proof. Assume that $\mathcal{A}\times_c\mathcal{I}$ satisfies the topological annihilator condition. Then there exists a dense subset $\widetilde{D} \subset \partial(\mathcal{A}\times_c\mathcal{I})$ such that $\ker\widetilde{\eta}$ admits a separating net for each $\widetilde{\eta} \in \widetilde{D}$. Set $D_{\mathcal{A}} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{D}$ or $\varphi^- \in \widetilde{D}\}$. Then D is a dense subset of $\partial\mathcal{A}$. Let $\varphi \in D_{\mathcal{A}}$. Then either $\varphi^+ \in \widetilde{D}$ or $\varphi^- \in \widetilde{D}$. If $\varphi^+ \in \widetilde{D}$, then $\ker(\varphi^+)$ admits a separating net, say $((a_\lambda, x_\lambda))_{\lambda\in\Lambda}$. Then $(a_\lambda + x_\lambda)$ is a separating net for $\ker\varphi$. If $\varphi^- \in \widetilde{D}$, then $\ker(\varphi^-)$ admits a separating net, say $((b_\lambda, y_\lambda))_{\lambda\in\Lambda}$. In that case, $(b_\lambda - y_\lambda)$ is a separating net for $\ker\varphi$. Thus $\mathcal{A}$ satisfies the topological annihilator condition.
Conversely, assume that $\mathcal{A}$ satisfies the topological annihilator condition. Then there exists a dense subset $D \subset \partial\mathcal{A}$ such that $\ker\varphi$ admits a separating net for each $\varphi \in D$. Then $\widetilde{D} = D^+ \cup D^-$ will be a dense subset of $\partial(\mathcal{A}\times_c\mathcal{I})$. Let $\widetilde{\eta} \in \widetilde{D}$. Then either $\widetilde{\eta} = \varphi^+$ or $\widetilde{\eta} = \varphi^-$ for some $\varphi \in D$. Suppose that $\widetilde{\eta} = \varphi^+$ for some $\varphi \in D$. Then, by the hypothesis, $\ker\varphi$ admits a separating net, say $(a_\lambda)_{\lambda\in\Lambda}$. Then $((a_\lambda, 0))$ is a separating net for the maximal ideal $\ker(\varphi^+) = \ker\widetilde{\eta}$. Similarly, if $\widetilde{\eta} = \varphi^-$ for some $\varphi \in D$, then $\ker\varphi$ admits a separating net, say $(b_\lambda)_{\lambda\in\Lambda}$. In this case also, $((b_\lambda, 0))_{\lambda\in\Lambda}$ is a separating net for the maximal ideal $\ker(\varphi^-) = \ker\widetilde{\eta}$. Thus, in each case, $\ker\widetilde{\eta}$ admits a separating net. So $\mathcal{A}\times_c\mathcal{I}$ satisfies the topological annihilator condition. ☐

Definition 2.9. *A commutative Banach algebra $\mathcal{B}$ is said to be a* commutative extension *of $\mathcal{A}$ if the algebra $\mathcal{A}$ is a (not necessarily closed) subalgebra of $\mathcal{B}$.*

Definition 2.10 ([4])**.** *A commutative Banach algebra $\mathcal{A}$ has* multiplicative Hahn-Banach property *if, for every commutative extension $\mathcal{B}$ of $\mathcal{A}$, each $\varphi \in \Delta(\mathcal{A})$ can be extended to some element of $\Delta(\mathcal{B})$.*

The concept of "multiplicative Hahn-Banach property" was also introduced and extensively studied by M. J. Meyer [7]. He has proved that a semisimple, commutative, Banach algebra has multiplicative Hahn-Banach property if and only if it has spectral extension property and the Šilov boundary coincides with the Gel'fand space. In particular, every semisimple, regular, commutative, Banach algebra has this property.

Theorem 2.11. *The Banach algebra $\mathcal{A}\times_c\mathcal{I}$ has multiplicative Hahn-Banach property if and only if $\mathcal{A}$ has multiplicative Hahn-Banach property.*

