Int. J. Math. And Appl., 7(4)(2019), 149–152
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On the Connectedness of the Complement of a Unit Graph of a Commutative Ring

Dr. Jaydeep Parejiya[1,*]

1 Department of Mathematics, Government Polytechnic, Rajkot, Gujarat, India.

Abstract: The rings considered in this article are commutative with identity. We denote the set of all maximal ideals of a ring R by $Max(R)$ and we denote the Jacobson radical of R by $J(R)$. Let R be a ring. Recall from [2] that the *unit graph* of R, denoted by $G(R)$, is an undirected graph whose vertex set of all elements of R and distinct vertices x, y are joined by an edge in this graph if and only if $x + y \in U(R)$. In this article, we studied Complement of unit graph and we denoted it $(UG(R))^c$. Hence, in this graph two elements x, y are joined by an edge in $(UG(R))^c$ if and only if $x + y \in NU(R)$. In this article we proved some results on connectedness of $(UG(R))^c$.

MSC: 13A15, 05C25.

Keywords: Connected graph, Max(R).

1. Introduction

Let R be a nonzero ring with identity. Let $U(R)$ denote the set of all units of R and let us denote the set of all nonunits of R by $NU(R)$. Recall from [2] that the *unit graph* of R, denoted by $G(R)$, is an undirected graph whose vertex set of all elements of R and distinct vertices x, y are joined by an edge in this graph if and only if $x + y \in U(R)$. Let $G = (V, E)$ be a simple graph. The complement G^c of G is defined by taking V(G^c)=V(G)=V and two distinct vertices u and v are adjacennt in G^c if and only if they are not adjacent in G. In this article, we studied Complement of unit graph and we denoted it $(UG(R))^c$. Hence, in this graph two elements x, y are joined by an edge in $(UG(R))^c$ if and only if $x + y \in NU(R)$. In this article we proved some results on connectedness of $(UG(R))^c$. Subgraph H of G is said to be a *spanning subgraph* of G, if $V(H) = V(G)$.

2. Some Results on the Connectedness of $(UG(R))^c$

Let R be a ring. The aim of this section is to discuss some results on the connectedness of $(UG(R))^c$.

Proposition 2.1. *Let R be a ring. The following statements are equivalent:*

(i). $(UG(R))^c$ is connected.

(ii). $|Max(R)| \geq 2$.

* *E-mail: parejiyajay@gmail.com*

Proof. $(i) \Rightarrow (ii)$ Assume that $(UG(R))^c$ is connected. Suppose that R is quasilocal with $\mathfrak{m}$ as its unique maximal ideal. Let $V_1 = \mathfrak{m}$ and let $V_2 = R \backslash \mathfrak{m}$. It is clear that $V_i \neq \emptyset$ for each $i \in \{1, 2\}$, $R = V((UG(R))^c) = V_1 \cup V_2$, and $V_1 \cap V_2 = \emptyset$. Let $x \in V_1$ and let $y \in V_2$. Then $x + y \in U(R)$. Hence, x and y are not adjacent in $(UG(R))^c$. Thus there exists no edge of $(UG(R))^c$ whose one end vertex is in V_1 and the other in V_2. Therefore, we obtain from [4, Theorem 2.1] that $(UG(R))^c$ is not connected. This is a contradiction and so, $|Max(R)| \geq 2$.

$(ii) \Rightarrow (i)$ We are assuming that $|Max(R)| \geq 2$. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be any two distinct maximal ideals of R. Note that $\mathfrak{m}_1 + \mathfrak{m}_2 = R$ and so, there exist $x \in \mathfrak{m}_1$ and $y \in \mathfrak{m}_2$ such that $x + y = 1$. Let $a, b \in R$ be such that $a \neq b$. We show that there exists a path of length at most two between a and b in $(UG(R))^c$. We can assume that a and b are not adjacent in $(UG(R))^c$. Then $a + b \in U(R)$. Observe that $a + (-ax - by) = a(1 - x) - by = ay - by = (a - b)y \in \mathfrak{m}_2$ and $b + (-ax - by) = -ax + b(1 - y) = -ax + bx = (b - a)x \in \mathfrak{m}_1$. Thus $a - (-ax - by) - b$ is a path of length two between a and b in $(UG(R))^c$. This proves that $(UG(R))^c$ is connected. □

Corollary 2.2. *Let R be a ring such that $|Max(R)| \geq 2$. Then $(UG(R))^c$ is connected and moreover, $diam((UG(R))^c) = r((UG(R))^c) = 2$.*

