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On a Zeuthen-type Problem

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Abstract: In this paper, we show that every degree d meromorphic function on a smooth connected projective curve $C \subset \mathbb{P}^2$ of degree $d > 4$ is isomorphic to a linear projection from a point $p \in \mathbb{P}^2 \setminus C$ to $\mathbb{P}^1$. We then pose a Zeuthen-type problem for calculating the plane Hurwitz numbers.

MSC: 14H30.

Keywords: Hurwitz numbers, Zeuthen numbers.

1. Introduction

Consider $C \subset \mathbb{P}^2$, a projective plane curve of degree d. An important geometric method for studying C, involves meromorphic functions arising from linear projections of C from a point $p \in \mathbb{P}^2$. For instance, B. Riemann established in his famous work [4], that the topological structure of a smooth curve $C \subset \mathbb{P}^2$ depends entirely on the nature of branch types of the branched covering π_p arising from a linear projection. To construct π_p, we choose a point $p \in \mathbb{P}^2$ which may or may not be lying on C and then identify $\mathbb{P}^1$ with the pencil of lines passing through $p \in \mathbb{P}^2$. If $p \in \mathbb{P}^2 \setminus C$, then a generic line through p meets the curve C in d distinct points. Thus, the linear projection from a point $p \in \mathbb{P}^2 \setminus C$ is a finite surjective morphism

$$\pi_p : C \longrightarrow \mathbb{P}^1 \tag{1}$$

of degree d. Namely, the morphism π_p is a branched covering of $\mathbb{P}^1$ and the points of $\mathbb{P}^1$ where several intersection points of the corresponding line with C coincide are the branch points of π_p. Therefore, it is a basic problem to characterize and enumerate those meromorphic functions f on C which can be realized as linear projections. First, note that in general not all meromorphic functions on a curve $C \subset \mathbb{P}^2$ can be realized as such. However, for $d > 4$ we have the following result which we will prove.

Theorem 1.1. *Suppose that $C \subset \mathbb{P}^2$ is a smooth projective plane curve of degree $d > 4$. Then any meromorphic function $f : C \longrightarrow \mathbb{P}^1$ of degree d can be realized as a linear projection $\pi_p : C \longrightarrow \mathbb{P}^1$.*

Hurwitz numbers [1, 3] count non-isomorphic meromorphic functions on curves with fixed genus g having a fixed branched profile. On the other hand, Zeuthen numbers [11] count nodal plane curves of a fixed degree d and geometric genus g passing through a general points and tangent to b general lines in $\mathbb{P}^2$, where $a + b = 3d + g - 1$. There is a class of Zeuthen

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numbers corresponding to what we call **plane Hurwitz numbers**. Zeuthen numbers have been interpreted by R.Vakil in the context of stable maps as positive degree Gromov-Witten invariants of $\mathbb{P}^2$. In section §5 below, following [9], we will sketch a derivation of a class of characteristic numbers of smooth plane curves which correspond to calculating plane Hurwitz numbers.

2. General Preliminaries

2.1. Notation and conventions

The base field is $\mathbb{C}$, the field of complex numbers and we denote by $\mathbb{P}^n$ the $n-$dimensional projective space over $\mathbb{C}$. By a **variety** we mean a reduced algebraic projective scheme over $\mathbb{C}$. The term **curve** means a complete connected variety of dimension 1. By a **smooth** or **nonsingular** curve we implicitly assume that it is irreducible. If $\Gamma \subset \mathbb{P}^n$ is a closed subscheme, we write $\mathcal{O}_\Gamma$ for the structure sheaf over Γ and $\mathscr{I}_\Gamma \subset \mathcal{O}_{\mathbb{P}^n}$ denotes the ideal sheaf of Γ. Let D be a divisor on a curve X, then $|D|$ is the complete linear system of D. We write K_X or K for the canonical class of a smooth curve X and we denote by $|K_X|$ or $|K|$ for the complete canonical series respectively. Suppose that $\mathscr{F}$ is a sheaf of vector spaces over a projective scheme X. Then we set

$$h^i(\mathscr{F}) := \dim \mathbf{H}^i(X, \mathscr{F}) \quad \text{and} \quad \chi(\mathscr{F}) := \sum_{i=0}^{\dim X} (-1)^i h^i(\mathscr{F}).$$