Proof. Assume that $\mathcal{A}\times_c\mathcal{I}$ has multiplicative Hahn-Banach property. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A}$. Then $\mathcal{C}\times_c\mathcal{C}$ is a commutative extension of $\mathcal{A}\times_c\mathcal{I}$. Let $\varphi \in \Delta(\mathcal{A})$. Then $\varphi^+ \in \Delta(\mathcal{A}\times_c\mathcal{I})$. Since $\mathcal{A}\times_c\mathcal{I}$ has has multiplicative Hahn-Banach property, φ^+ can be extended to an element $\widetilde{\eta} \in \Delta(\mathcal{C}\times_c\mathcal{C}) = \Delta^+(C) \uplus \Delta^-(C)$ - the disjoint union of $\Delta^+(C)$ and $\Delta^-(C)$ equipped with the sum topology. Define $\overline{\varphi}(c) = \widetilde{\eta}((c, 0))\,(c \in \mathcal{C})$. Then *Opt. Lett.* $\varphi \in \Delta(C)$ and $\overline{\varphi} = \varphi$ on $\mathcal{A}$. Thus, $\overline{\varphi}$ is an extension of φ. Hence, $\mathcal{A}$ has multiplicative Hahn-Banach property.
Conversely, assume that $\mathcal{A}$ has multiplicative Hahn-Banach property. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A}\times_c\mathcal{I}$. Then $\mathcal{C}$ will be a commutative extension of $\mathcal{A}$. Let $\widetilde{\eta} \in \Delta(\mathcal{A}\times_c\mathcal{I})$. Then $\widetilde{\eta} \in \Delta^+(\mathcal{A})$ or $\widetilde{\eta} \in \Delta^-(\mathcal{I})$. First, suppose that $\widetilde{\eta} \in \Delta^+(\mathcal{A})$. Then $\widetilde{\eta} = \varphi^+$ for some $\varphi \in \Delta(\mathcal{A})$. Since $\mathcal{C}$ is a commutative extension of $\mathcal{A}$, by the hypothesis, φ can be extended to some element $\overline{\varphi}$ of $\Delta(\mathcal{C})$. Then $\widetilde{\eta}((a, x)) = \varphi^+((a, x)) = \varphi(a) + \varphi(x) = \overline{\varphi}(a) + \overline{\varphi}(x) = \overline{\varphi}^+((a, x)) = \overline{\varphi}((a, x))$ because $\overline{\varphi} \in \Delta(\mathcal{C})$ and $\mathcal{A}\times_c\mathcal{I} \subset \mathcal{C}$. Next, suppose that $\widetilde{\eta} \in \Delta^-(\mathcal{I})$. Then $\widetilde{\eta} = \varphi^-$ for some $\varphi \in \Delta(\mathcal{I})$. So, by the assumption, there exists $\overline{\varphi} \in \Delta(\mathcal{C})$ such that $\overline{\varphi} = \varphi$ on $\mathcal{A}$. Now, $\widetilde{\eta}((a, x)) = \varphi^-((a, x)) = \varphi(a) - \varphi(x) = \overline{\varphi}(a) - \overline{\varphi}(x) = \overline{\varphi}^-((a, x)) = \overline{\varphi}((a, x))$. Thus $\widetilde{\eta} = \overline{\varphi}$. In both cases, we get an extension of $\widetilde{\eta}$ over $\mathcal{C}$. Therefore $\mathcal{A}\times_c\mathcal{I}$ has multiplicative Hahn-Banach property. ☐

Definition 2.12 ([5, Definition 8.5.1])**.** *Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ satisfies*

1. Ditkin's condition at $\varphi \in \Delta(\mathcal{A})$ *if, for every $a \in \ker\varphi$, there exists a sequence (a_n) in $\mathcal{A}$ such that $\widehat{a_n} \in C_c(\Delta(\mathcal{A}))$, $\varphi \notin supp\widehat{a_n}$ and $a_n a \longrightarrow a$ as $n \longrightarrow \infty$.*

2. Ditkin's condition at infinity *if, for every* $a \in \mathcal{A}$*, there exists a sequence* (a_n) *in* $\mathcal{A}$ *such that* $\widehat{a_n} \in C_c(\Delta(\mathcal{A}))$ *and* $a_n a \longrightarrow a$ *as* $n \longrightarrow \infty$.

3. Ditkin's condition *if* $\mathcal{A}$ *satisfies the Ditkin's condition at every* $\varphi \in \Delta(\mathcal{A})$ *as well as at infinity.*

A Banach algebra $\mathcal{A}$ satisfying Ditkin's condition is also called a Ditkin's algebra. We should note that both the definitions of Ditkin algebras given above as well as in [2, Definition 4.1.31] are equivalent. Every Ditkin algebra is a regular algebra [2, Page 417]. Thus the class of Ditkin algebras is smaller than the class of regular Banach algebras. The commutative group algebra $L^1(G)$ and every commutative C^*-algebra are Ditkin algebras. The readers should refer to [2] for more on this concept. It would be interesting to know the converse of the following result.

Theorem 2.13. *If the Banach algebra* $\mathcal{A} \times_c \mathcal{I}$ *satisfies the Ditkin's condition, then* $\mathcal{A}$ *also satisfies the Ditkin's condition.*