Proof. Since $|Max(R)| \geq 2$, we know from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(UG(R))^c$ is connected and moreover, we know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((UG(R))^c) \leq 2$. Let $r \in R$. We claim that $e(r) \geq 2$ in $(UG(R))^c$. Suppose that $r \in U(R)$. Then r and 0 are not adjacent in $(UG(R))^c$) and so, $d(r, 0) \geq 2$ in $(UG(R))^c$. Hence, $e(r) \geq 2$ in $(UG(R))^c$. Suppose that $r \in NU(R)$. Note that $r \neq 1 - r$ and $r + 1 - r = 1 \in U(R)$. Hence, r and $1 - r$ are not adjacent in $(UG(R))^c$. Therefore, $d(r, 1 - r) \geq 2$ in $(UG(R))^c$ and so, $e(r) \geq 2$ in $(UG(R))^c$. Since $diam((UG(R))^c) \leq 2$, we obtain that $e(r) = 2$ in $(UG(R))^c$ for any $r \in R$. This proves that $diam((UG(R))^c) = r((UG(R))^c) = 2$. □

Corollary 2.3. *Let R be a ring. Then $(UG(R[X]))^c$ is connected and $diam((UG(R[X]))^c)$ $= r((U(R[X]))^c) = 2$, where $R[X]$ is the polynomial ring in one variable X over R.*

Proof. Let $\mathfrak{m}$ be any maximal ideal of R. Observe that $\frac{R[X]}{\mathfrak{m}[X]} \cong \frac{R}{\mathfrak{m}}[X]$ as rings. Let us denote the field $\frac{R}{\mathfrak{m}}$ by F. Since $F[X]$ has an infinite number of maximal ideals, it follows that $Max(R[X])$ is infinite. Therefore, we obtain from Corollary 2.2 that $(UG(R[X]))^c$ is connected and $diam((UG(R[X]))^c) = r((UG(R[X]))^c) = 2$. □

Proposition 2.4. *Let R be a ring. The following statements are equivalent:*

(i). $(UG(R))^c$ *is connected.*

(ii). $|Max(R)| \geq 2$.

(iii). $NU(R)$ *is a dominating set of* $(UG(R))^c$.

Proof. $(i) \Rightarrow (ii)$ This follows from $(i) \Rightarrow (ii)$ of Proposition 2.1.

$(ii) \Rightarrow (iii)$ Let $\mathfrak{m}_1, \mathfrak{m}_2$ be any two distinct maximal ideals of R. From $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, we obtain that there exist $x \in \mathfrak{m}_1$ and $y \in \mathfrak{m}_2$ such that $x + y = 1$. Let $a \in U(R)$. Note that $-ax \in \mathfrak{m}_1 \subseteq NU(R)$ and $a + (-ax) = a(1 - x) = ay \in \mathfrak{m}_2$. Hence, a and $-ax$ are adjacent in $(UG(R))^c$. This proves that $NU(R)$ is a dominating set of $(UG(R))^c$.

$(iii) \Rightarrow (i)$ We are assuming that $NU(R)$ is a dominating set of $(UG(R))^c$. We want to prove that $(UG(R))^c$ is connected. In view of $(ii) \Rightarrow (i)$ of Proposition 2.1, it is enough to show that $|Max(R)| \geq 2$. Suppose that R is quasilocal with $\mathfrak{m}$ as its unique maximal ideal. Observe that $NU(R) = \mathfrak{m}$. For any $m \in \mathfrak{m}$, $1 + m \in U(R)$. Hence, 1 is not adjacent to any nonunit m of R in $(UG(R))^c$. This is in contradiction to the assumption that $NU(R)$ is a dominating set of $(UG(R))^c$. Therefore, we obtain that $(UG(R))^c$ is connected. □

Let $(R, \mathfrak{m})$ be a quasilocal ring. We know from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $(UG(R))^c$ is not connected. In Theorems 2.5 and 2.6, we determine the number of components of $(UG(R))^c$ under the assumption that $\frac{R}{\mathfrak{m}}$ is finite.

