Int. J. Math. And Appl., 7(4)(2019), 71–76
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Space Using Occasionally Weakly Compatible Maps with Integral Type Inequality

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Abstract: In this paper, we obtain common fixed point theorems in intuitionistic fuzzy metric spaces using occasionally weakly compatible maps with Integral Type Inequality.

MSC: 47H10, 54H25.

Keywords: Intuitionistic fuzzy metric space, Occasionally weakly compatible mappings, Common fixed point, Integral type.

1. Introduction

Atanassove [2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [13]. In 2004, Park [10] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. Recently, in 2006, Alaca [1] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [6]. Branciari [3] obtained a fixed point theorem for a single mapping satisfying an analogue Banach's contraction principle for an integral type inequality. Sedghi [12] established a common fixed point theorem for weakly compatible mappings in intuitionistic fuzzy metric space satisfying a contractive condition of integral type.. In this paper, we obtain common fixed point theorems in intuitionistic fuzzy metric spaces using occasionally weakly compatible maps with integral type inequality.

2. Preliminaries

Definition 2.1 ([11])**.** *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous t-norm if it satisfy the following condition:*

(1). $*$ *is associative and commutative.*

(2). $*$ *is continous function.*

(3). $a * 1 = a$ *for all* $a \in [0,1]$.

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(4). $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *and* $a, b, c, d \in [0,1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2 ([11]). *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if it satisfies the following conditions:*

(1). $\Diamond$ *is commutative and associative;*

(2). $\Diamond$ *is continuous;*

(3). $a \Diamond b = a$ *for all* $a \in [0,1]$*;*

(4). $a \Diamond b \leq c \Diamond d$ *whenever* $a \leq c$ *and* $b \leq d$*, for each* $a, b, c, d \in [0,1]$.

Alaca [1] using the idea of intuitionistic fuzzy sets, introduced the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [6] as :

Definition 2.3 ([11]). *A 5-tuple* $(X, M, N, *, \Diamond)$ *is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\Diamond$ *is a continuous t-conorm and* M*,* N *are fuzzy sets on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X$ *and* $s, t > 0$*;*

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$*;*

(IFM-2) $M(x, y, 0) = 0$*;*

(IFM-3) $M(x, y, t) = 1$ *if and only if* $x = y$*;*

(IFM-4) $M(x, y, t) = M(y, x, t)$*;*

(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*;*

(IFM-6) $M(x, y, .) : [0, \infty) \to [0,1]$ *is left continuous;*

(IFM-7) $\lim_{t\to\infty} M(x, y, t) = 1$*;*

(IFM-8) $N(x, y, 0) = 1$*;*

(IFM-9) $N(x, y, t) = 0$ *if and only if* $x = y$*;*

(IFM-10) $N(x, y, t) = N(y, x, t)$*;*

(IFM-11) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$*;*

(IFM-12) $N(x, y, .) : [0, \infty) \to [0,1]$ *is right continuous;*

(IFM-13) $\lim_{t\to\infty} N(x, y, t) = 0$*;*

Then (M, N) *is called an intuitionistic fuzzy metric on* X*. The functions* $M(x, y, t)$ *and* $N(x, y, t)$ *denote the degree of nearness and degree of nonnearness between* x *and* y *with respect to* t*, respectively.*

Remark 2.4. *Every fuzzy metric space* $(X, M, *)$ *is an intuitionistic fuzzy metric space if* X *of the form* $(X, M, 1 - M, *, \Diamond)$ *such that t-norm* $*$ *and t-conorm* $\Diamond$ *are associated,that is,* $x \Diamond y = 1 - ((1 - x) * (1 - y))$ *for any* $x, y \in X$*. But the converse is not true.*

Remark 2.5. *In intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, $M(x, y, *)$ is non-decreasing and $N(x, y, \Diamond)$ is non-increasing for all $x, y \in X$.*

Alaca, Turkoglu and Yildiz [1] introduced the following notions:

