Int. J. Math. And Appl., 7(4)(2019), 175–184

ISSN: 2347-1557

Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Finite Dimensional Approximation of Simplified Gauss-Newton Scheme for Nonlinear Ill-Posed Problems

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Abstract: We consider the finite dimensional approximation of simplified Gauss-Newton iterative scheme presented in [14] for solving nonlinear ill-posed problems. The convergence and convergent analysis of this scheme is carried out with both an a priori and an a posteriori parameter choice strategies. The error estimates are derived accordingly. We propose an order optimal parameter choice strategy for the regularization parameter, which gives the optimal convergence rate. Finally, we present numerical examples to verify the theoretical results.

MSC: 65J10, 65J20, 65J22, 47L10.

Keywords: Nonlinear Ill-posed Problems, Regularization, Inverse Problems, Iterative Method, Parameter Choice Rule.

1. Introduction

Many inverse problems in science and engineering can be modeled as an operator equation of the form

$$F(x) = y \tag{1}$$

where $F : D \to H$ is a nonlinear operator, H is a real Hilbert space, $D \subset H$. Nonlinear ill-posed problems of the form (1) arise in a number of applications, see e.g., [1, 3, 8, 9, 15]. In practical situations, one may not have the accurate data y rather has to deal with inexact data $\tilde{y}$. Hence we assume that we have only approximate data $\tilde{y}$ of y satisfying

$$\|y - \tilde{y}\| \leq \delta \tag{2}$$

where $\delta > 0$ is the known noise level. In such circumstances, we consider the operator equation as

$$F(x) = \tilde{y} \tag{3}$$

with $\|y - \tilde{y}\| \leq \delta$, $\delta > 0$. In general, the operator equation (3) is ill-posed in the sense that the continuous dependence on the data cannot be guaranteed. We assume that (1) has a solution $x^\dagger$ and F possesses a locally uniformly bounded Fréchet derivative in $B_r(x_0) := \{x \in X : \|x - x_0\| < r\}$, where x_0 is an initial guess of $x^\dagger$. Then the computation of a stable solution of (1) from noisy data $\tilde{y}$ becomes an important topic of ill-posed problems, and the regularization [1, 3] have to be taken into

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account. Many regularization techniques are available in the literature [2, 4, 6, 9, 11–14]. The iterative scheme considered in [14] is

$$\tilde{x}_{n+1} = \tilde{x}_n + (A + \alpha I)^{-1}(K^*(\tilde{y} - F(\tilde{x}_n)) - \alpha(\tilde{x}_n - x_0)) \tag{4}$$

where $K = F'(x_0)$, $A = K^*K$ and $\alpha > 0$ be the regularization parameter. In this paper we are considering the finite dimensional approximation of (4) of the form,

$$\tilde{x}_{n+1,h} = \tilde{x}_{n,h} + (P_hAP_h + \alpha_n I)^{-1}(P_hK^*(\tilde{y} - F(\tilde{x}_{n,h})) - \alpha_n(\tilde{x}_{n,h} - x_0)), \tag{5}$$

where $\{\alpha_n\}$ is a positive sequence, $\tilde{x}_{0,h} := P_hx_0$, P_h defined by $P_h(x) = \sum_{j=1}^{dimH_h} \langle x, e_j\rangle e_j,\ x \in H$ is a sequence of orthogonal projections on the finite dimensional subspaces $H_0 \subset H_1 \subset .... \subset H_j ... \subset H$ with $\dim H_j \sim 2^{sj}, s \geq 1$ and $\{e_1, e_2, ...\}$ be the orthonormal basis for H. Here apart from an a priori method, we also propose an a posteriori stopping rule and an adaptive parameter choice strategy for choosing the regularization parameter.

This paper is organized as follows. In Section 2, we prove convergence analysis of the scheme and convergence rate using an a priori parameter choice rule. In Section 3, we propose an a posteriori parameter for the choice of regularization parameter combined with the discrepancy principle. Numerical examples are discussed in Section 4 to illustrate the theoretical results.

2. Convergence Analysis and Error Estimate

In order to establish the convergence of the method and deriving the convergence rate, we make use of the following assumptions.

Assumption I (A-I): There exists a positive constant k_0 and $\phi(x, x_0, v) \in H$ satisfying $(F'(x) - F'(x_0))v = F'(x_0)\phi(x, x_0, v)$ with $\|\phi(x, x_0, v)\| \leq k_0\|v\|\|x - x_0\|, \forall x, v$ in $B_{\frac{r}{2}}(x_0)$. Moreover, $\frac{k_0 r}{2}(1 + \gamma_{h,n}) < 1$ with $\gamma_{h,n} = \frac{\eta_h}{2\sqrt{\alpha_n}}$, where $\eta_h = \|K(I - P_h)\|$.

Assumption II (A-II): $x^\dagger - x_0 = A\hat{u}$ for some $\hat{u} \in H$.

We make use of the following results for the convergence analysis.