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\frac{R}{\mathfrak{m}}$ is finite and $char(\frac{R}{\mathfrak{m}}) \neq 2$. Then the number of components of $(UG(R))^c$ equals $\frac{|(\frac{R}{\mathfrak{m}})^*|}{2} + 1$.*

Proof. Observe that $V((UG(R))^c) = R = \mathfrak{m} \cup (R \backslash \mathfrak{m})$. For any $x, y \in \mathfrak{m}$, $x + y \in \mathfrak{m} = NU(R)$. Hence, the subgraph of $(UG(R))^c$ induced on $\mathfrak{m}$ is a clique. Moreover, if $x \in \mathfrak{m}$ and $y \in U(R) = R \backslash \mathfrak{m}$, $x + y \in U(R)$ and so, x and y are not adjacent in $(UG(R))^c$. This shows that the subgraph of $(UG(R))^c$ induced on $\mathfrak{m}$ is a component of $(UG(R))^c$ and let us denote it by H. We are assuming that $\frac{R}{\mathfrak{m}}$ is finite and $char(\frac{R}{\mathfrak{m}}) \neq 2$. Therefore, $|\frac{R}{\mathfrak{m}}| = p^n$ for some odd prime number p and $n \geq 1$. Hence, $|(\frac{R}{\mathfrak{m}})^*| = 2t$ for some $t \geq 1$. Therefore, there exist $u_i \in U(R)$ for each $i \in \{1, \ldots, t\}$ with $u_1 = 1$ such that $(\frac{R}{\mathfrak{m}})^* = \{u_i + \mathfrak{m}, -u_i + \mathfrak{m} | i \in \{1, \ldots, t\}\}$. Let $i \in \{1, \ldots, t\}$ and let us denote $\{x \in U(R) |$ either $x \equiv u_i (mod \mathfrak{m})$ or $x \equiv -u_i (mod \ \ \mathfrak{m})\}$ by W_i. It is clear that $U(R) = \cup_{i=1}^{t} W_i$ and $W_i \cap W_j = \emptyset$ for all distinct $i, j \in \{1, \ldots, t\}$. For each $i \in \{1, \ldots, t\}$, let us denote by H_i, the subgraph of $(UG(R))^c$ induced on W_i. We claim that H_i is a component of $(UG(R))^c$. First, we show that H_i is connected. Let $x, y \in W_i$ be such that $x \neq y$. Suppose that x and y are not adjacent in $(UG(R))^c$. Then $x + y \in U(R)$. Therefore, either both x and y are congruent to u_i modulo $\mathfrak{m}$ or both x and y are congruent to $-u_i$ modulo $\mathfrak{m}$. We consider the following cases.

Case (i)**:** $x \equiv u_i (mod \ \ \mathfrak{m})$ and $y \equiv u_i (mod \ \ \mathfrak{m})$.

Note that $-u_i \in W_i$ and $x - u_i, y - u_i \in \mathfrak{m}$. Hence, $x - (-u_i) - y$ is a path in H_i between x and y.

Case (ii)**:** $x \equiv -u_i (mod \ \ \mathfrak{m})$ and $y \equiv -u_i (mod \ \ \mathfrak{m})$.

Observe that $u_i \in W_i$ and $x + u_i, y + u_i \in \mathfrak{m}$. Hence, $x - u_i - y$ is a path in H_i between x and y.

This proves that H_i is connected. We next verify that there is no edge of $(UG(R))^c$ whose one end vertex is in W_i and the other end vertex not in W_i. Suppose that $x - y$ is an edge of $(UG(R))^c$ such that $x \in W_i$ and $y \notin W_i$. Hence, $x + y \in NU(R) = \mathfrak{m}$. As $x \in U(R)$, it follows that $y \in U(R)$. Therefore, $y \in W_j$ for some $j \in \{1, \ldots, t\}$ with $j \neq i$. Note that $x \equiv \pm u_i (mod \ \ \mathfrak{m})$ and $y \equiv \pm u_j (mod \ \ \mathfrak{m})$. As $\pm u_i \pm u_j \in U(R)$ and $x + y \equiv \pm u_i \pm u_j (mod \ \ \mathfrak{m})$, we obtain that $x + y \in U(R)$. This is a contradiction. Therefore, there exists no edge of $(UG(R))^c$ whose one vertex is in W_i and the other end vertex not in W_i. This proves that H_i is a component of $(UG(R))^c$.