Example 2.6 ([10]). *Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows;*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$
$$N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Example 2.7. *Let $X = N$. Define $a * b = \max\{0, a + b - 1\}$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows;*

$$M(x, y, t) = \left\{ \begin{array}{ll} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x \end{array} \right\}$$
$$N(x, y, t) = \left\{ \begin{array}{ll} \frac{y-x}{x} & \text{if } x \leq y \\ \frac{x-y}{x} & \text{if } y \leq x \end{array} \right\}$$

for all $x, y, z \in X$ and $t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

Definition 2.8 ([1]). *Let $(X, M, N, *, \Diamond)$ be an Intuitionistic fuzzy metric space.*

(a). A sequence $\{x_n\}$ in X is called cauchy sequence if for each $t > 0$ and $P > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.

(b). A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$ for each $t > 0$.

(c). An Intuitionistic fuzzy metric space is said to be complete if every Cauchy sequence is convergent.

Lemma 2.9 ([10]). *In an intuitionistic fuzzy metric space X, $M(x, y, .)$ is Non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.*

In 1976, Jungck [4] introduced the notion of weakly compatible maps as follows:

Definition 2.10 ([4]). *A pair of self mappings (f, g) of a metric space is said to be weakly compatible if they commute at the coincidence points i.e. $fu = gu$ for some $u \in X$, then $fgu = gfu$.*

Definition 2.11 ([4]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. f and g be self maps on X. A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.*

Definition 2.12 ([4]). *A pair of self mappings (f, g) of intuitionistic fuzzy metric space is said to be weakly compatible if they commute at the coincidence points i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.*

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.13 ([5]). *Two self mappings f and g of intuitionistic fuzzy metric space are said to be occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.*

Lemma 2.14 ([5]). *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. f and g be self maps on X and let f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.*

Proof. Since f and g are owc, there exists a point x in X such that $fx = gx = w$ and $fgx = gfx$. Thus, $ffx = fgx = gfx$, which says that fx is also a point of coincidence of f and g. Since the point of coincidence $w = fx$ is unique by hypothesis, $gfx = ffx = fx$, and $w = fx$ is a common fixed point of f and g.
Moreover, if z is any common fixed point of f and g, then $z = fz = gz = w$ by the uniqueness of the point of coincidence. □

Alaca [1] proved the following results:

Lemma 2.15 ([1]). *Let $(X, M, N, *, \Diamond)$ be intuitionistic fuzzy metric space and for all $x, y \in X$. $t > 0$ and if for a number $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x; y; t)$ and $N(x, y, kt) \leq N(x, y, t)$, then $x = y$.*

3. Main Results

Theorem 3.1. *Let $(X, M, N, *, \Diamond)$ intuitionistic fuzzy metric space and A, B, S & T be the self mapping of X. Let the pair (A, S) and (B, T) be Occasionally Weakly Compatible maps. If there exist $k \in (0, 1)$ such that*

$$\int_0^{M(Ax,By,kt)} \varphi(t)\,dt \geq \int_0^{\min\left\{M(Sx,Ty,t), M(Sx,Ax,t)M(By,Ty,t), \frac{aM(Ax,Ty,t)+bM(By,Sx,t)+cM(Sx,Ty,t)}{a+b+c}\right\}} \varphi(t)\,dt$$
$$\int_0^{N(Ax,By,kt)} \varphi(t)\,dt \leq \int_0^{max\left\{N(Sx,Ty,t), N(Sx,Ax,t), N(By,Ty,t), \frac{aN(Ax,Ty,t)+bN(By,Sx,t)+cN(Sx,Ty,t)}{a+b+c}\right\}} \varphi(t)\,dt \tag{1}$$

for all $x, y \in X$, $t > 0$ and $a, b, c, d \geq 0$ with and a & b (c & d) cannot be simultaneous 0 and where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, nonnegative, and such that $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon > 0$ then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point, $z \in X$ such that, $Bz = Tz = z$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S & T.