2.2. General Definitions

Let C be a nonsingular curve of genus g. A surjective morphism $f : C \to \mathbb{P}^1$ is called a **meromorphic function**. More precisely, a meromorphic function f gives a finite morphism to the complex projective line $\mathbb{P}^1$ whose degree d by definition is the degree of the morphism $f : C \longrightarrow \mathbb{P}^1$. Thus for a meromorphic function f and any fixed point $q \in \mathbb{P}^1$ we have the divisor $f^{-1}(q) = \mu_1 p_1 + \ldots + \mu_n p_n$, where $p_1, \ldots, p_n$ are pairwise distinct points on C and $\mu_1, \ldots, \mu_n$ are positive integers summing up to d. In particular, we can assume $\mu_1 \geq \ldots \geq \mu_n$. The partition $(\mu_1, \ldots, \mu_n) \vdash d$ is called the **branch type** of f at a point q. For instance, f is unbranched over q, if the branch type equal to $(1, 1, \ldots, 1)$. The branch type for a **simple** branch point is $(2, 1, \ldots, 1)$. The set of all branch points is called the **branching locus** of f. In this way, every nonconstant meromorphic function on a curve C is a **branched covering**. The basic problem is then the classification and enumeration of such maps $f : C \to \mathbb{P}^1$ for a given g and d for a prescribed branch type over each branch point of f. The set of all branch types for f will be called **branch profile** of f.

2.3. Hurwitz Numbers

Branched coverings were first described in the famous paper [4] by Riemann who developed the idea of representing non-singular curves as branched coverings of $\mathbb{P}^1$ in order to study their moduli. However, systematic investigation of branched coverings was initiated by Hurwitz in [1, 2] more than thirty years later.

Definition 2.1. *Let $f_1 : C_1 \to \mathbb{P}^1$, $f_2 : C_2 \to \mathbb{P}^1$ be two branched coverings. Then f_1 and f_2 are said to be* **equivalent** *if there exists an isomorphism $h : C_1 \to C_2$ such that $f_1 = f_2 \circ h$.*

Hurwitz observed that if we fix the degree d of the branched coverings $f : C \to \mathbb{P}^1$ and the number w of branch points and branch profile, then equivalence classes of branched coverings form a covering space $\mathscr{H}_{d,g}$ (we suppress the branch profile to avoid notational clutter) of the configuration space of w points in $\mathbb{P}^1$. These parameter spaces $\mathscr{H}_{d,g}$ are called **Hurwitz spaces**. The fundamental group of the configuration space of w branch points in $\mathbb{P}^1$ acts on the fibers of $\mathscr{H}_{g,d}$ and the orbits of this action are in one-one correspondence with the connected components of $\mathscr{H}_{g,d}$. A very special case is when all the branch points are simple. Hurwitz proved that in this case there is only one orbit. Consequently, it follows that the Hurwitz space parametrizing simple branched coverings of $\mathbb{P}^1$ of degree d is an irreducible smooth algebraic variety (see §21 of [5] or [12]) called the **small Hurwitz space** denoted by

$$\mathcal{H}_{g,d} = \left\{ f : C \longrightarrow \mathbb{P}^1 \;\middle|\; \begin{array}{c} C \text{ has genus } g \text{ and } f \text{ is a branched covering} \\ \text{of degree } d \text{ with } w \text{ simple branch points} \end{array} \right\} \Big/ \sim . \tag{2}$$

It turns out that $\mathcal{H}_{d,g}$ is a covering space. In fact it is shown in [1] that $\mathcal{H}_{g,d}$ comes with a natural finite étale covering

$$\begin{aligned} \Phi : \mathcal{H}_{g,d} &\longrightarrow \operatorname{Sym}^w \mathbb{P}^1 \backslash \Delta \\ (f : C \longrightarrow \mathbb{P}^1) &\longmapsto \{\text{branch locus of } f\} \end{aligned} \tag{3}$$

where $\operatorname{Sym}^w \mathbb{P}^1$ is the space of unordered $w-$tuples of points in $\mathbb{P}^1$ and Δ is the discriminant hypersurface corresponding to sets of cardinality strictly less than w. The Riemann-Hurwitz formula tells us that the degree of the branch divisor for $f : C \longrightarrow \mathbb{P}^1$ in $\mathcal{H}_{g,d}$, equals $w = 2g + 2d - 2$. The morphism Φ is called the **branching morphism** and its degree is called the **simple Hurwitz number** $h_{d,g}$. Since the map Φ is finite-to-one, the branch points can be regarded as local coordinates on $\mathcal{H}_{g,d}$ and it follows that the dimension of the Hurwitz space is equal to $w = 2g + 2d - 2$.

3. Proof of Theorem 1.1

Given a smooth curve C, specifying a meromorphic function $f : C \longrightarrow \mathbb{P}^1$ of degree d on C corresponds to identifying an effective degree d divisor D of f such that the linear system $|D|$ has no base points and $\dim |D| \geq 1$.