Lemma 2.1. *Suppose that F possesses a locally bounded Fréchet derivative in $B_{\frac{r}{2}}(x_0)$. Then by using assumption A-I*

$$\|(A + \alpha I)^{-1}K^* (F(x) - F(y) - K(x - y))\| \leq \frac{k_0 r}{2}\|x - y\| \tag{6}$$

for all x,y in $B_{\frac{r}{2}}(x_0)$.

Proof. See [14]. □

Lemma 2.2. *Let a_n be the sequence satisfying, $0 \leq a_n \leq a$ and $\lim a_n \leq \overline{a}$. Moreover, we assume that ϑ_n be the sequence satisfying*

$$0 \leq \vartheta_{n+1} \leq a_n + b\vartheta_n + c\vartheta_n^2$$

with $n \in N$ and $\vartheta_0 \geq 0$, holds for some $b, c \geq 0$. Let $\vartheta\prime$ and $\overline{\vartheta}$ be defined as

$$\vartheta\prime = \frac{2a}{1 - b + \sqrt{(1-b)^2 - 4ac}}$$

and

$$\overline{\vartheta} = \frac{1 - b + \sqrt{(1-b)^2 - 4ac}}{2c}.$$

If $b + \sqrt{ac} < 1$ and if $\vartheta_0 \leq \overline{\vartheta}$, then

$$\vartheta_n \leq max\{\vartheta_0, \vartheta'\}.$$

Proof. See [2, 10]. □

Now by above lemma we claim that all the successive iterations belong to the ball $B_{r/2}(x_0)$. We assume that $x^\dagger \in B_{r/4}(x_0) \subset B_{r/2}(x_0) \subset D(F)$.

Lemma 2.3. *The successive iterations $\tilde{x}_{n+1,h}$ of the iterative scheme (5) belong to the ball $B_{\frac{r}{2}}(x_0)$.*

Proof.

$$
\begin{aligned}
\tilde{x}_{n+1,h} - x^\dagger &= (P_hAP_h + \alpha_n I)^{-1}\Big((P_hAP_h + \alpha_n I)(\tilde{x}_{n,h} - x^\dagger) + P_hK^*(\tilde{y} - F(\tilde{x}_{n,h})) - \alpha_n(\tilde{x}_{n,h} - x_0)\Big) \\
&= (P_hAP_h + \alpha_n I)^{-1}P_hK^*\Big(K(\tilde{x}_{n,h} - P_hx^\dagger) + \tilde{y} - F(\tilde{x}_{n,h})\Big) + (P_hAP_h + \alpha_n I)^{-1}\alpha_n(x_0 - x^\dagger) \\
&= (P_hAP_h + \alpha_n I)^{-1}P_hK^*\Big(K(\tilde{x}_{n,h} + x^\dagger - x^\dagger - P_hx^\dagger) + \tilde{y} - F(\tilde{x}_{n,h})\Big) + (P_hAP_h + \alpha_n I)^{-1}\alpha_n(x_0 - x^\dagger) \\
&= (P_hAP_h + \alpha_n I)^{-1}P_hK^*\Big(K(\tilde{x}_{n,h} - x^\dagger) + \tilde{y} - F(\tilde{x}_{n,h}) + F(x^\dagger) - F(x^\dagger)\Big) \\
&\quad + (P_hAP_h + \alpha_n I)^{-1}P_hK^*(K(x^\dagger - P_hx^\dagger)) + (P_hAP_h + \alpha_n I)^{-1}\alpha_n(x_0 - x^\dagger) \\
&= (P_hAP_h + \alpha_n I)^{-1}P_hK^*\Big(K(\tilde{x}_{n,h} - x^\dagger) + \tilde{y} - F(\tilde{x}_{n,h}) + F(x^\dagger) - F(x^\dagger)\Big) \\
&\quad + (P_hAP_h + \alpha_n I)^{-1}P_hK^*(K(x^\dagger - P_hx^\dagger)) + (P_hAP_h + \alpha_n I)^{-1}\alpha_n(x_0 - x^\dagger) \\
&= (P_hAP_h + \alpha_n I)^{-1}P_hK^*\Big(K(\tilde{x}_{n,h} - x^\dagger) - F(\tilde{x}_{n,h}) + F(x^\dagger)\Big) + (P_hAP_h + \alpha_n I)^{-1}P_hK^*(\tilde{y} - F(x^\dagger)) \\
&\quad + (P_hAP_h + \alpha_n I)^{-1}P_hK^*(K(x^\dagger - P_hx^\dagger)) + (P_hAP_h + \alpha_n I)^{-1}\alpha_n(x_0 - x^\dagger). \qquad (7)
\end{aligned}
$$

