Int. J. Math. And Appl., 7(4)(2019), 63–70

ISSN: 2347-1557

Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Estimating the Approximate Solutions of the Fornberg-Whitham and Oskolkov-Benjamin-Bona-Mahony Equations

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Abstract: In this paper we study the initial value problems of Fornberg-Whitham(FW) and Oskolkov-Benjamin-Bona-Mahony(OBBM) equations which are locally wellposed in the Sobolev space H^s for $s > \frac{3}{2}$. we define the approximate solutions of FW and OBBM equations and compute the errors. Then we estimate the H^σ -norm of this errors.

Keywords: Sobolev space, Approximate solutions, Well-posedness, Non-local form.

1. Introduction

The Fornberg-Whitham(FW) equation introduced as a model to study breaking of non-linear dispersive water waves. In mathematical physics, the Whitham equation is a non-local model for non-linear dispersive waves. We consider the initial value problem for the Fornberg-Whitham equation

$$u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}uu_{xxx} - \frac{3}{2}uu_x + u_x = 0 \tag{1}$$
$$u(x,0) = u_0(x), x \in T, t \in \Re$$

where $u(x,t)$ is the fluid velocity, t is the time and x is the spatial co-ordinate. The FW equation was written by Fornberg and Whitham in 1978 as a model for breaking waves. The FW equation can be (and is more conveniently) written in the following non-local form

$$u_t + \frac{3}{2}uu_x = (1-\partial_x^2)^{-1}\partial_x u \tag{2}$$

The non-local form can be obtained from FW equation as follows,

$$\begin{aligned} u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}uu_{xxx} - \frac{3}{2}uu_x + u_x &= 0 \\ u_t + \frac{3}{2}uu_x - u_{txx} - \frac{3}{2}[uu_{xxx} + 3u_x u_{xx}] &= \partial_x u \end{aligned}$$

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$$\begin{aligned} u_t + \frac{3}{2}uu_x - u_{txx} - \frac{3}{2}\frac{\partial}{\partial_x}\left[\frac{\partial}{\partial_x}(uu_x)\right] &= \partial_x u \\ \left(1 - \frac{\partial^2}{\partial_x^2}\right)\left(u_t + \frac{3}{2}uu_x\right) &= \partial_x u \end{aligned}$$

Multiply bothsides by $(1-\partial_x^2)^{-1}$, we get

$$u_t + \frac{3}{2}uu_x = (1-\partial_x^2)^{-1}\partial_x u$$

Next, We consider the initial value problem (i.v.p) of the Oskolkov-Benjamin-Bona-Mahony Equation

$$u_t - u_x - u_{xxt} + uu_x = 0 \tag{3}$$

$$u(x,0) = u_0(x), x \in T, t \in \Re$$

The OBBM equation derived from the water wave model. Non-local form of the OBBM Equation can be written as,

$$u_t + uu_x = (1-\partial_x^2)^{-1}\partial_x(u - u_x^2 - uu_{xx}) \tag{4}$$

The non-local form can be obtained from OBBM equation as follows. Adding and subtracting the terms uu_{xxx} and $3u_xu_{xx}$, we get

$$\begin{aligned} u_t - u_x - u_{xxt} + uu_x + uu_{xxx} - uu_{xxx} + 3u_xu_{xx} - 3u_xu_{xx} &= 0 \\ (1-\partial_x^2)(u_t + uu_x) &= u_x - 3u_xu_{xx} - uu_{xxx} \\ u_t + uu_x &= (1-\partial_x^2)^{-1}\partial_x(u - u_x^2 - uu_{xx}) \end{aligned}$$

Written this way, the FW and OBBM equations are become a special case in the family of nonlinear wave equations of the form

$$u_t + auu_x = L(u)$$

This work studies the initial value problems of FW and OBBM equations. The paper is structured as follows. In section 1 we give the introduction and preliminaries. In section 2 we define approximate solutions and compute the error, while in section 3 we estimate the H^σ -norm of this error.