Definition 3.1. *Let $D = p_1 + \ldots + p_d$ be a divisor on a smooth curve C. If $|D|$ has no base point and $\dim |D| = 1$, we say that D* **moves in a linear pencil** *$|D|$. Equivalently, we have a meromorphic function of degree d*

$$f : C \longrightarrow \mathbb{P}^1$$

such that $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathscr{L}$, where $\mathscr{L} \cong \mathcal{O}_C(D)$ for $\mathcal{O}_C(D)$ the invertible sheaf over C determined by the divisor D and $h^0(\mathscr{L}) = 2$, so that we may choose a basis say $\{f_0, f_1\}$ for $\mathbf{H}^0(C, \mathscr{L})$ such that $f = [f_0 : f_1]$.

Remark 3.2. *The assertion of Theorem 1.1 fails if $d = 3$ and $d = 4$.*

Example 3.3. *If $C \subset \mathbb{P}^2$ is a smooth projective quartic, then there is a meromorphic function on C of degree 4 which is not isomorphic to a linear projection π_p. Indeed let $D = p_1 + \ldots + p_4$ be a divisor given by any 4 points on C such that no three of them are collinear. In our case $h^0(\mathscr{L}) = 2$ by Riemann-Roch's theorem. Recall that an invertible sheaf $\mathscr{L}$ on C is base point free if $h^0(\mathscr{L}) - h^0(\mathscr{L}(-p)) = 1$ for all $p \in C$. Then $h^0(\mathscr{L}(-p)) = \deg(\mathscr{L}(-p)) - g + 1 = 1$ again by Riemann-Roch. So we obtain $h^0(\mathscr{L}(-p)) = 1 = h^0(\mathscr{L}) - 1$ and we conclude that the linear system $|p_1 + p_2 + p_3 + p_4|$ has no base points. Hence the four points move in a linear pencil but a meromorphic function specified by this divisor on a smooth quartic cannot be realized as a linear projection as this 4 points are not in a line.*

The proof of Theorem 1.1 will be derived from the following result.

Theorem 3.4. *Let $\Gamma = \{p_1, \ldots, p_d\} \subset \mathbb{P}^2$, be any collection of $d \geq 5$ distinct points. If Γ fails to impose independent linear conditions on $|\mathcal{O}_{\mathbb{P}^2}(d-3)|$ then at least $d-1$ of the points are collinear.*

To see why the proof of Theorem 1.1 follows from that of Theorem 3.4, recall from the introduction that to specify a meromorphic function of degree d on C, we specify a divisor D of degree d on C such that the linear system $|D|$ has no base points and $\dim |D| \geq 1$, where

$$\dim |D| := h^0(D) - 1.$$

In the case the divisor D on C has a linear system as above, we say that D moves.

Definition 3.5. *The finite set* $\Gamma = \{p_1, \ldots, p_d\} \subset \mathbb{P}^2$ *of distinct points* **imposes linear independent conditions** *on plane curves of degree m if for every point $P \in \Gamma$ there exist plane curves of degree m that contains $\Gamma \setminus P$ and does not contain the point $P \in \Gamma$.*

Consider the subset $\Gamma \subset \mathbb{P}^2$ as a closed zero-dimensional subscheme of $\mathbb{P}^2$. Then we have the standard exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow \mathcal{O}_\Gamma(m) \longrightarrow 0, \tag{4}$$

where $\mathscr{I}_\Gamma \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of the zero dimensional variety Γ. Note that $\mathcal{O}_\Gamma(m) \cong \oplus_{i=1}^{d} \mathcal{O}_{p_i} \cong \mathbb{C}^d$, and that surjectivity of

$$\alpha : \mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow \mathbf{H}^0(\Gamma, \mathcal{O}_\Gamma(m))$$

exactly means that there is for each p_i, $i = 1, \ldots, d$ a plane curve of degree m that contains $\Gamma \setminus \{p_i\}$ but not p_i. Hence $\Gamma \subset \mathbb{P}^2$ fails to impose independent conditions on curves of degree m if and only if α is not surjective. Namely if and only if

$$h^0(\mathscr{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^2}(m)) > h^0(\mathcal{O}_{\mathbb{P}^2}(m)) - d = \frac{(m+1)(m+2)}{2} - d.$$

Equivalently since $\mathbf{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) = 0$, Γ fails to impose independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(m)|$ if we have $h^1\big(\mathscr{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^2}(m)\big) > 0$.

Let $D = p_1 + \ldots + p_d$ be a divisor of degree d on a smooth curve $C \subset \mathbb{P}^2$. A criterion for determining when D moves is given by the Riemann-Roch theorem for curves. Denote by H the divisor of a general linear section. The adjunction formula tells us that

$$K_C \sim (d-3)H.$$

By the Bézout theorem the degree of the divisor $(d-3)H$ is equal to $d(d-3)$. So we obtain that

$$2g - 2 = (d-3)d \quad \text{or} \quad g = \frac{(d-1)(d-2)}{2}.$$

The Riemann-Roch formula implies that

$$h^0(D) = d - g + 1 + h^0\big(K_C - D\big),$$

and hence $\dim |D| \geq 1$ if and only if

$$\dim |K_C - D| \geq \frac{(d-1)(d-2)}{2} - d. \tag{5}$$

Now the ideal sheaf $\mathscr{I}_C$ of C in $\mathbb{P}^2$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-C)$, and so