Proof. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequalities (1)

$$\int_0^{M(Ax,By,kt)} \varphi(t)\,dt \geq \int_0^{min\left\{M(Ax,By,t), M(Ax,Ax,t)M(By,By,t), \frac{aM(Ax,By,t)+bM(By,Ax,t)+cM(Ax,By,t)}{a+b+c}\right\}} \varphi(t)\,dt$$
$$\int_0^{N(Ax,By,kt)} \varphi(t)\,dt \leq \int_0^{max\left\{NAx,By,t), N(Ax,Ax,t), N(By,By,t), \frac{aN(Ax,By,t)+bN(By,Ax,t)+cN(Ax,By,t)}{a+b+c}\right\}} \varphi(t)\,dt$$
$$\int_0^{M(Ax,By,kt)} \varphi(t)\,dt \geq \int_0^{min\{M(Ax,By,t),1,1,M(Ax,By,t)\}} \varphi(t)\,dt$$
$$\int_0^{N(Ax,By,kt)} \varphi(t)\,dt \leq \int_0^{max\{N(Ax,By,t),0,0,N(Ax,By,t)\}} \varphi(t)\,dt$$
$$\int_0^{M(Ax,By,kt)} \varphi(t)\,dt \geq \int_0^{M(Ax,By,t)} \varphi(t)\,dt \tag{2}$$
$$\int_0^{N(Ax,By,kt)} \varphi(t)\,dt \leq \int_0^{N(Ax,By,t)} \varphi(t)\,dt \tag{3}$$

From (2) & (3)

$$Ax = By$$

Suppose that there is a another point z such that $Az = Sz$ then by (1) we have $Az = Sz = By = Ty$, So $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$ then by (1)

$$\int_0^{M(w,z,kt)} \varphi(t)\,dt = \int_0^{M(Aw,Bz,kt)} \varphi(t)\,dt$$

$$\geq \int_0^{min\left\{M(Sw,Tz,t),M(Sw,Aw,t)M(Bz,Tz,t),\frac{aM(Aw,Tz,t)+bM(Bz,Sw,t)+cM(Sw,Tz,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{M(w,z,kt)} \varphi(t)\,dt \geq \int_0^{min\left\{M(Aw,Bz,t),M(Aw,Aw,t)M(Bz,Bz,t),\frac{aM(Aw,Bz,t)+bM(Bz,Aw,t)+cM(Aw,Bz,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{M(w,z,kt)} \varphi(t)\,dt \geq \int_0^{min\{M(w,z,t),1,1,M(w,z,t)\}} \varphi(t)\,dt$$

$$\int_0^{M(w,z,kt)} \varphi(t)\,dt \geq \int_0^{M(w,z,t)} \varphi(t)\,dt \tag{4}$$

Again

$$\int_0^{N(w,z,kt)} \varphi(t)\,dt = \int_0^{N(Aw,Bz,kt)} \varphi(t)\,dt$$

$$\leq \int_0^{max\left\{M(Sw,Tz,t),M(Sw,Aw,t)M(Bz,Tz,t),\frac{aM(Aw,Tz,t)+bM(Bz,Sw,t)+cM(Sw,Tz,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{N(w,z,kt)} \varphi(t)\,dt \leq \int_0^{max\left\{N(Aw,Bz,t),N(Aw,Aw,t),N(Bz,Bz,t),\frac{aN(Aw,Bz,t)+bN(Bz,Aw,t)+cN(Aw,Bz,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{N(w,z,kt)} \varphi(t)\,dt \leq \int_0^{max\{N(w,z,t),0,0,N(w,z,t)\}} \varphi(t)\,dt$$

$$\int_0^{N(w,z,kt)} \varphi(t)\,dt \leq \int_0^{N(w,z,t)} \varphi(t)\,dt \tag{5}$$

From (4) & (5), we have $w = z$.