We have, using spectral theory result and assumptions,

$$
\begin{aligned}
\|(P_hAP_h + \alpha_n I)^{-1}P_hK^*(K(x^\dagger - P_hx^\dagger))\| &\leq \frac{1}{2\sqrt{\alpha_n}}\|K(I - P_h)x^\dagger\|, \qquad (8) \\
\|\alpha_n(P_hAP_h + \alpha_n I)^{-1}(x_0 - x^\dagger)\| &= \alpha_n\|(P_hAP_h + \alpha_n I)^{-1}(P_h + I - P_h)A\hat{u}\| \\
&\leq \alpha_n\|(P_hAP_h + \alpha_n I)^{-1}P_hA\hat{u}\| + \alpha_n\|(P_hAP_h + \alpha_n I)^{-1}(I - P_h)A\hat{u}\| \\
&\leq \alpha_n\|(P_hAP_h + \alpha_n I)^{-1}P_hA(P_h + I - P_h)\hat{u}\| + \|(I - P_h)A\hat{u}\| \\
&\leq \alpha_n\|(P_hAP_h + \alpha_n I)^{-1}P_hAP_h\hat{u}\| + \alpha_n\|(P_hAP_h + \alpha_n I)^{-1}P_hA(I - P_h)\hat{u}\| \\
&\quad + \|(I - P_h)A\hat{u}\| \\
&\leq \alpha_n\|\hat{u}\| + \|A(I - P_h)\hat{u}\| + \|(I - P_h)A\hat{u}\| \\
&\leq \alpha_n\|\hat{u}\| + 2\|A(I - P_h)\|\|\hat{u}\| \qquad (9)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|(P_hAP_h + \alpha_n I)^{-1}P_hK^*(K(\tilde{x}_{n,h} - x^\dagger) - F(\tilde{x}_{n,h}) + F(x^\dagger))\Big\| \\
&\leq \|(P_hAP_h + \alpha_n I)^{-1}P_hK^*K\phi(\tilde{x}_{n,h} + t(x^\dagger - \tilde{x}_{n,h}), x_0, (x^\dagger - \tilde{x}_{n,h}))\| \\
&\leq \|(P_hAP_h + \alpha_n I)^{-1}P_hK^*K(I - P_h + P_h)\phi(\tilde{x}_{n,h} + t(x^\dagger - \tilde{x}_{n,h}), x_0, (x^\dagger - \tilde{x}_{n,h}))\| \\
&\leq \|(P_hAP_h + \alpha_n I)^{-1}P_hK^*KP_h\phi(\tilde{x}_{n,h} + t(x^\dagger - \tilde{x}_{n,h}), x_0, (x^\dagger - \tilde{x}_{n,h}))\| \\
&+ \|(P_hAP_h + \alpha_n I)^{-1}P_hK^*K(I - P_h)\phi(\tilde{x}_{n,h} + t(x^\dagger - \tilde{x}_{n,h}), x_0, (x^\dagger - \tilde{x}_{n,h}))\| \\
&\leq \frac{k_0 r}{2}\|x^\dagger - \tilde{x}_{n,h}\| + \frac{1}{2\sqrt{\alpha_n}}\|K(I - P_h)\|\|\phi(\tilde{x}_{n,h} + t(x^\dagger - \tilde{x}_{n,h}), x_0, (x^\dagger - \tilde{x}_{n,h}))\| \\
&\leq \frac{k_0 r}{2}\|x^\dagger - \tilde{x}_{n,h}\| + \frac{k_0 r}{2}\frac{\eta_h}{2\sqrt{\alpha_n}}\|x^\dagger - \tilde{x}_{n,h}\|
\end{aligned}
$$

$$\leq \frac{k_0 r}{2}(1+\gamma_{h,n})\|\tilde{x}_{n,h}-x^\dagger\|. \tag{10}$$

Therefore,

$$\|\tilde{x}_{n+1,h}-x^\dagger\| \leq \frac{k_0 r}{2}(1+\gamma_{h,n})\|\tilde{x}_{n,h}-x^\dagger\| + \frac{\delta}{2\sqrt{\alpha_n}} + \frac{1}{2\sqrt{\alpha_n}}\|K(I-P_h)x^\dagger\| + \alpha_n\|\hat{u}\| + 2\|A(I-P_h)\|\|\hat{u}\|. \tag{11}$$

This is of the form

$$\vartheta_{n+1} \leq c\vartheta_n^2 + b\vartheta_n + a_n$$

with $\vartheta_n = \|\tilde{x}_{n,h}-x^\dagger\|$, $a_n = \frac{\delta}{2\sqrt{\alpha_n}} + \frac{1}{2\sqrt{\alpha_n}}\|K(I-P_h)x^\dagger\| + \alpha_n\|\hat{u}\| + 2\|A(I-P_h)\|\|\hat{u}\|$, $b = \frac{k_0 r}{2}(1+\gamma_{h,n})$, $c = 0$. We have $b + 2\sqrt{ac} = \frac{k_0 r}{2}(1+\gamma_{h,n}) < 1$.

