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Fixed Point Results for Single Self Maps in Probabilistic Metric Space

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Abstract: This paper deals with some fixed-point results for single self-maps in probabilistic metric space which is based on two contraction conditions, one is B-contraction and another is C- contraction in Probabilistic metric space.

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1. Introduction

Stefen Banach Proved a fixed-point theorem in 1922, which ensures under appropriate conditions, the existence and uniqueness of a fixed point. This result is called Banach Fixed point theorem or Banach contraction Principle. Many authors like A.T. Bharucha- Reid and V.M. Sehgal [1] and T.L.Hicks [2] have extended , generalized and improved the Banach contraction. Using B-contraction and C- contraction established so many fixed-point results in Probabilistic Metric space. K. Menger [3] introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has been developed in many directions, especially in nonlinear analysis and applications. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. B. Schweizer and A. Sklar [5] studied this concept and gave some fundamental results on this space. The important development of fixed point theory in PM space was due to Sehgal and Bharucha and Hicks. The sub division of this paper as follows: In section 2, some related notions and concept in probabilistic Metric space, and probabilistic contractions are recalled. In section 3, Some results in PM space in single maps are stated as main results.

2. Preliminary Notes

Definition 2.1. *Metric space is a pair (S, d), where S is a non-empty set and d is a distance function or metric of the space defined by $d : S \times S \to [0, \infty)$, satisfies the following conditions:*

(1). $d(p, q) = 0$ if $p = q$ (Indiscrinibles)

(2). $d(p, q) > 0$ if $p \neq q$ (Positivity)

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(3). $d(p,q) = d(q,p), \forall\ p,q \in S$ *(Symmetry)*

(4). $d(p,r) \leq d(p,q) + d(q,r), \forall\ p,q,r \in S$ *(Triangle Inequality)*

Example 2.2. *Let X be a non-empty set. For $x, y \in X$, we define*

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Then, d is discrete metric and the space (X, d) is discrete metric space.

Definition 2.3 ([4]). *Let $f : X \to X$ be a map. Then, an element $x \in X$ is said to be fixed point of f if $f(x) = x$. Geometrically, the fixed point of a function $f(x)$ are the point of intersection of the curve $y = f(x)$ and the line $y = x$.*

Example 2.4. *Let $y = f(x) = x^3 - 4x^2 + x + 6 = 0$, cubic equation. Then, it can be transferred to as*

$$x = f(x) = \frac{x^3 + 6}{4x - 1}$$

Here, $f(-1) = -1$, $f(2) = 2$ & $f(3) = 3$. So, by definition $x = -1, x = 2$ and $x = 3$ are fixed points of f.

Definition 2.5 ([4]). *Let (X, d) be a metric space and let $f : X \to X$ be a mapping. Then, f is called contraction if there exists a fixed constant $h \in [0, 1)$, such that*

$$d(f(x), f(y)) \leq hd(x,y), \forall\ x,y \in X$$

Example 2.6. *Let $f\,[0,2] \to [0,2]$ be defined by,*

$$f(x) = \begin{cases} 0 & x \in [0,1] \\ 1 & x \in (1,2] \end{cases}$$

Then, $f^2(x) = 0$ for all $x \in [0,\ 2]$. So, f^2 is a contraction on $[0, 2]$. But f is not continuous and thus not a contraction map.

Definition 2.7 ([4]). *For the set $\mathbb{R}$ of real numbers, a function $F : \mathbb{R} \to [0, 1]$ is called a distribution function if*

(1). F is non-decreasing,

(2). F is left continuous, and

(3). $\inf\limits_{x\in\mathbb{R}} F(x) = 0$ *and* $\sup\limits_{x\in\mathbb{R}} F(x) = 1$.

If X is a non-empty set, $F : X \times X \to \Delta$ is called probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}. We will denote by Δ the family of all distribution function on $(-\infty, \infty)$ and Δ^+ on $[0, \infty)$.