$$\mathbf{H}^0(\mathbb{P}^2, \mathscr{I}_C \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)) \cong \mathbf{H}^1(\mathbb{P}^2, \mathscr{I}_C \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)) = 0$$

since $\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbf{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$. Twisting the exact sequence

$$0 \longrightarrow \mathscr{I}_C \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

by $\mathcal{O}_{\mathbb{P}^2}(d-3)$, we find that $\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \cong \mathbf{H}^0(C, \mathcal{O}_C(d-3))$. Furthermore we have that

$$\mathbf{H}^0(\mathbb{P}^2, \mathscr{I}_\Gamma \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)) = \ker\big(\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) \longrightarrow \mathbf{H}^0(\Gamma, \mathcal{O}_\Gamma(d-3))\big).$$

On the other hand, $K_C \sim (d-3)H$ and $\mathcal{O}_C(D)$ is the ideal of D in C which implies that

$$\mathbf{H}^0(C, \mathcal{O}_C(K_C - D)) = \ker\big(\mathbf{H}^0(C, \mathcal{O}_C(d-3)) \longrightarrow \mathbf{H}^0(\Gamma, \mathcal{O}_\Gamma(d-3))\big),$$

so we find that $h^0(\mathcal{O}_C(K_C - D)) = h^0\big(\mathscr{I}_D \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)\big)$. Hence (5) is equivalent to the inequality

$$h^0\big(\mathscr{I}_D \otimes \mathcal{O}_{\mathbb{P}^2}(d-3)\big) > \frac{(d-1)(d-2)}{2} - d. \tag{6}$$

In other words, the divisor $D = p_1 + \ldots + p_d$ satisfies $\dim|D| \geq 1$ if and only if the set $\Gamma = \{p_1, \ldots, p_d\}$ fails to impose independent conditions on the canonical linear system $|K_C|$. We will now see that we may use this to derive Theorem 1.1 from Theorem 3.4.

To complete the proof of Theorem 1.1, it suffices to show that either all the d points of D are collinear, or if only the $d-1$ points of D lie on a line then the d-th point is a base point of the linear system $|D|$. In the first case $D \sim H$ and we are done. In the second case, suppose that $D = p_1, \ldots, p_{d-1} + q$, where the points $p_1, \ldots, p_{d-1}$ lie on a line ℓ and $q \notin \ell$. We must show that q is a base point of the linear system $|D|$ or equivalently that we have

$$\dim|p_1 + \ldots + p_{d-1}| = \dim|p_1 + \ldots + p_{d-1} + q|.$$

But as the degree of the divisor $p_1 + \ldots + p_{d-1}$ is equal to $\deg D - 1$, the Riemann-Roch then implies that it is enough to show that the following equality:

$$\dim|K_C - p_1 - \ldots - p_{d-1} - q| = \dim|K_C - p_1 - \ldots - p_{d-1}| - 1 \tag{7}$$

holds. Since $\deg C = d$, we can write the divisor cut by C on ℓ as $C \cdot \ell = p_1 + \ldots + p_{d-1} + b$, where $b \neq q$ because $q \notin \ell$. If a curve C_1 of degree $d-3$ passes through $d-1$ collinear points $p_1, \ldots, p_{d-1}$, it must contain ℓ as a component. Thus, the linear system in equation (7) on left-hand side

$$|K_C - p_1 - \ldots - p_{d-1} - q| \cong |\mathscr{I}_q \otimes \mathcal{O}_{\mathbb{P}^2}(d-4)|,$$

whereas the linear system on right-hand side in (7)

$$|K_C - p_1 - \ldots - p_{d-1}| \cong |\mathcal{O}_{\mathbb{P}^2}(d-4)|$$

which follows from the fact that $\dim|\mathscr{I}_q \otimes \mathcal{O}_{\mathbb{P}^2}(d-4)| = \dim|\mathcal{O}_{\mathbb{P}^2}(d-4)| - 1$. And this implies (7), which completes the proof. $\square$

It is worthy to remark that if $p_1, \ldots, p_{d-1}$ are distinct points in $\mathbb{P}^2$, then they will always impose independent conditions on curves of degree $d \geq 4$. In particular, the divisor $D = p_1 + \ldots + p_{d-1}$ moves in a linear pencil if and only if the points $p_1, \ldots, p_{d-1}$ lie on a line. It follows that for a smooth plane curve $C \subset \mathbb{P}^2$ of degree d, there is no nonconstant meromorphic function of degree less than $d-1$.

4. Proof of Theorem 3.4

To shorten the proof of theorem 3.4, we first reformulate it below in a slightly different but equivalent form.