By using lemma 2.2

$$\|\tilde{x}_{n+1,h}-x^\dagger\| \leq \|x_0 - x^\dagger\|. \tag{12}$$

$$\begin{aligned} \text{Therefore} \quad \|\tilde{x}_{n+1,h}-x_0\| &\leq \|\tilde{x}_{n+1,h}-x^\dagger\| + \|x^\dagger - x_0\| \\ &\leq 2\|x^\dagger - x_0\| \\ &\leq \frac{r}{2}. \end{aligned} \tag{13}$$

Hence the theorem is proved. □

Theorem 2.4 (An a priori method)**.** *Let the assumptions A-I and A-II hold. The regularization parameter is chosen a priori as* $\alpha_n = \alpha_0(1+p^n)$ *with* $0 < p < 1$, $n = 1, 2, \ldots$ *and* $\alpha_0 \sim (\delta+\eta_h)^{2/3}$*. If we choose the iteration number* n *and dimension* h *such that* $(\frac{k_0 r}{2})^{n+1} \leq (\delta+\eta_h)^{\frac{2}{3}}$ *then,*

$$\|x^\dagger - \tilde{x}_{n+1,h}\| = O((\delta+\eta_h)^{2/3}). \tag{14}$$

Proof. We have, using (11),

$$\begin{aligned} \|\tilde{x}_{n+1,h}-x^\dagger\| &\leq \frac{k_0 r}{2}(1+\gamma_{h,n})\|\tilde{x}_{n,h}-x^\dagger\| + \frac{\delta}{2\sqrt{\alpha_n}} + \frac{\|K(I-P_h)x^\dagger\|}{2\sqrt{\alpha_n}} + \alpha_n\|\hat{u}\| + 2\|A(I-P_h)\|\|\hat{u}\| \\ &\leq \frac{k_0 r}{2}\|\tilde{x}_{n,h}-x^\dagger\| + \frac{k_0 r}{2}\frac{\eta_h}{2\sqrt{\alpha_n}}\|\tilde{x}_{n,h}-x^\dagger\| + \frac{\delta}{2\sqrt{\alpha_n}} + \frac{\|K(I-P_h)x_0\|}{2\sqrt{\alpha_n}} \\ &\quad + \frac{\|K(I-P_h)(x^\dagger - x_0)\|}{2\sqrt{\alpha_n}} + \alpha_n\|\hat{u}\| + 2\|K\|\|K(I-P_h)\|\|\hat{u}\|. \end{aligned}$$

We have,

$$\begin{aligned} \frac{\|K(I-P_h)x_0\|}{2\sqrt{\alpha_n}} &= \frac{\|K(I-P_h)^2 x_0\|}{2\sqrt{\alpha_n}} \\ &\leq \frac{\|K(I-P_h)\|\|(I-P_h)x_0\|}{2\sqrt{\alpha_n}} \\ &\leq \frac{\eta_h\|(I-p_h)x_0\|}{2\sqrt{\alpha_n}} \end{aligned}$$

Therefore,

$$\|\tilde{x}_{n+1,h}-x^\dagger\| \leq \frac{k_0 r}{2}\|\tilde{x}_{n,h}-x^\dagger\| + \frac{k_0 r}{2}\frac{\eta_h}{2\sqrt{\alpha_n}}\|x_0 - x^\dagger\| + \frac{\delta}{2\sqrt{\alpha_n}} + \frac{\eta_h\|(I-P_h)x_0\|}{2\sqrt{\alpha_n}}$$

$$
\begin{aligned}
&+\frac{\eta_h^2}{2\sqrt{\alpha_n}}\|K\|\|\hat{u}\| + \alpha_n\|\hat{u}\| + 2\|K\|\|\hat{u}\|\eta_h \\
\leq\ & \frac{k_0 r}{2}\|\tilde{x}_{n,h} - x^\dagger\| + \frac{k_0 r}{2}\frac{\delta+\eta_h}{2\sqrt{\alpha_n}}\|x_0 - x^\dagger\| + \frac{\delta+\eta_h}{2\sqrt{\alpha_n}} + \frac{(\delta+\eta_h)\|(I-P_h)x_0\|}{2\sqrt{\alpha_n}} \\
&+\frac{\delta+\eta_h}{2\sqrt{\alpha_n}}\eta_h\|K\|\|\hat{u}\| + \alpha_n\|\hat{u}\| + 2\|K\|\|\hat{u}\|(\delta+\eta_h) \\
\leq\ & \frac{k_0 r}{2}\|\tilde{x}_{n,h} - x^\dagger\| + \frac{\delta+\eta_h}{\sqrt{\alpha_n}}\left\{\frac{k_0 r}{4}\frac{r}{4} + \frac{1}{2} + \frac{\|(I-P_h)x_0\|}{2} + \frac{\eta_h\|K\|\|\hat{u}\|}{2}\right\} + 2\|K\|\|\hat{u}\|(\delta+\eta_h) + \alpha_n\|\hat{u}\|
\end{aligned}
$$