1.1. Preliminaries

Lemma 1.1. *Let $\sigma \in \Re$. If $\lambda \in Z^+$ and $\lambda >> 1$ then*

$$\|cos(\lambda x - \alpha)\|_{H^\sigma(T)} \approx \lambda^\sigma, \quad \alpha \in \Re$$

The above relation is also true if $cos(\lambda x - \alpha)$ is replaced by $sin(\lambda x - \alpha)$. Finally, for any $s \geq 0$ we have

$$\|u^{\omega,\lambda}(t)\|_{H^\sigma(T)} = \|\omega\lambda^{-1} + \lambda^{-s}cos(\lambda x - \omega t)\|_{H^\sigma(T)}$$

$$\|u^{\omega,\lambda}(t)\|_H^\sigma(T) \leq \lambda^{-1} + \lambda^{-s+\sigma}$$

where $\lambda >> 1$.

Definition 1.2. *For any $s \in \Re$ the operator $D^s = (1-\partial^2 x)^{\frac{s}{2}}$ is defined by*

$$\widehat{D^s f}(\xi) = (1+\xi^2)^{\frac{s}{2}} \hat{f}(\xi),$$

where $\hat{f}$ is the Fourier transform

$$\hat{f}(\xi) = \int_T e^{-ix\xi} f(x) dx$$

The inverse relation is given by

$$f(x) = \frac{1}{2\pi} \sum_{\xi \in Z} \hat{f}(\xi) e^{ix\xi}$$

Then, for $f \in H^s(T)$ we have

$$\begin{aligned} \|f\|^2_{H^s(T)} &= \frac{1}{2\pi} \sum_{\xi \in Z} (1+\xi^2)^s |\hat{f}(\xi)|^2 \\ &= \|D^2 f|^2_{L^2(T)} \end{aligned}$$

2. Approximate FW and OBBM Solutions

We shall consider approximate solutions of the form

$$u^{\omega,\lambda}(x,t) = \omega\lambda^{-1} + \lambda^{-s} cos(\lambda x - \omega t) \tag{5}$$

Where ω is bounded subset of $\Re$ and λ is in the set of positive integers Z^+. Next, we compute the error of the approximate solution of (4). Differentiating (4) with respect to t,

$$\begin{aligned} \partial_t u^{\omega,\lambda} &= 0 + \lambda^{-s}(-sin(\lambda x - \omega t))(-\omega) \\ \partial_t u^{\omega,\lambda} &= \omega\lambda^{-s} sin(\lambda x - \omega t) \end{aligned} \tag{6}$$

Differentiating (4) with respect to x,

$$\begin{aligned} \partial_x u^{\omega,\lambda} &= 0 + \lambda^{-s}(-sin(\lambda x - \omega t))(\lambda) \\ &= -\lambda\lambda^{-s} sin(\lambda x - \omega t) \\ \partial_x u^{\omega,\lambda} &= -\lambda^{-s+1} sin(\lambda x - \omega t) \end{aligned} \tag{7}$$

Approximate FW Solutions: Substituting (6) and (7) into the Burgers part of the FW equation, we get

$$\begin{aligned} \partial_t u^{\omega,\lambda} + \frac{3}{2} u^{\omega,\lambda} \partial_x u^{\omega,\lambda} &= \omega\lambda^{-s} sin(\lambda x - \omega t) + \frac{3}{2}[\omega\lambda^{-1} + \lambda^{-s} cos(\lambda x - \omega t)][-\lambda^{-s+1} sin(\lambda x - \omega t)] \\ &= \omega\lambda^{-s} sin(\lambda x - \omega t) - \frac{3}{2}\omega\lambda^{-s} sin(\lambda x - \omega t) - \frac{3}{2}\lambda^{-2s+1} cos(\lambda x - \omega t) sin(\lambda x - \omega t) \\ &= \frac{-1}{2}\omega\lambda^{-s} sin(\lambda x - \omega t) - \frac{3}{4}\lambda^{-2s+1} sin2(\lambda x - \omega t) \\ &= F_1 + F_2 \end{aligned}$$

Next, we compute the error resulting from applying the non-local perturbation part of the FW equation to the approximate solution. That is, we form the quantity $(1-\partial_x^2)^{-1}\partial_x u^{\omega,\lambda}$. We shall write this term by using the operator D^s.