Theorem 4.1. *Let $\Gamma = \{p_0, \dots, p_d\} \subset \mathbb{P}^2$, be any collection of $d+1 \geq 5$ distinct points. If Γ fails to impose independent linear conditions on $|\mathcal{O}_{\mathbb{P}^2}(d-2)|$ then at least d of the points in Γ are collinear.*

Proof. By assumption there exists at least one point (without loss of generality) say $p_0 \in \Gamma$ such that any curve of degree $d-2$ passing through the points in $\Gamma \setminus p_0$ also passes through p_0. Note that if we have a curve C of degree $n \leq d-2$ that passes through $\Gamma \setminus p_0$, then it follows by assumption that C also must pass through p_0.

Let $p_0, p_1 \dots, p_j$ be the minimal number of points in Γ lying on a line ℓ containing the point p_0. Rename the remaining points as $q_1, \dots q_{d-j}$. By construction, any line through a point $p_i \neq p_0$ and a point q_i, will not pass through p_0. We now construct a curve C being a product of such lines. We let ℓ_i be the line through p_i and q_i if $1 \leq i \leq \min\{j, d-j\}$. For the possible remaining points, we either let ℓ_i denote the lines through p_i and q_1 (if $d-j < i \leq j$) or the line through q_i and p_1 (if $j < i \leq d-j$). The curve

$$C = \ell_1 \dots \ell_n \qquad (\text{where } n = \max\{j, d-j\})$$

passes through all the points of $\Gamma \setminus p_0$, but not though p_0. If we have $2 \leq j \leq d-2$ then we get that the degree $n \leq d-2$, which is a contradiction to our assumption.

If we have $j = 1$, then any line ℓ' through two points $\Gamma \setminus p_0$ would not contain p_0. Observe that, to cover $\Gamma \setminus p_0$, we need at most $n \leq d/2$ lines $\ell'_1, \dots, \ell'_n$ if d is even, and at most $n \leq (d+1)/2$ lines to cover $\Gamma \setminus p_0$, if d is odd. Note that $d \geq 5$ is equivalent to $(d+1)/2 \leq d-2$, and if $d = 4$ then we have that $d/2 \leq d-2$. Hence for any d, in our range, we have the curve

$$C' = \ell'_1 \dots \ell'_n$$

of degree $n \leq d-2$ that passes through all points of $\Gamma \setminus p_0$, but not through p_0. This is impossible by assumption.

Finally, we are left with the only possibility that $j > d-2$. However if $j \geq d-1$, then we have at least $j+1 \leq d$ point $p_0, \dots, p_j$ aligned on the line ℓ. This completes the proof. $\square$

5. Plane Hurwitz numbers and Zeuthen Numbers

5.1. Plane Hurwitz Numbers

Generally in calculating Hurwitz numbers, we make no reference to the embedding of curves. For example, one can not expect for instance a branched covering of $\mathbb{P}^1$ whose domain is genus 2 to be planar and smooth, since a smooth plane curve of degree d, has $g = \binom{d-1}{2}$. Additionally, we expect that not all curves of genus $g = \binom{d-1}{2}$ can be embedded in $\mathbb{P}^2$ as smooth curves. For instance, among all smooth curves of genus 3 (for $d = 4$), there are hyperelliptic curves, which are not planar.

Fix $d > 0$; the space parametrizing all degree d algebraic curves in $\mathbb{P}^2$ is a complete system $|\mathcal{O}_{\mathbb{P}^2}(d)|$, which forms a projective space

$$\mathbb{P}(\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))) \cong \mathbb{P}^{\mathbf{N}},$$

where $\mathbf{N} = \binom{d+2}{2} - 1 = d(d+3)/2$. In particular, the set of all smooth plane curves of a given degree d is an open subset of $\mathbb{P}^{\mathbf{N}}$. The group $\mathbb{P}\mathbf{GL}(3, \mathbb{C})$ of all projective automorphisms of $\mathbb{P}^2$ acts on $\mathbb{P}^{\mathbf{N}}$ in a natural way. Of particular interest is the subgroup $\mathcal{G}_p \subset \mathbb{P}\mathbf{GL}(3, \mathbb{C})$ fixing p and preserving the pencil of lines through p. Given a smooth curve $C \subset \mathbb{P}^2$, for instance if $p = [0:1:0] \in \mathbb{P}^2 \setminus C$ for some choice of coordinate system of $\mathbb{P}^2$ an element of the group $\mathcal{G}_p$ has the form

$$g = \begin{bmatrix} g_0 & 0 & 0 \\ g_1 & g_2 & g_3 \\ 0 & 0 & g_0 \end{bmatrix} \quad \text{with} \quad g_0 g_2 \neq 0.$$