With the choice of α_n and α_0 we have,

$$
\frac{\delta+\eta_h}{\sqrt{\alpha_n}} \leq (\delta+\eta_h)^{2/3} \quad \textit{and}
$$

$$
\alpha_n = \alpha_0(1+p^n) \leq 2\alpha_0 \leq (\delta+\eta_h)^{2/3}.
$$

Therefore,

$$
\begin{aligned}
\leq\ & \frac{k_0 r}{2}\|\tilde{x}_{n,h} - x^\dagger\| + c_1(\delta+\eta_h)^{2/3} \\
\leq\ & \frac{k_0 r}{2}\left(\frac{k_0 r}{2}\|\tilde{x}_{n-1,h} - x^\dagger\| + c_1(\delta+\eta_h)^{2/3}\right) + c_1(\delta+\eta_h)^{2/3} \\
=\ & (\frac{k_0 r}{2})^2\|\tilde{x}_{n-1,h} - x^\dagger\| + (\frac{k_0 r}{2}c_1 + c_1)(\delta+\eta_h)^{2/3} \\
\leq\ & (\frac{k_0 r}{2})^{n+1}\|\tilde{x}_0 - x^\dagger\| + \frac{2(1-(\frac{k_0 r}{2})^n)}{2-k_0 r}c_1(\delta+\eta_h)^{2/3} \\
\leq\ & \left(\frac{k_0 r}{2}\right)^{n+1}\|x_0 - x^\dagger\| + c_2(\delta+\eta_h)^{2/3} \\
\leq\ & c(\delta+\eta_h)^{2/3},
\end{aligned}
$$

where $c_1 = \frac{k_0 r}{4}\frac{r}{4} + \frac{1}{2} + \frac{\|(I-P_h)x_0\|}{2} + \frac{\eta_h\|K\|\|\hat{u}\|}{2} + 2\|K\|\|\hat{u}\|(\delta+\eta_h) + \|\hat{u}\|$ and $c_2 = \frac{2(1-(\frac{k_0 r}{2})^n)}{2-k_0 r}c_1$. □

3. An a posteriori stopping rule

In this section, we propose an a posteriori rule for stopping the iteration using a discrepancy principle. Throughout this section we assume that $\|K\| \leq 1$. We terminate the iteration in such a way that there is a number N for which the following relation holds. For $n = 0, 1, 2, \ldots,$

$$
\alpha_N\|P_h K^*(\tilde{y} - F(\tilde{x}_{N,h}))\| < C(\delta+\eta_h)^{2/3} \leq \alpha_n\|P_h K^*(\tilde{y} - F(\tilde{x}_{n,h}))\| \tag{15}
$$

where C is an appropriately chosen positive number. We make use of the following principle to obtain a decreasing sequence of regularization parameter α_n. Choose an initial approximation α_0 by solving the equation

$$
\alpha_0\|(P_h A P_h + \alpha_0 I)^{-1}P_h K^*(\tilde{y} - F(x_0))\| = C(\delta+\eta_h)^{2/3}. \tag{16}
$$

Define α_n such that

$$
\alpha_n = \|K\|^2(1+q^{n+1})\alpha_0
$$

where $q := \frac{k_0 r}{2}$. Similar kind of stopping rule has been employed in literature for iterative schemes [11, 13, 14, 16]. In this paper, we claim that as a consequence of the stopping rule, the parameter obtained through this strategy and the resultant error estimate are of the order $O(\delta+\eta_h)^{2/3}$.

Proposition 3.1. *Suppose iteration (5) is stopped according to the stopping criterion (15) and α_n are chosen as mentioned above. Then*

$$\alpha_{N-1} \sim (\delta + \eta_h)^{2/3}, \tag{17}$$

where N is the stopping index.

Proof. The iteration formula is

$$\tilde{x}_{N,h} = \tilde{x}_{N-1,h} + (P_h A P_h + \alpha_{N-1} I)^{-1}\Big(P_h K^*(\tilde{y} - F(\tilde{x}_{N-1,h})) - \alpha_{N-1}(\tilde{x}_{N-1,h} - x_0)\Big).$$

Therefore

$$P_h K^*(\tilde{y} - F(\tilde{x}_{N-1,h})) = (P_h A P_h + \alpha_{N-1} I)(\tilde{x}_{N,h} - \tilde{x}_{N-1,h}) + \alpha_{N-1}(\tilde{x}_{N-1,h} - x_0).$$

By using (13) and $\alpha_{N-1} = \|K\|^2(1+q^N)\alpha_0$,

$$\begin{aligned}
\|P_h K^*(\tilde{y} - F(\tilde{x}_{N-1,h}))\| &\le \|(P_h A P_h + \alpha_{N-1} I)(\tilde{x}_{N,h} - \tilde{x}_{N-1,h})\| + \alpha_{N-1}\|\tilde{x}_{N-1,h} - x_0\| \\
&\le (\|K\|^2 + \alpha_{N-1})\|\tilde{x}_{N,h} - \tilde{x}_{N-1,h}\| + \alpha_{N-1}\|\tilde{x}_{N-1,h} - x_0\| \\
&\le (\|K\|^2 + \alpha_{N-1})r + \alpha_{N-1}\|\tilde{x}_{N-1,h} - x_0\| \\
&\le \Big(\frac{2\alpha_{N-1}}{(1+q^N)\alpha_0} + 2\alpha_{N-1}\Big)\frac{r}{2} + \alpha_{N-1}\frac{r}{2} \\
&\le \Big(\frac{2}{(1+q^N)\alpha_0} + 3\Big)\frac{r}{2}\alpha_{N-1} \\
&\le \Big(\frac{2 + 3(1+q^N)\alpha_0}{(1+q^N)\alpha_0}\Big)\frac{r}{2}\alpha_{N-1}.
\end{aligned}$$