$$[-(1-\partial_x^2)^{-1}]\partial_x u^{\omega,\lambda} = [-D^{-2}\partial_x(\omega\lambda^{-1} + \lambda^{-s} cos(\lambda x - \omega t)]$$

$$
\begin{aligned}
&= D^{-2}\lambda^{-s+1}sin(\lambda x - \omega t) \\
&= F_3
\end{aligned}
$$

To summarize the error F of the approximate solution (5) is,

$$
F = \partial_t u^{\omega,\lambda} + \frac{3}{2}u^{\omega,\lambda}\partial_x u^{\omega,\lambda} - D^{-2}\partial_x u^{\omega,\lambda} \tag{8}
$$

$$
= F_1 + F_2 + F_3 \tag{9}
$$

Where F_j are defined as above.

Approximate OBBM Solutions: we substituting (6) and (7) into the'Burgers part of the OBBM equation we get,

$$
\begin{aligned}
\partial_t u^{\omega,\lambda} + u^{\omega,\lambda}\partial_x u^{\omega,\lambda} &= \omega\lambda^{-s}sin(\lambda x - \omega t) + [\omega\lambda^{-1} + \lambda^{-s}cos(\lambda x - \omega t)][-\lambda^{-s+1}sin(\lambda x - \omega t)] \\
&= \omega\lambda^{-s}sin(\lambda x - \omega t) - \omega\lambda^{-s}sin(\lambda x - \omega t) - \lambda^{-2s+1}cos(\lambda x - \omega t)sin(\lambda x - \omega t) \\
&= \lambda^{-2s+1}cos(\lambda x - \omega t)sin(\lambda x - \omega t) \\
\partial_t u^{\omega,\lambda} + u^{\omega,\lambda}\partial_x u^{\omega,\lambda} &= -\frac{1}{2}\lambda^{-2s+1}sin2(\lambda x - \omega t) \\
&= F_1
\end{aligned}
$$

Next,we compute the error resulting from applying the non local perturbation part of the OBBM equation to the approximate solution. That is we form the quantity

$$
\begin{aligned}
-(1-\partial_x^2)^{-1}\partial_x(u - u_x^2 - uu_{xx}) &= -D^{-2}\partial_x(u^{\omega,\lambda} - (u_x^{\omega,\lambda})^2 - u^{\omega,\lambda}u_{xx}^{\omega,\lambda}) \\
&= -D^{-2}\partial_x(u^{\omega,\lambda} + D^{-2}\partial_x(u_x^{\omega,\lambda})^2 + D^{-2}\partial_x(u^{\omega,\lambda}u_{xx}^{\omega,\lambda})
\end{aligned}
$$

Now consider

$$
\begin{aligned}
-D^{-2}\partial_x(u^{\omega,\lambda}) &= -D^{-2}(-\lambda^{-S+1})sin(\lambda x - \omega t) \\
&= D^{-2}(\lambda^{-S+1})sin(\lambda x - \omega t) \\
&= F_2
\end{aligned}
$$

Now consider

$$
\begin{aligned}
(\partial_x(u^{\omega,\lambda}))^2 &= (-\lambda^{-S+1}sin(\lambda x - \omega t))^2 \\
&= \lambda^{-2S+2}sin^2(\lambda x - \omega t) \\
&= \lambda^{-2S+2}[\frac{1}{2} - \frac{1}{2}cos2(\lambda x - \omega t)] \\
\partial_x(\partial_x(u^{\omega,\lambda}))^2 &= \lambda^{-2S+2}[0 - \frac{1}{2}(-sin2(\lambda x - \omega t))2\lambda] \\
&= \lambda^{-2S+2}\lambda sin2(\lambda x - \omega t) \\
&= \lambda^{-2S+3}sin2(\lambda x - \omega t)
\end{aligned}
$$