The group of automorphisms $\mathcal{G}_p$ acts equivalently on $\mathbb{P}^{\mathbf{N}}$ keeping the branching points of the projection $\pi_p : C \to \mathbb{P}^1$ fixed. Recall from Definition 2.1, that two branched coverings $\pi_p^1 : C_1 \to \mathbb{P}^1$ and $\pi_p^2 : C_2 \to \mathbb{P}^1$ are called equivalent if there exists an isomorphism $g : C_1 \to C_2$ such that $\pi_p^2 \circ g = \pi_p^1$. Then we have:

Proposition 5.1. *Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth projective plane curves of the same degree $d > 1$ and not passing through $p \in \mathbb{P}^2$. Two projections $\pi_p^1 : C_1 \longrightarrow \mathbb{P}^1$ and $\pi_p^2 : C_2 \longrightarrow \mathbb{P}^1$ are equivalent if and only if there exists an automorphism $g \in \mathcal{G}_p$ such that $g(C_1) = C_2$.*

Proof. Let $C_1, C_2 \subset \mathbb{P}^2$ be smooth projective curves not passing through $p \in \mathbb{P}^2$. If there exists an automorphism $g \in \mathcal{G}_p$ such that $C_2 = g(C_1)$, then the morphisms π_p and π_p' are equivalent by an isomorphism given by g. For the 'only if ' direction, suppose that π_p^1 and π_p^2 are equivalent and that this equivalence is determined by an isomorphism $g : C_1 \to C_2$. For each line $\ell \ni p$ the isomorphism g maps $C_1 \cap \ell$ to $C_2 \cap \ell$; thus, g maps hyperplane sections of C_1 to hyperplane sections of C_2. Since both C_1 and C_2 are embedded in $\mathbb{P}^2$ by complete linear system of hyperplane sections $\mathbf{H}^0(\mathbb{P}^2, \mathcal{O}_{C_i}(1))$, for $i = 1, 2$, this implies that g is induced by projective automorphism $\mathbb{P}\mathbf{GL}(3, \mathbb{C})$. To complete the proof, it only remains to check that $g \in \mathcal{G}_p$; to that end, consider a generic line $\ell \ni p$; this line intersects C_i for $i = 1, 2$ at $d = \deg C_i > 1$ points and this points are mapped by g to d distinct points on ℓ. So $g(\ell) = \ell$ for the generic line and thus for any $\ell \ni p$. If ℓ_1, ℓ_2 containing p then

$$g(p) = g(\ell_1 \cap \ell_2) = g(\ell_1) \cap g(\ell_2) = \ell_1 \cap \ell_2 = p.$$

Hence $g \in \mathcal{G}_p$ as expected and this completes the proof. □

A **generic projection** of smooth curve $C \subset \mathbb{P}^2$ from a point $p \in \mathbb{P}^2$ which is not on a bitangent line or a flex line we obtain a linear projection $\pi_p : C \to \mathbb{P}^1$ with only simple branch points. This leads us to the orbit space parametrizing all generic linear projections. Denote this space of generic linear projections by:

$$\mathcal{PH}_d = \left\{ \pi_p : C \to \mathbb{P}^1 \;\middle|\; \begin{array}{c} \pi_p \text{ is a simple linear projection from} \\ p \in \mathbb{P}^2 \setminus C \text{ of a smooth curve } C \subset \mathbb{P}^2 \end{array} \right\} \Big/ \sim . \tag{8}$$

where $\sim$ is the equivalence of projections from a point $p \in \mathbb{P}^2$ up to the $\mathcal{G}_p$- action.

Note that for $g = \binom{d-1}{2}$, we have a natural inclusion $\mathcal{PH}_d \subseteq \mathcal{H}_{d,g}$ of small Hurwitz spaces for $d > 1$. The information about the dimension of $\mathcal{PH}_d$ is a direct consequence of proposition 5.1 we summarize as follows.

Corollary 5.2. *The dimension of the space $\mathcal{PH}_d$ is equal to $\mathbf{N} - 3 = \frac{d(d+3)}{2} - 3$.*

The number of branch points of a generic projection $\pi_p : C \to \mathbb{P}^1$ of a smooth curve of degree d from $p \in \mathbb{P}^2 \setminus C$ is determined by the Riemann-Hurwitz formula as $w = d(d-1)$. We refer to the number of 3-dimensional $\mathcal{G}$-orbits with the same set of w tangents lines as the **d-th plane Hurwitz number** and denote it by $\mathfrak{h}_d$. Thus, to compute $\mathfrak{h}_d$ as indicated in (3), we need to calculate the degree of the branch morphism

$$\mathcal{PH}_d \longrightarrow \mathrm{Sym}^w \mathbb{P}^1 \backslash \Delta, \tag{9}$$

restricted to its image. Notice that by Corollary 5.2 the $\dim \mathcal{PH}_d < d(d-1)$ for $d \geq 4$. Next we will give two examples of known plane Hurwitz numbers.