By using discrepancy principle

$$\begin{aligned}
C(\delta + \eta_h)^{2/3} &\le \alpha_{N-1}\|P_h K^*(\tilde{y} - F(\tilde{x}_{N-1,h}))\| \\
&\le \alpha_{N-1}\Big(\frac{2 + 3(1+q^N)\alpha_0}{(1+q^N)\alpha_0}\Big)\frac{r}{2}\alpha_{N-1} \\
&\le \Big(\frac{2 + 3(1+q^N)\alpha_0}{(1+q^N)\alpha_0}\Big)\frac{r}{2}\alpha_{N-1}^2.
\end{aligned}$$

We know that

$$\alpha_0\|(P_h A P_h + \alpha_0 I)^{-1} P_h K^*(\tilde{y} - F(x_0))\| = C(\delta + \eta_h)^{2/3}.$$

This would imply that $\alpha_0\|\tilde{x}_{1,h} - x_0\| = C(\delta + \eta_h)^{2/3}$. Let $\|\tilde{x}_{1,h} - x_0\| = r'$. Therefore $\alpha_0 = c'(\delta + \eta_h)^{2/3}$, where $c' = C/r$. Hence,

$$\begin{aligned}
\frac{2C(1+q^N)\alpha_0}{r(2 + 3(1+q^N)\alpha_0)}(\delta + \eta_h)^{2/3} &\le \alpha_{N-1}^2 \\
\frac{2C(1+q^N)c'(\delta + \eta_h)^{2/3}}{r(2 + 3(1+q^N)\alpha_0)}(\delta + \eta_h)^{2/3} &\le \alpha_{N-1}^2 \\
\frac{2C(1+q^N)c'}{r(2 + 3(1+q^N)\alpha_0)}(\delta + \eta_h)^{4/3} &\le \alpha_{N-1}^2 \\
\Big(\frac{2C(1+q^N)c'}{2r + 3r(1+q^N)\alpha_0}\Big)^{1/2}(\delta + \eta_h)^{2/3} &\le \alpha_{N-1}.
\end{aligned}$$

On the other hand, from the definition of $\alpha_n = \|K\|^2(1+q^{n+1})\alpha_0$, we have

$$\alpha_{N-1} \le c'\|K\|^2(1+q^N)(\delta + \eta_h)^{2/3}. \tag{18}$$

Hence by (18) and (18) $\alpha_{N-1} \sim (\delta + \eta_h)^{2/3}$. □

Theorem 3.2. *Suppose iterative scheme (5) satisfies the assumption (A-I), (A-II) and is stopped according to the stopping criterion (15) with parameter choice described above, then*

$$\|x^\dagger - \tilde{x}_{N,h}\| = O(\delta + \eta_h)^{\frac{2}{3}}. \tag{19}$$

Proof. Throughout this proof, for simplify the notation we set $e_n := \|\tilde{x}_{n,h} - x^\dagger\|$ for $n = 1, 2, ...,$ and $e_0 = \|x_0 - x^\dagger\|$. Using (11) and (9),