Uniqueness: Let u be another common fixed point of A, B, S & T. Then put $x = z$ and $y = u$ in (1)

$$\int_0^{M(z,u,kt)} \varphi(t)\,dt = \int_0^{M(Az,Bu,kt)} \varphi(t)\,dt$$

$$\geq \int_0^{min\left\{M(Sz,Tu,t),M(Sz,Az,t),M(Bu,Tu,t),\frac{aM(Az,,Tu,t)+bM(Bu,Sz,t)+cM(Sz,,Tu,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{M(z,u,kt)} \varphi(t)\,dt \geq \int_0^{min\left\{M(z,u,t),M(z,z,t),M(u,u,t),\frac{aM(z,,u,t)+bM(u,z,t)+cM(z,u,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{M(z,u,kt)} \varphi(t)\,dt \geq \int_0^{min\{M(z,u,t),1,1,M(z,u,t)\}} \varphi(t)\,dt$$

$$\int_0^{M(z,u,kt)} \varphi(t)\,dt \geq \int_0^{M(z,u,t)} \varphi(t)\,dt \tag{6}$$

Again

$$\int_0^{N(z,u,kt)} \varphi(t)\,dt = \int_0^{N(Az,Bu,kt)} \varphi(t)\,dt$$

$$\leq \int_0^{max\left\{N(Sz,Tu,t),N(Sz,Az,t),N(Bu,Tu,t),\frac{aN(Az,,Tu,t)+bN(Bu,Sz,t)+cN(Sz,,Tu,t)}{a+b+c}\right\}} \emptyset(t)\,dt$$

$$\int_0^{N(z,u,kt)} \varphi(t)\,dt \leq \int_0^{max\left\{N(z,u,t),N(z,z,t),N(u,u,t),\frac{aN(z,,u,t)+bN(u,z,t)+cN(z,u,t)}{a+b+c}\right\}} \varphi(t)\,dt$$

$$\int_0^{N(z,u,kt)} \varphi(t)\,dt \leq \int_0^{max\{N(z,u,t),0,0,N(z,u,t)\}} \varphi(t)\,dt$$

$$\int_0^{N(z,u,kt)} \varphi(t)\,dt \leq \int_0^{N(z,u,t)} \varphi(t)\,dt \tag{7}$$

From (6) and (7) and Lemma 2.15 we have $z = u$. Hence z is the unique common fixed point of A, B, S & T. □

Theorem 3.2. *Let $(X, M, N, *, \diamondsuit)$ Intuitionistic fuzzy metric space and A, B, S & T be the self mapping of X. Let the pair (A, S) and (B, T) be Occasionally Weakly Compatible maps .If there exist $k \in (0,1)$ such that*

$$\int_0^{M(Ax,By,kt)} \varphi(t)\,dt \geq \int_0^{\Phi\left[min\left\{M(Sx,Ty,t),M(Sx,Ax,t)M(By,Ty,t),\frac{aM(Ax,Ty,t)+bM(By,Sx,t)+cM(Sx,Ty,t)}{a+b+c}\right\}\right]} \varphi(t)\,dt$$
$$\int_0^{N(Ax,By,kt)} \varphi(t)\,dt \leq \int_0^{\Psi\left[max\left\{N(Sx,Ty,t),N(Sx,Ax,t),N(By,Ty,t),\frac{aN(Ax,Ty,t)+bN(By,Sx,t)+cN(Sx,Ty,t)}{a+b+c}\right\}\right]} \varphi(t)\,dt \tag{8}$$

for all $x, y \in X$, $t > 0$ and $a, b, c, d \geq 0$ with and a &b (c & d) cannot be simultaneous 0, where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, nonnegative, and such that $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon > 0$, and Φ, $\Psi : [0,1] \to [0,1]$ such that $\Phi(t) > t$ and $\Psi(t) < t$ for all $t > 0$ then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point, $z \in X$ such that, $Bz = Tz = z$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S & T.

Proof. The Proof follows from Theorem 3.1. □

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