Degree 3-plane Hurwitz Numbers

The first nontrivial case involves projections of smooth plane cubics. The remark following Theorem 1.1 asserts that if $d = 3$ not all meromorphic function of degree 3 on smooth plane cubics are realizable as projections. However, degree 3 simple plane Hurwitz numbers coincides with the usually Hurwitz number. Namely, over $w = 6$ pairwise distinct points on the projective line $\mathbb{P}^1$ there are exactly 40 three-dimensional orbits of smooth cubics branched over them, see [1]. To see this, recall that Hurwitz numbers count branched covering up to equivalence, the equivalence of plane Hurwitz with the usual Hurwitz number is a consequence of the fact that every meromorphic function of degree 3 on a smooth cubic is a composition of a group shift of C followed by a linear projection from $p \in \mathbb{P}^2 \setminus C$. This is a well-known consequence of the fact that any smooth plane cubic curve is an abelian group. We give the details below.

Proposition 5.3. *Every meromorphic function of degree* 3 *on a smooth cubic curve* $C \in \mathbb{P}^2$ *can be represented as a composition of a group shift on* C *by a fixed point on* C *with a linear projection from a point* $p \in \mathbb{P}^2$.

Proof. Let C be a smooth projective cubic and let $f : C \longrightarrow \mathbb{P}^1$ be a meromorphic function of degree 3. If we write $f^{-1}(0) = z_1 + z_2 + z_3$, $f^{-1}(\infty) = p_1 + p_2 + p_3$ for the zero divisor and polar divisor of f respectively (where z_i and p_i for all $i = 1, 2, 3$ are not necessarily distinct). The linear equivalence of divisors $f^{-1}(0) \sim f^{-1}(\infty)$ implies the equality

$$p_1 + p_2 + p_3 = z_1 + z_2 + z_3$$

as divisors, where "+" denotes the addition from group law on the cubic curve. Fix a point $P_0 \in C$ such that $p_1 + p_2 + p_3 + 3P_0 = 0$ and define

$$Q_i = p_i + P_0, \quad \text{and} \quad R_i = z_i + P_0 \quad \text{for all } i = 1, 2, 3.$$

Then we have

$$Q_1 + Q_2 + Q_3 = p_1 + p_2 + p_3 + 3P_0 = 0$$
$$R_1 + R_2 + R_3 = z_1 + z_2 + z_3 + 3P_0 = 0.$$

In particular, $\{Q_1, Q_2, Q_3\}$ and $\{R_1, R_2, R_3\}$ lie on distinct lines in $\mathbb{P}^2$, Since otherwise these sets would be equal and so $f^{-1}(0) = f^{-1}(\infty)$, which is impossible. Denote the lines given by the translates $\{Q_1, Q_2, Q_3\}$ and $\{R_1, R_2, R_3\}$ by $\ell_1 \subset \mathbb{P}^2$ and $\ell_2 \subset \mathbb{P}^2$ respectively. If $l_1(x, y, z)$ and $l_2(x, y, z)$ are equations for the lines ℓ_1 and ℓ_2, the meromorphic function given by composition of the group shift and projection is the quotient l_1/l_2: $f(P - P_0) = \frac{\ell_1(P)}{\ell_2(P)} \iff f(P) = \frac{\ell_1(P+P_0)}{\ell_2(P+P_0)}$, (where $P = (x, y, x)$) after possibly multiplying with a constant using the fact that a meromorphic function without poles will be constant. $\square$

Degree 4-plane Hurwitz Numbers

The case $d = 4$ is more exciting. Note that the space parametrizing projections $\mathcal{PH}_4$ has dimension $\frac{4(4+3)}{2} - 3 = 11$. As branched coverings, this 11-dimensional family $\mathcal{PH}_4$ admits a natural inclusion into the small Hurwitz space $\mathcal{H}_{4,3}$ defined in (2) which is a smooth irreducible variety of dimension 12. The inclusion $\mathcal{PH}_4 \subset \mathcal{H}_{4,3}$ implies that the branch locus defines an hypersurface $\mathbf{B} \subset \mathrm{Sym}^{12}\,\mathbb{P}^1$. R. Vakil in [10] has computed its degree to be equal to 3762. Moreover, he establishes that there are essentially 120 smooth plane quartic branched over admissible 12 points in $\mathbb{P}^1$. Thus, it follows that the plane Hurwitz number of degree 4 is

$$\mathfrak{h}_4 = 120 \times \frac{(3^{10} - 1)}{2}. \tag{10}$$

The corresponding Hurwitz number is known to be equal to $h_{3,4} = 255 \times \frac{(3^{10}-1)}{2}$.