$$\begin{aligned}
\|e_N\| &\leq \frac{k_0 r}{2}\Big(1 + \frac{\eta_h}{2\sqrt{\alpha_{N-1}}}\Big)\|e_{N-1}\| + \frac{\delta}{2\sqrt{\alpha_{N-1}}} + \frac{\|K(I-P_h)x^\dagger\|}{2\sqrt{\alpha_{N-1}}} + \alpha_{N-1}\|\hat{u}\| + 2\|A(I-P_h)\|\|\hat{u}\| \\
&\leq q\|e_{N-1}\| + \frac{k_0 r}{2}\frac{\eta_h}{2\sqrt{\alpha_{N-1}}}\|e_{N-1}\| + \frac{\delta}{2\sqrt{\alpha_{N-1}}} + \frac{\|K(I-P_h)x_0\|}{2\sqrt{\alpha_{N-1}}} \\
&\quad + \frac{\|K(I-P_h)(x^\dagger - x_0)\|}{2\sqrt{\alpha_{N-1}}} + \alpha_{N-1}\|\hat{u}\| + 2\|A(I-P_h)\|\|\hat{u}\| \\
&\leq q\|e_{N-1}\| + \frac{k_0 r}{2}\frac{\eta_h}{2\sqrt{\alpha_{N-1}}}\|e_0\| + \frac{\delta}{2\sqrt{\alpha_{N-1}}} + \frac{\eta_h\|(I-P_h)x_0\|}{2\sqrt{\alpha_{N-1}}} \\
&\quad + \frac{\eta_h^2\|K\|\|\hat{u}\|}{2\sqrt{\alpha_{N-1}}} + \alpha_{N-1}\|\hat{u}\| + 2\|K\|\|\hat{u}\|\eta_h \\
&\leq q\|e_{N-1}\| + \frac{k_0 r}{2}\frac{\delta+\eta_h}{2\sqrt{\alpha_{N-1}}}\frac{r}{4} + \frac{\delta+\eta_h}{2\sqrt{\alpha_{N-1}}} + \frac{\delta+\eta_h}{2\sqrt{\alpha_{N-1}}}\|(I-P_h)x_0\| \\
&\quad + \frac{\delta+\eta_h}{2\sqrt{\alpha_{N-1}}}\eta_h\|K\|\|\hat{u}\| + \alpha_{N-1}\|\hat{u}\| + 2\|K\|\|\hat{u}\|(\delta+\eta_h) \\
&\leq q\|e_{N-1}\| + \frac{\delta+\eta_h}{\sqrt{\alpha_{N-1}}}\Big\{\frac{k_0 r}{4}\frac{r}{4} + \frac{1}{2} + \frac{\|(I-P_h)x_0\|}{2} + \frac{\eta_h\|K\|\|\hat{u}\|}{2}\Big\} + \alpha_{N-1}\|\hat{u}\| + 2\|K\|\|\hat{u}\|(\delta+\eta_h) \\
&\leq q\|e_{N-1}\| + c_1(\delta+\eta_h)^{2/3}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|e_N\| &\leq q^N\|e_0\| + \frac{1-q^N}{1-q}c_1(\delta+\eta_h)^{2/3} \\
&\leq \Big\{\frac{\alpha_{N-1}}{\|K\|^2\alpha_0} - 1\Big\}\|e_0\| + c_2(\delta+\eta_h)^{2/3} \\
&\leq \frac{\alpha_{N-1}}{\|K\|^2\alpha_0}\|e_0\| + c_2(\delta+\eta_h)^{2/3} \\
&\leq \frac{\alpha_{N-1}}{\|K\|^2\alpha_0}\|K\|^2\|\hat{u}\| + c_2(\delta+\eta_h)^{2/3} \\
&\leq \frac{\alpha_{N-1}}{\alpha_0}\|\hat{u}\| + c_2(\delta+\eta_h)^{2/3} \\
&\leq c_3\{\alpha_{N-1} + (\delta+\eta_h)^{2/3}\} \\
&\leq c(\delta+\eta_h)^{2/3}
\end{aligned}$$

where c is a positive constant. Hence the theorem follows from $\alpha_{N-1} \sim (\delta+\eta_h)^{2/3}$. □

4. Numerical Examples

Example 4.1 (cf. [5])**.** *In the following, we consider a parameter identification problem to illustrate the proposed scheme. The problem is to evaluate the parameter a in the two-point boundary problem*

$$-u'' + au = f, \quad t \in (0,1), \quad u(0) = u_0, \quad u(1) = u_1, \tag{20}$$

from the perturbed data $\tilde{u}$ of the u, where u_0, u_1 and $f \in L^2[0,1]$ are given. Now the nonlinear operator $F : D(F) \subseteq L^2[0,1] \to L^2[0,1]$ for computing a is defined as the parameter-to-solution mapping $F(a) = u(a)$ with $u(a)$ being the unique solution of (20).

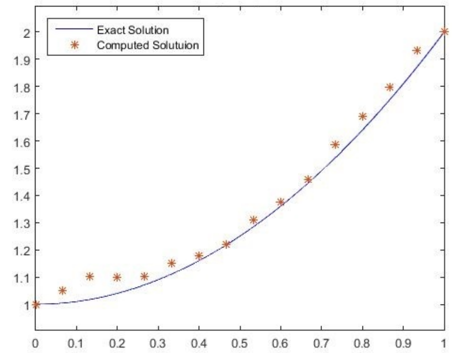

Figure 1. Soln. when $\delta = .1\%$ (h=16)

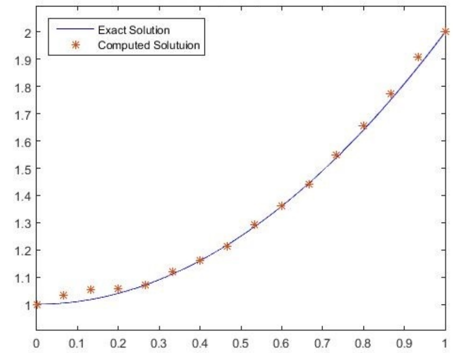

Figure 2. Soln. when $\delta = .01\%$ (h=16)