Example 2.8. *Let H is a maximal element for Δ^+ then, distribution function H is defined by*

$$H(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

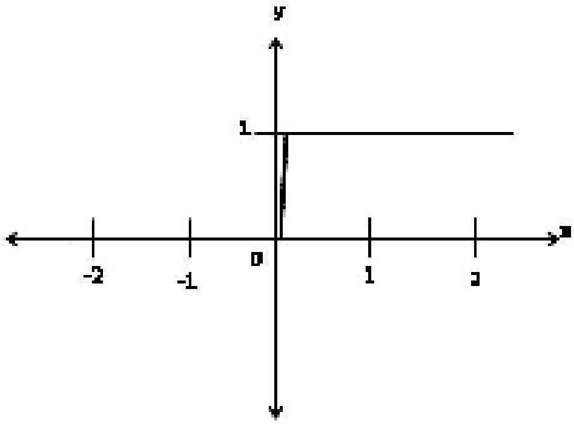

Figure 1. **Distribution Function**

Definition 2.9 ([5])**.** *A probabilistic metric space (brief, PM-space) is an order pair (X, F) where X is a non-empty set and F is a function defined by $F : X \times X \to \Delta^+$ (the set of all distribution functions) that is F associates a distribution function $F(p, q)$ with every pair (p, q) of points in X. The distribution function $F(p, q)$ is denoted by $F_{p,q}$, whence the symbol $F_{p,q}(x)$ will represent the value of $F_{p,q}$ at $x \in \mathbb{R}$. And the function $F_{p,q},\ p, q \in X$ are assumed to satisfy following conditions:*

(1). $F_{p,q}(0) = 0$;

(2). $F_{p,q} = F_{q,p}$,

(3). $F_{p,q}(x) = 1$, for every $x > 0 \Leftrightarrow p = q$.

(4). For every $p, q, r \in X$ and for every

$$x, y > 0,\ \ F_{p,q}(x) = 1,\ \ F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x + y) = 1$$

The interpretation of $F_{p,q}(x)$ as the probability that the distance from p to q is less than x, it is clear that PM condition (3), (1) and (2) are straight forward generalizations of the corresponding metric space conditions (1), (2) and (3). The PM condition (4) is a 'minimal' generalization of the triangle inequality of metric space condition (4). If it is certain that the distance of p and q is less than x, and like wise certain that the distance of q and r is less than y, then it is certain that the distance of p and r is less than $x + y$. The PM condition (iv) is always satisfied in metric spaces, where it reduces to the ordinary triangle inequality.

Definition 2.10 ([6])**.** *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (shortly t-norm) if for all $a, b, c, d, \in [0, 1]$ the following conditions are satisfied:*

(1). $T(a, 1) = a$ for every $a \in [0, 1]$, (Neutral Element 1)

(2). $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$, (Commutativity)

(3). $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ (Monotonicity)

(4). $T(a, T(b, c)) = T(T(a, b), c))(a, b, c \in [0, 1])$ (Associativity).

Example 2.11. *Example of t-norms $T(a, b) = \max\{(a + b) - 1, 0\}$ and $T(a, b) = \min\{a, b\}$. The four basic standard t-norms are:*

(1). The minimum t-norm, T_M, is defined by $T_M(x, y) = \min\{x, y\}$,

(2). The product t-norm, T_p, is defined by $T_p(x, y) = x, y$,

(3). The Lukasiewicz t-norm, T_L, is defined by $T_L(x, y) = \max\{x + y - 1, 0\}$,

(4). The weakest t-norm, the drastic product, T_D, is defined by

$$T_D(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

With references to the point wise ordering, we have the following inequalities $T_D < T_L < T_P < T_M$.

Definition 2.12 ([5]). *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (S, F, T), where (S, F) is a probabilistic metric space, T is a triangular norm and also satisfies the following conditions, for all $x, y, z \in X$ and $t, s > 0$, (v) $F_{xy}(t + s) \geq T(F_{xz}(t), F_{zy}(s))$. This is the extension of triangle inequality. This inequality is called Menger's triangle inequality.*

Example 2.13. *Let $X = \mathbb{R}$, $a * b = \min(a, b)$ $\forall\ a, b \in (0, 1)$ and*

$$f_{u,w}(x) = \begin{cases} H(x) & \text{for } u \neq v \\ 1 & \text{for } u = v \end{cases}$$

where

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 0 \end{cases}$$

then $(X, F, *)$ is Menger Space.

Definition 2.14 ([4]). *Let (X, F, T) be a Menger Space and T be a continuous t-norm*

(1). A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \to x$) iff for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N$.

(2). A sequence $\{x_n\}$ in X is called a Cauchy if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$.

(3). A Menger space in which every Cauchy sequence is convergent is said to be Complete Menger Space.