5.2. Zeuthen numbers

This notion of plane Hurwitz numbers has a strong analogy to the special case of Zeuthen's classical problem which asks to calculate the number of irreducible plane curves of degree $d > 0$ and geometric genus $g \geq 0$ passing through a general points and b tangent lines in $\mathbb{P}^2$, where $a + b = 3d + g - 1$. More precisely, assuming that the only singularities of an irreducible curve $C \subset \mathbb{P}^2$ are δ nodes, since each node reduces the freedom of the curve by 1, we expect the set of irreducible degree d curves with δ nodes depends on

$$\dim |\mathcal{O}_{\mathbb{P}^2}(d)| - \delta = \frac{d(d+3)}{2} - \delta = 3d + g - 1$$

parameters. Indeed, for all fixed integers $d > 0$ and $g \geq 0$ as first observed by F. Severi [6] and proved by J. Harris [7], the Severi variety $V_{g,\delta}$ parametrizing irreducible plane curves of degree d with δ nodes is a quasiprojective variety of dimension $3d + g - 1$. It follows that for a fixed $d > 0, g \geq 0$ the numbers $N_d(g)$ of curves passing through $3d + g - 1$ general points is finite and does not depend on the generic configuration of points chosen. This $N_d(g)$ number is commonly referred to as **Severi degree** of plane curves.

In general, fix integers $d > 0$ and $a, b, g \geq 0$. The number of irreducible curves of geometric genus g and degree d passing through a general points and tangent to b general lines in $\mathbb{P}^2$ is finite provided $a + b = 3d + g - 1$. These numbers are called **characteristic numbers** of plane curves and we denote them by $N_g(a, d)$. The question of calculating characteristic numbers is the **classical problem of Zeuthen** and thus we usually refer to the numbers $N_g(a, d)$ as **Zeuthen Numbers**. In [11], H.G. Zeuthen calculated the characteristic numbers of smooth curves in $\mathbb{P}^2$ of degree at most 4 and [9] has verified Zeuthen's results using modern results on moduli spaces of stable maps, for an exposé see e.g. [13].

5.3. Homological interpretation of Zeuthen numbers

Let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)$ be the Kontsevich moduli space of maps to $\mathbb{P}^2$ of fixed degree $d > 0$ and arithmetic genus $g \geq 0$. Consider the open substack of maps of smooth curves $\mathcal{M}_{g,0}(\mathbb{P}^2, d)$. The closure of $\mathcal{M}_{g,0}(\mathbb{P}^2, d)$ is a unique component of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)$ of dimension $3d + g - 1$ we denote by $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$. The Zeuthen number $N_g(a, d)$ can be interpreted in the language of stable maps.

Let α and β denote the divisors in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$ representing classes of a point and a line respectively. The characteristic number $N_g(a, d)$ is given by the degree of $\alpha^a\beta^b$ and is denoted by $\alpha^a\beta^b \cap \left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger\right]$. For example, it is known there is a unique smooth cubic through 9 general points, then we will write $\alpha^9 \cap \left[\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)^\dagger\right] = 1$.

The following existence result is the key point for this interpretation.

Proposition 5.4. *There exist two divisors α and β such that the number $N_g(a, d)$ is $\alpha^a\beta^b \cap \left[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger\right]$.*

Proof. See [8], Theorem 3.15. $\square$

We finish with an open problem. As above let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$ be the closure of the open substack $\mathcal{M}_{g,0}(\mathbb{P}^2, d)$ of maps of smooth curves of degree d. Among the boundary divisors representing the closure of loci of maps (see [8] for precise descriptions) of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2, d)^\dagger$, we have a divisor $\mathbb{I}_d$ the closure of the locus of degree $d : 1$ maps of smooth curves of degree d into a line in $\mathbb{P}^2$. Such generic maps are necessarily branched at $d(d-1)$ points by Riemann-Hurwitz formula. Thus the divisor $\mathbb{I}_d$ enumerates a special class of Zeuthen numbers whose calculation is related to that of Hurwitz numbers. Namely, the Zeuthen numbers $\beta^{3d+g-2}[\mathbb{I}_d]$ for $g = \binom{d-1}{2}$. For instance, R. Vakil in [9] calculates that $\beta^8[\mathbb{I}_3] = 40 \times 210$ and $\beta^{13}[\mathbb{I}_4] = 120 \cdot 2535$. It makes sense to consider the divisor $\mathbb{I}_d$ up to the $\mathcal{G}_p$-action.

6. Problem

Is there a natural homology class $\beta \in \mathbf{H}_{2(3d+g-4)}\big(\overline{\mathcal{M}}_{g,0}(\mathbb{P}^2,d)\big)^{\dagger}/\mathcal{G}_p,\mathbb{Q})$ such that

$$\mathfrak{h}_d = \beta^{3d+g-5} \cap \big[\mathbb{I}_d/\mathcal{G}_p\big].$$

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