It is clear from the above discussion that $\{H, H_i | i \in \{1, \ldots, t\}\}$ is the set of all components of $(UG(R))^c$. Therefore, the number of components of $(UG(R))^c$ equals $t + 1 = \frac{|(\frac{R}{\mathfrak{m}})^*|}{2} + 1$. □

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\frac{R}{\mathfrak{m}}$ is finite and $char(\frac{R}{\mathfrak{m}}) = 2$. Then the number of components of $(UG(R))^c$ equals $|\frac{R}{\mathfrak{m}}|$.*

Proof. It follows as in the proof of Theorem 2.5 that the subgraph of $(UG(R))^c$ induced on $\mathfrak{m}$ is a component of $(UG(R))^c$. Let us denote this component by H. As $\frac{R}{\mathfrak{m}}$ is finite and $char(\frac{R}{\mathfrak{m}}) = 2$, we obtain that $|\frac{R}{\mathfrak{m}}| = 2^n$ for some $n \geq 1$. Let $\{u_i \in U(R) | i \in \{1, \ldots, 2^n - 1\}\}$ with $u_1 = 1$ be such that $(\frac{R}{\mathfrak{m}})^* = \{u_1 + \mathfrak{m}, \ldots, u_{2^n - 1} + \mathfrak{m}\}$. Let $i \in \{1, \ldots, 2^n - 1\}$. Let us denote $\{x \in U(R) | x \equiv u_i (mod \ \ \mathfrak{m}\}\}$ by V_i. Let us denote the subgraph of $(UG(R))^c$ induced on V_i by H_i. Note that $U(R) = \cup_{i=1}^{2^n - 1} V_i$ and $V_i \cap V_j = \emptyset$ for all distinct $i, j \in \{1, \ldots, 2^n - 1\}$. Let $i \in \{1, \ldots, 2^n - 1\}$. We claim that H_i is a component of $(UG(R))^c$. First, we show that H_i is connected. Let $x, y \in V_i$ be such that $x \neq y$. Observe that $x + y \equiv 2u_i (mod \ \ \mathfrak{m})$ and as $2 \in \mathfrak{m}$, we get that $x + y \in \mathfrak{m}$ and so, x and y are adjacent in H_i. This shows that H_i is complete. We next verify that there is no edge of $(UG(R))^c$ whose one end vertex is in V_i and the other end vertex not in V_i. Suppose that $x - y$ is an edge of $(UG(R))^c$ such that $x \in V_i$ and $y \notin V_i$. Hence, $x + y \in NU(R)$ and so, $y \in U(R)$. Therefore, $y \in V_j$ for some $j \in \{1, \ldots, 2^n - 1\}$ with $j \neq i$. Now, $x + y \equiv u_i + u_j (mod \ \ \mathfrak{m})$. As $u_i - u_j \in U(R)$ and $2 \in \mathfrak{m}$, we obtain that $u_i + u_j = u_i - u_j + 2u_j \in U(R)$. Therefore, it follows that $x + y \in U(R)$. This is a contradiction. Hence, there is no edge of $(UG(R))^c$ of the form $x - y$ with $x \in V_i$ and $y \notin V_i$. This proves that H_i is a component of $(UG(R))^c$ for each $i \in \{1, \ldots, 2^n - 1\}$. It is clear from the above discussion that $\{H, H_i | i \in \{1, \ldots, 2^n - 1\}\}$ is the set of all components of $(UG(R))^c$. Therefore, the number of components of $(UG(R))^c$ equals $2^n = |\frac{R}{\mathfrak{m}}|$. □

References

[1] S. Kavithaa and R. Kalab, *A note on comaximal ideal graph of commutative rings*, AKCE International Journal of Graphs and Combinatorics, (2019), https://doi.org/10.1016/j.akcej.2019.06.004.

[2] N. Ashrafi, H. R. Mainmani, M. R. Pournaki and S. Yassemi, *Unit graph associated with rings* , Comm. Algebra, 38(2010), 2851-2871.

[3] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley publishing Company, (1969).

[4] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall of India Private Limited, New Delhi, (1994).

[5] M. I. Jinnah and S. C. Mathew, *When is the comaximal graph split?*, Comm. Algebra, 40(7)(2012), 2400-2404.

[6] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra, 319(2008), 1801-1808.

[7] S. M. Moconja and Z. Z. Petrovic, *On the structure of comaximal graphs of commutative rings with identity*, Bull. Aust. Math. Soc., 83(2011), 11-21.

[8] K. Samei, *On the comaximal graph of a commutative ring*, Canad. Math. Bull., 57(2)(2014), 413-423.

[9] P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, 176(1995), 124-127.

[10] S. Visweswaran and Jaydeep Parejiya, *When is the complement of the comaximal graph of a commutativering planar?*, ISRN algebra, 2014(2014), 8 pages.

[11] M. Ye and T. Wu, *Comaximal ideal Graphs of commutative rings*, J. Algebra Appl., 6(2012), DOI : 10.1142/S0219498812501149.