2.1. Banach Contraction Condition in Metric Space

The most basic fixed-point theorem in analysis known as the Banach Contraction Principle (BCP). It is due to S. Banach and appeared in his Ph.D. thesis (1920, published in 1922). The BCP was first stated and proved by Banach for the Contraction maps in setting of complete normed linear spaces. At about the same time the concept of an abstract metric space was introduced by Hausdorff for the set valued mappings, which then provided the general framework for the principle for contraction mappings in a complete metric space. The BCP can be applied to mappings which are differentiable, or more generally, Lipschitz continuous.

Theorem 2.15. *Let (X, d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.*

Example 2.16. *$T:\mathbb{R}\to\mathbb{R}$, $T(x)=\frac{x}{2}+3$, $x\in\mathbb{R}$. Obviously T is a Banach contraction and Fix$(T)=\{6\}$ where Fix(T) denotes the fixed point of the mapping T.*

The following definition of a contraction mapping was suggested and studied by V.M. Seghal and A.T. Bharucha-Reid in 1972, which is very natural probabilistic version of the notion of Banach contraction in metric space.

Definition 2.17 ([1]). *Let $(X,\ F)$ be a probabilistic metric space. A mapping $T:X\to X$ is a contraction mapping (or a SB - Contraction mapping or B-contraction) on (X,F) if and only if there is a $k\in(0,1)$ such that*

$$F_{Tp,Tq}(t)\geq F_{p,q}(t/k), \tag{1}$$

where $p,q\in X$ and $t>0$. It is also known as probabilistic k-contraction.

The geometrical interpretation expression (1) is that the probability that the distance between the image points F_p, F_q being less than kt, is at least equal to the probability that the distance between p,q that is less than t. T.L. Hicks in 1996, defined the following C-contraction mapping in PM space.

Definition 2.18 ([2]). *Let (X,T) be a probabilistic metric space and $T:X\to X$. The mapping T is called Hicks C-contraction (or, C-contraction) if there exists $k\in(0,1)$ such that the following implication holds for every $p,q\in X$: and for every $t>0$*

$$T_{pq}(t)>1-t\Rightarrow T_{T(p)T(q)}(kt)>1-kt.$$

D.Mihet in 2005, introduced the weak- hicks contraction in PM Space as follows:

Definition 2.19 ([7]). *Let S be a nonempty set and F be a probabilistic distance on S. A mapping $f:S\to S$ is said to be weak - Hicks contraction (w-H contraction) if there exists $k\in(0,1)$ such that, for all $p,q\in S$.*

$$(w-H):t\in(0,1)\,,F_{pq}(t)>1-t\Rightarrow F_{f(p)f(q)}(kt)>1-kt.$$

Example 2.20. *Let $X=[0,\infty)$ and*

$$F_{xy}(t)=\frac{\min(x,y)}{\max(x,y)},\ \forall\ t\in(0,\infty)\,,\ \forall\ x,y\in X,\ x\neq y.$$

It is known that (X,F,T) is a complete Menger space under the triangular norm $T=T_p>T_L$. Also, it can easily be seen that the mapping $g:X\to X$,

$$g(x)=\begin{cases}0 & \text{if } x=0\\ 1 & \text{if } x>0\end{cases}$$

is a w-H contraction for every $k\in(0,1)$.
As a generalization of the notion of a probabilistic B-contraction, we shall introduce the notion of a probabilistic (m,k)-B-contraction where $m\geq1$ and $k\in(0,1)$.

Definition 2.21 ([6]). *If (S,F) is a PM - space, $m\geq1$ and $k\in(0,1)$, a function $f:S\to S$ is called probabilistic (m,k)-B-contraction if for any $p,q\in S$ there is an i with $1\leq i\leq m$ such that for every $t>0$,*

$$F_{f^ip,f^iq}\left(k^it\right)\geq F_{p,q}(t)\,.$$

If $m=1$ and $k\in(0,1)$ then a probabilistic $(1-k)$-B-contraction f is a probabilistic B-contraction.