F is Fréchet differentiable and it is given by $F'(a)h = -A(a)^{-1}(hu(a))$ and its adjoint is given by $F'(a)^*w = -u(a)A(a)$, where $A(a) : H^2 \cap H_0^1 \to L^2$ is defined by $A(a)u = -u'' + au$. For numerical calculation we assume $f = 1 + t^2$, $u_0 = u_1 = 1$ and if $u(a^\dagger) = 1$ then the the true solution is $a^\dagger = 1 + t^2$. Let iteration start with the initial guess $a_0 = 1 + t^2 + 2(t - 2t^3 + t^4)$ and perturbed data $u_\delta = 1 + \delta\sqrt{2}\sin(2\pi t/\delta)$. The assumption A-I follows from [17] and [5]. We used Haar orthonormal basis of $L^2[0,1]$ and employed finite difference scheme to solve (20). The iteration (5) is stopped using the stopping criteria (15). The result associated with various deltas are given in Table 1. The computed solutions associated with $\delta = 0.1\%$ and $\delta = 0.01\%$ are shown in Figure 1 and Figure 2.

δ	h	n	Relative Error	$\frac{\text{Relative Error}}{((\delta+\eta_h)^{2/3})}$
0.1	2^4	1	0.0375	0.1740
	2^5	1	0.0348	0.1615
	2^6	1	0.0271	0.1256
0.05	2^4	1	0.0292	0.2152
	2^5	1	0.0279	0.2055
	2^6	1	0.0220	0.1620
0.01	2^4	2	0.0141	0.3043
	2^5	2	0.0123	0.2649
	2^6	4	0.0097	0.2091
0.001	2^4	15	0.0090	0.8899
	2^5	18	0.0067	0.6661
	2^6	21	0.0047	0.4658

Table 1. Numerical Results For Example 1

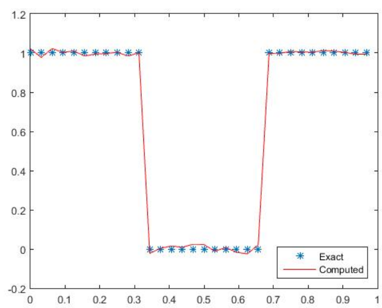

Figure 3. Soln. when $\delta = 10\%$ (h=32)

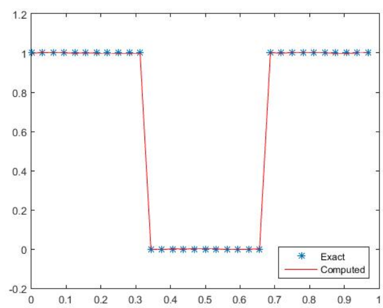

Figure 4. Soln. when $\delta = 1\%$ (h=32)

Example 4.2. *Consider the following nonlinear integral operator equation [9] defined on* $H = L^2[0,1]$.

$$F(u) := B(u) + (\arctan u)^3 := \int_0^1 e^{(-|x-y|)} u(y)dy + (\arctan u)^3. \tag{21}$$

We consider the data

$$v = \begin{cases} 0 & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 2 + ((\arctan(1))^3) - \exp(x-1) - \exp(-x) & \textit{otherwise.} \end{cases} \tag{22}$$

So that the actual solution will be

$$u(x) = \begin{cases} 0 & \frac{1}{3} \leq x \leq \frac{2}{3} \\ 1 & \textit{otherwise.} \end{cases} \tag{23}$$

The Fréchet derivative of F *is*

$$F'(u)h = \frac{3(\arctan u)^2}{1+u^2} h + \int_0^1 \exp(-|x-y|)h(y)dy. \tag{24}$$

δ	h	n	Relative Error	$\frac{\text{Relative Error}}{((\delta+\eta_h)^{2/3})}$
0.1	2^4	12	0.0242	0.1125
	2^5	14	0.0237	0.1099
	2^6	14	0.0158	0.0155
0.01	2^4	28	0.0024	0.0508
	2^5	30	0.0018	0.0378
	2^6	31	0.0010	0.0225
0.001	2^4	46	0.000204	0.0202
	2^5	48	0.000258	0.0259
	2^6	50	0.000163	0.0164

Table 2. **Numerical Results For Example 2**

We started the iteration (5) with initial guess as $u_0 = 1.23$ and randomly perturbed data $\tilde{v}$ of v with a data error δ such that $\|v - \tilde{v}\| \leq \delta$. The assumptions A-I and A-II follows from [14]. The computational results are summarized in Table 4. The computed solutions associated with $\delta = 10\%$ and $\delta = 1\%$ are shown in Figure 3 and Figure 4. It shows that numerical estimate is consistent with theoretical estimate.

5. Conclusion

In this paper, we have considered a finite dimensional approximation of simplified Gauss-Newton iterative scheme. We proposed an a priori and an a posteriori stopping rules, that guarantees convergence of the iterates to a solution, as the noise level goes to zero. We found the error estimates are optimal order. The numerical results are consistent with the theoretical estimates.

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