Therefore

$$
D^{-2}\partial_x(\partial_x(u^{\omega,\lambda}))^2 = D^{-2}\lambda^{-2S+3}sin2(\lambda x - \omega t)
$$

$$= F_3$$

Now consider $D^{-2}\partial_x(u^{\omega,\lambda}u^{\omega,\lambda}_{xx})$

$$\begin{aligned}
u^{\omega,\lambda}_{xx} &= -\lambda^{-s+1}cos(\lambda x-\omega t)\lambda \\
&= -\lambda^{-s+2}cos(\lambda x-\omega t) \\
u^{\omega,\lambda}u^{\omega,\lambda}_{xx} &= [\omega\lambda^{-1}+\lambda^{-s}cos(\lambda x-\omega t)][-\lambda^{-s+2}cos(\lambda x-\omega t)] \\
&= -\omega\lambda^{-s+1}cos(\lambda x-\omega t)-\lambda^{-2s+2}\left[\frac{1}{2}+\frac{1}{2}cos2(\lambda x-\omega t)\right]
\end{aligned}$$

Since $cos^2\theta = \frac{1}{2}+\frac{1}{2}cos2\theta$

$$\begin{aligned}
\partial_x(u^{\omega,\lambda}u^{\omega,\lambda}_{xx}) &= -\omega\lambda^{-s+1}(-sin(\lambda x-\omega t)\lambda)+0-\lambda^{-2s+2}\left(\frac{1}{2}(-sin2(\lambda x-\omega t)2\lambda)\right) \\
\partial_x(u^{\omega,\lambda}u^{\omega,\lambda}_{xx}) &= \omega\lambda^{-s+2}(sin(\lambda x-\omega t))+\lambda^{-2s+3}sin2(\lambda x-\omega t)) \\
D^{-2}\partial_x(u^{\omega,\lambda}u^{\omega,\lambda}_{xx}) &= D^{-2}\omega\lambda^{-s+2}sin(\lambda x-\omega t)+D^{-2}\lambda^{-2s+3}sin2(\lambda x-\omega t) \\
&= F_4+F_5
\end{aligned}$$

To summarize, the error F of the approximate solution (5) is

$$F = F_1+F_2+F_3+F_4+F_5 \tag{10}$$

Where F_j's are derived as above.

3. Estimating the Error of Approximate solutions of FW and OBBM equations

Next we shall estimate the H^σ-norm of the error F in (9) and (10) by estimating separately the H^σ norm of each term F_j.

Estimating the H^σ-norm of error of FW equation:

Estimating the H^σ-norm of F_1

$$\begin{aligned}
\|F_1(t)\|_{H^\sigma(T)} &= \|\frac{-1}{2}\omega\lambda^{-s}sin(\lambda x-\omega t)\|_{H^\sigma(T)} \\
&= \frac{1}{2}\omega\lambda^{-s}\|sin(\lambda x-\omega t)\|_{H^\sigma(T)} \\
&\leq \frac{1}{2}\omega\lambda^{-s+\sigma}
\end{aligned}$$

since by Lemma 1.1, $\lambda >> 1$.

Estimating the H^σ-norm of F_2

$$\begin{aligned}
\|F_2(t)\|_{H^\sigma(T)} &= \|-\frac{3}{4}\lambda^{-2s+1}sin2(\lambda x-\omega t)\|_{H^\sigma(T)} \\
&= \frac{3}{4}\lambda^{-2s+1}\|sin2(\lambda x-\omega t)\|_{H^\sigma(T)} \\
&\leq \frac{3}{4}\lambda^{-2s+1+\sigma}
\end{aligned}$$

since by Lemma 1.1, $\lambda >> 1$.

Estimating the H^σ-norm of F_3

$$\begin{aligned}\|F_3(t)\|_{H^\sigma(T)} &= \|D^{-2}\lambda^{-s+1}sin(\lambda x-\omega t)\|_{H^\sigma(T)}\\ &= \|\lambda^{-s+1}sin(\lambda x-\omega t)\|_{H^{\sigma-2}(T)}\\ &\leq \lambda^{-s+1}\|sin(\lambda x-\omega t)\|_{H^{\sigma-2}(T)}\\ &\leq \lambda^{-s+1}\lambda^{\sigma-2}\\ &= \lambda^{-s+\sigma-1}\end{aligned}$$

since by Lemma 1.1, $\lambda >> 1$. Putting the above estimates together gives the following H^σ-estimate for the error of the FW approximate solution.

$$\|F\|_{H^\sigma(T)} \leq \frac{1}{2}\omega\lambda^{-s+\sigma} + \frac{3}{4}\lambda^{-2s+1+\sigma} + \lambda^{-s+\sigma-1} \tag{11}$$

Estimating the H^σ-norm of error of OBBM equation: Next we shall estimate the H^σ -norm of the error F in (6) by estimating separately the H^σ norm of each term F_j.