As a generalization of C-contraction, we have

Definition 2.22 ([6]). *If (S, φ) is a PM-space, $m \geq 1$ and $k \in (0,1)$, a function $f : S \to S$ is called a (m,k)-C-contraction if for any $p, q \in S$ there is an i with $l \leq i \leq m$ such that for every $t > 0$.*

$$F_{p,q}(t) > 1 - t \Rightarrow F_{f^i p, f^i q}\left(k^i t\right) > 1 - k^i t.$$

If $m = 1$ and $k \in (0,1)$ then a probabilistic $(1,k)$-C-contraction f is a probabilistic C-contraction.

Definition 2.23 ([2]). *Let (S, F) be a Probabilistic Metric Space, $\varphi \in \emptyset$ and $k \in (0,1)$ be given. A mapping $f : S \to S$ is called a $(\varphi - k) - B$ contraction on S if the following condition hold*

$$x, y \in S, \varepsilon \in (0,1), \lambda \in (0,1), F_{x,y}(\epsilon) > 1 - \lambda \Rightarrow F_{f(x),f(y)}(k\varepsilon) > 1 - \varphi(\lambda).$$

Definition 2.24 ([8]). *Let F be a probabilistic distance on S. A mapping $f : S \to S$ is called continuous if for every $\varepsilon > 0$ there exist $\delta > 0$ such that $F_{u,v}(\delta) > 1 - \delta \Rightarrow F_{fu,f_v}(\varepsilon) > 1 - \varepsilon$.*

3. Main Results

This section consists fixed point results in single maps in Probabilistic Metric Space. In 1972, V.M. Sehgal and A.T. Bharucha-Reid extended the famous Banach Contraction Principle in Probabilistic Metric Space as follows:

Theorem 3.1 ([1]). *Let (X, T, Δ) be a complete Menger space, where Δ is a continuous function satisfying $\Delta(x,x) \geq x$ for each $x \in [0,\ 1]$, if T is any contraction mapping of X into itself, then there is unique $p \in X$ such that $Tp = p$. Moreover, $T^n q \to p$ for each $q \in X$.*

Theorem 3.2 ([4]). *Let (X, F, Δ) be a complete Menger space, Δ a t-norm of H-type and $f : X \to X$ a probabilistic q-contraction. Then there exists a unique Fixed point $x \in X$ of the mapping f and $x = \lim\limits_{x \to \infty} f^n p$ for every $p \in X$.*

Theorem 3.3 ([2]). *Let $(X, T, \min)$ be a complete Menger Probabilistic Metric Space. $T : X \to X$ be a contraction mapping for every $p, q \in X$, $k \in (0,1)$ and for every $t > 0$*

$$T_{pq}(t) > 1 - t \Rightarrow T_{T(p)T(q)}(kt) > 1 - kt$$

Then, T has a fixed point.

Theorem 3.4 ([9]). *Let $(X; d)$ be a complete metric space and $T : X \to X$ be a mapping satisfying the following condition: There exists a constant k, $0 < k < 1$, such that $d(Tp, Tq) \leq kd(p; q)$; $p, q \in X$. Then, T has a Fixed point $p* \in X$, and for any $p_0 \in X$; $T^n P_0 \to p*$.*

Theorem 3.5 ([10]). *The Mapping $f : X \to X$ is an H-contraction on the PM space $((X, F, \tau)$ with $\tau \geq \tau_M$ if and only if there is a $\gamma \in (0,1)$ such that $\beta(f_p, f_q) \leq \gamma\beta(p,q) \quad \forall \quad p, q \in X$.*

Theorem 3.6 ([7]). *Let (X, F, T) be a complete Menger space with $\Delta \geq \Delta\ L$ or T is of Hadzic type and let us suppose that $f : X \to X$ is a weak Hicks-contraction with the property that $Fpf(p)(t) > 0$ for some $p \in X$ and some $t \in (0;1)$. Then f has a Fixed point.*

Theorem 3.7 ([10]). *Let (S, F, T) be a complete Menger space such that Range $(F) \subset D^+$ and $\sup a < 1T(a;a) = 1$. Then, every C-contraction f on S has unique fixed point which is the limit of the sequence $(f^n(p))n \in N$ for every $p \in S$.*

Theorem 3.8 ([6]). *Let (S, F, T) be a complete Menger space with t-norm T such that $\sup a < 1t(a,a) = 1$ and let $f : S \to S$ be a generalized C-contraction such that $h(0) \in R$. Then $x = \lim_{n\to\infty} f^n p$ is the unique fixed point of the mapping f for an arbitrary $p \in S$.*