Proof. Suppose that $\mathcal{A} \times_c \mathcal{I}$ satisfies the Ditkin's condition. Let $\varphi \in \Delta(\mathcal{A})$ and $a \in \ker\varphi$. Then $\varphi^+ \in \Delta(\mathcal{A} \times_c \mathcal{I})$ and $(a, 0) \in \ker(\varphi^+)$. Since $\mathcal{A} \times_c \mathcal{I}$ satisfies the Ditkin's condition at φ^+, there exists a sequence $((a_n, x_n))$ in $\mathcal{A} \times_c \mathcal{I}$ such that $(a_n, x_n)^\wedge \in C_c(\Delta(\mathcal{A} \times_c \mathcal{I}))$, $\varphi^+ \notin supp(a_n, x_n)^\wedge$ and $(a_n, x_n)(a, 0) \longrightarrow (a, 0)$ as $n \longrightarrow \infty$. But then $(a_n + x_n)$ is a sequence in $\mathcal{A}$ such that $(a_n + x_n)^\wedge \in C_c(\Delta(\mathcal{A}))$, $\varphi \notin supp(a_n + x_n)^\wedge$ and $(a_n + x_n)a \longrightarrow a$ as $n \longrightarrow \infty$. Thus $\mathcal{A}$ satisfies the Ditkin's condition at every point $\varphi \in \Delta(\mathcal{A})$.
Next we show that $\mathcal{A}$ satisfies Ditkin's condition at infinity. Let $a \in \mathcal{A}$. Since $\mathcal{A} \times_c \mathcal{I}$ satisfy the Ditkin's condition at infinity, there exists a sequence (a_n, x_n) in $\mathcal{A} \times_c \mathcal{I}$ such that $(a_n, x_n)^\wedge \in C_c(\Delta(\mathcal{A} \times_c \mathcal{I}))$ and $(a_n, x_n)(a, 0) \longrightarrow (a, 0)$ as $n \longrightarrow \infty$. But then $(a_n + x_n)$ is a sequence in $\mathcal{A}$ such that $(a_n + x_n)^\wedge \in C_c(\Delta(\mathcal{A}))$ and $(a_n + x_n)a \longrightarrow a$ as $n \longrightarrow \infty$. Thus $\mathcal{A}$ satisfies the Ditkin's condition at infinity. This completes the proof. □

Lemma 2.14. *Let* $(a, x) \in \mathcal{A} \times_c \mathcal{I}$*. Then* $(a, x)^\wedge \in C_c(\Delta(\mathcal{A} \times_c \mathcal{I}))$ *iff both* $(a + x)^\wedge$ *and* $(a - x)^\wedge$ *belong to* $C_c(\Delta(\mathcal{A}))$*.*

Proof. This follows using $supp(a, x)^\wedge = supp(a + x)^\wedge \cup supp(a - x)^\wedge$. □

Definition 2.15 ([5, Definition 8.1.2])**.** *A commutative Banach algebra* $\mathcal{A}$ *is said to be* a Tauberian algebra *if the set* $\{a \in \mathcal{A} : \widehat{a} \in C_c(\Delta(\mathcal{A}))\}$ *is dense in* $\mathcal{A}$*.*

If $\mathcal{A}$ is unital, then every proper, closed ideal in $\mathcal{A}$ is contained in some maximal regular ideal in $\mathcal{A}$. In general, this is not true. However, if a regular, semisimple, commutative Banach algebra $\mathcal{A}$ is a Tauberian algebra, then every proper, closed ideal in $\mathcal{A}$ is contained in some maximal, regular ideal in $\mathcal{A}$ [5, Corollary 8.1.1]. In particular, the group algebra $L^1(G)$ and the C^*-algebra $C_0(X)$ are Tauberian algebras. The following result supplies some more examples of Tauberian algebras.

Theorem 2.16. *The Banach algebra* $\mathcal{A} \times_c \mathcal{I}$ *is a Tauberian algebra if and only if* $\mathcal{A}$ *is a Tauberian algebra.*

Proof. Assume that $\mathcal{A} \times_c \mathcal{I}$ is a Tauberian algebra. Let $a \in \mathcal{A}$ and $\epsilon > 0$. Since the Banach algebra $\mathcal{A} \times_c \mathcal{I}$ is a Tauberian algebra, there exits $(a_0, x_0) \in \mathcal{A} \times_c \mathcal{I}$ such that $(a_0, x_0)^\wedge \in C_c(\Delta(\mathcal{A} \times_c \mathcal{I}))$ and $\|(a, 0) - (a_0, x_0)\|_1 = \|a - a_0\| + \|x_0\| < \epsilon$. Then, by Lemma 2.14, $(a_0 + x_0)^\wedge \in C_c(\Delta(\mathcal{A}))$ and $\|(a_0 + x_0) - a\| \leq \|a - a_0\| + \|x_0\| < \epsilon$. So $\mathcal{A}$ is a Tauberian algebra.
Conversely, suppose that $\mathcal{A}$ is a Tauberian algebra. Let $(a, x) \in \mathcal{A} \times_c \mathcal{I}$ and $\epsilon > 0$ be arbitrary. Since $\mathcal{A}$ is a Tauberian algebra, there exist $a_0, x_0 \in \mathcal{A}$ such that $\widehat{a_0}, \widehat{x_0} \in C_c(\Delta(\mathcal{A}))$, $\|a - a_0\| < \epsilon/2$ and $\|x - x_0\| < \epsilon/2$. Since $\widehat{a_0}, \widehat{x_0} \in C_c(\Delta(\mathcal{A}))$, by Lemma 2.14, we have $(a_0, x_0)^\wedge \in C_c(\Delta(\mathcal{A} \times_c \mathcal{I}))$. Then, by the definition of $\|\cdot\|_1$, we have $\|(a, x) - (a_0, x_0)\|_1 = \|a - a_0\| + \|x - x_0\| < \epsilon$. Thus $\mathcal{A} \times_c \mathcal{I}$ is a Tauberian algebra. □

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