Estimating the H^σ -norm of F_1

$$\begin{aligned}\|F_1(t)\|_{H^\sigma(T)} &= \|-\frac{1}{2}\lambda^{-2s+1}sin2(\lambda x-\omega t)\|_{H^\sigma(T)}\\ &= \frac{1}{2}\lambda^{-2s+1}\|sin2(\lambda x-\omega t)\|_{H^\sigma(T)}\\ &\leq \frac{1}{2}\lambda^{-2s+1}\lambda^\sigma\\ &\leq \lambda^{-2s+\sigma+1}\end{aligned}$$

since by Lemma 1.1, $\lambda >> 1$.

Estimating the H^σ-norm of F_2

$$\begin{aligned}\| F_2(t) \|_{H^\sigma} &= \| D^{-2}\lambda^{-s+1}sin(\lambda x-\omega t) \|_{H^\sigma}\\ &= \lambda^{-s+1} \| sin(\lambda x-\omega t) \|_{H^{\sigma-2}}\\ &\leq \lambda^{-s+1+\sigma}\end{aligned}$$

Estimating the H^σ-norm of F_3

$$\begin{aligned}\| F_3(t) \|_{H^\sigma} &= \| D^{-2}\lambda^{-2s+3}sin2(\lambda x-\omega t) \|_{H^\sigma}\\ &= \lambda^{-2s+3} \| D^{-2}sin2(\lambda x-\omega t) \|_{H^\sigma}\\ &= \lambda^{-2s+3} \| sin2(\lambda x-\omega t) \|_{H^{\sigma-2}}\\ &\leq \lambda^{-2s+3+\sigma}\end{aligned}$$

since by Lemma 1.1, $\lambda >> 1$.

Estimating the H^σ-norm of F_4

$$\| F_4(t) \|_{H^\sigma} = \| D^{-2}\omega\lambda^{-s+2}sin(\lambda x-\omega t) \|_{H^\sigma}$$

$$
\begin{aligned}
&= \omega\lambda^{-s+2} \parallel D^{-2} sin(\lambda x - \omega t) \parallel_{H^\sigma} \\
&= \omega\lambda^{-s+2} \parallel sin(\lambda x - \omega t) \parallel_{H^{\sigma-2}} \\
&\leq \omega\lambda^{-s+2+\sigma}
\end{aligned}
$$

since by Lemma 1.1, $\lambda >> 1$.

Estimating the H^σ-norm of F_5

$$
\begin{aligned}
\parallel F_5(t) \parallel_{H^\sigma} &= \parallel D^{-2}\lambda^{-2s+3} sin2(\lambda x - \omega t) \parallel_{H^\sigma} \\
&= \lambda^{-2s+3} \parallel D^{-2} sin2(\lambda x - \omega t) \parallel_{H^\sigma} \\
&= \lambda^{-2s+3} \parallel sin2(\lambda x - \omega t) \parallel_{H^{\sigma-2}} \\
&\leq \lambda^{-2s+3+\sigma}
\end{aligned}
$$

since by Lemma 1.1, $\lambda >> 1$. Now putting the above estimates together gives the following H^σ-Estimate for the error of the OBBM approximate solution.

$$
\parallel F(t) \parallel_{H^\sigma} \leq \lambda^{-2s+1+\sigma} + \lambda^{-s+1+\sigma} + \lambda^{-2s+3+\sigma} + \omega\lambda^{-s+2+\sigma} + \lambda^{-2s+3+\sigma} \tag{12}
$$

Hence, we derived the approximate solutions of FW and OBBM equations and compute the error of the approximate sloution of FW and OBBM equation. Also we estimated the H^σ norm of the error.

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