Theorem 3.9 ([11]). *Let $(X, \varphi, \min)$ be a complete Menger Space and $f : S \to S$ a probabilistic (m, k)-B-contraction mapping, then f has a fixed point.*

Lemma 3.10 ([11]). *Let $(S, \varphi, \min)$ be a Menger space and $f : S \to S$ a probabilistic (m, k)-B-contraction. If $x \in S$ is such that $f^n p = p$ for some $n \geq 1$, then $fp = p$.*

Theorem 3.11 ([12]). *Let (S, F, T) be a complete Menger Space, T be a t-norm such that $\sup 0 \leq t < 1$, $T(t, t) = 1$ and $f : S \to S$ a $(\varphi - k)$-B-contraction if $\lim_{t\to\infty} F_{x_0 f^m_{x_0}}(t) = 1$ for some $x_0 \in S$, $m \in N$, then there exist a unique fixed point $x = \lim_{n\to\infty} f^n_{x_0}$.*

Theorem 3.12 ([12]). *Let (S, F, T) be a complete Menger Space, T be a t-norm such that $\sup 0 \leq t < 1$, $T(t, t) = 1$ and $f : S \to S$ a $(\varphi - k)$-B-contraction if $\lim_{t\to\infty} F_{x_0 f^m_{x_0}}(t) = 1$ for some $p \in S$, and $j > 0$, $\sup_{x>j} x^j(1 - F_{p,f_p(x)}) < \infty$ if t-norm T is φ convergent, then there exist a unique fixed-point z of mapping f and $z = \lim_{l\to\infty} f^l_{x_0}$.*

4. Applications of Probabilistic Metric Space

Probabilistic metric space can apply to estimate the rate of convergence of probability density estimation and other function like estimates of quantities, hazard rated etc. A PM Space appears to be well adapted for the investigation of physiological thresholds and physical quantities particularly. Its importance in probabilistic functional analysis due to its extensive applications in random differential as well as random integral. Probabilistic Metric Space permits the initial Formulation a greater flexibility than offered by a deterministic approach. It also permits the inclusion of probabilistic features in the equations, which may play an essential role in making the connection between operator equations and the real phenomena. It also purports to describe to how the triangular norms in probabilistic metric spaces as well as how to represent many valued equalities.

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References

[1] V. M. Sehgal and A. T. Bharucha-Reid, *Fixed Point contraction mapping in Probabilistic Metric Space*, Math System Theory, 6(1972), 97-102.

[2] T. L. Hicks, *Fixed Point Theory In Probabilistic Metric Space*, Zb. Rad. Prirod. Mat Fak. Ser. Mat., 13(1983), 63-72.

[3] K. Menger, *Statistical Matrices*, Proceedings of National Academy of Sciences of USA, 28(1942), 535-537.

[4] A. K. Chaudhary, *A Study on Contraction conditions in Probabilistic Metric Space*, M. Phil. Thesis submitted to Kathmandu University, Nepal, (2018).

[5] A. Sklar and B. Schweizer, *Probabilistic Metric Space*, Dover Publ., INC, Mineola, New York, (2005).

[6] O. Hadzic and E. Pap, *Fixed point theory in probabilistic metric spaces*, Kluver Academic Publishers, London, (2010).

[7] D. Mihet, *A note on Hicks type contractions on generalized menger space*, West university of Timisoara, STPA, (2002), 131-133.

[8] P. Azhdari, *Probabilistic Metric Space and Some Contraction Mappings*, International Journal of Computer Science and Network Society, 17(2)(2017), 56-60.

[9] J. S, Jung, S. M. Jang, Y. J. Cho, S. S. Chang and X. Chu, *Minimax Theorems in Probabilistic Metric Space*, Bull. Austral. Math. Soc., 51(1995), 103-119.

[10] V. Radu, *On Some fixed-point theorems in probabilistic metric spaces*, Lecture Notes on Maths, Vol. 1233, Springer-Verlag, New York, (1987), 125-133.

[11] L. Benarodo and R. Mohd, *Generalized Banach Contraction in Probabilistic Metric Spaces/Normed Spaces*, arXiv:0706.1418v1 [math.FA], (2007).

[12] A. Beitollahi and P. Azhdari, *Singled Valued Contraction Theorems in Probabilistic Metric Space*, International Mathematical Forum, 6(9)(2011), 439-443.