Int. J. Math. And Appl., 7(4)(2019), 193–199
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Gel'fand Theory of the Commutative Banach Algebra $\mathcal{A} \times_c \mathcal{I}$ with the Convolution Product

H. V. Dedania[1,*] and H. J. Kanani[2]

1 Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, Gujarat, India.

2 Department of Mathematics, Bahauddin Science College, Junagadh, Gujarat, India.

Abstract: Let $\mathcal{A}$ be an algebra and $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then $\mathcal{A} \times \mathcal{I}$ is an algebra with pointwise linear operations and the convolution product $(a, x)(b, y) = (ab + xy, ay + xb)$ $((a, x), (b, y) \in \mathcal{A} \times \mathcal{I})$; it will be denoted by $\mathcal{A} \times_c \mathcal{I}$. If $\mathcal{A}$ is a commutative Banach algebra and $\mathcal{I}$ is a closed ideal in $\mathcal{A}$, then $\mathcal{A} \times_c \mathcal{I}$ is also a commutative Banach algebra with some suitable norm. In this paper, we shall study the Gel'fand theory, uniqueness properties, and regularity of $\mathcal{A} \times_c \mathcal{I}$.

MSC: Primary 46J05; Secondary 46K05.

Keywords: Convolution product, UUNP, UC*NP, regular algebra, uniform algebra

1. Introduction

Consider the group $\mathbb{Z}_2 = \{0, 1\}$ with the binary operation addition modulo 2. Then $\ell^1(\mathbb{Z}_2)$ is a Banach algebra with convolution product. For $f, g \in \ell^1(\mathbb{Z}_2)$, the convolution product of f and g is defined as

$$f * g = (f(0)g(0) + f(1)g(1), f(0)g(1) + f(1)g(0)).$$

This motivates the following product. Let $\mathcal{A}$ be an algebra and $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then $\mathcal{A} \times_c \mathcal{I}$ is an algebra with pointwise linear operations and the *convolution product* defined as $(a, x)(b, y) = (ab + xy, ay + xb)$ $((a, x), (b, y) \in \mathcal{A} \times_c \mathcal{I})$. It is commutative (resp. unital) iff $\mathcal{A}$ is commutative (resp. unital). Further, If $\mathcal{A}$ is a normed algebra (resp. Banach algebra), then $\mathcal{A} \times_c \mathcal{I}$ is a normed algebra (resp. Banach algebra) with the norm $\|(a, x)\|_1 = \|a\| + \|x\|$ $((a, x) \in \mathcal{A} \times_c \mathcal{I})$.

2. Basic Results

Throughout the paper, let $\mathcal{A}$ be an algebra and $\mathcal{I}$ be an ideal in $\mathcal{A}$. Let $\mathcal{A}_{-1}$ denote the set of all quasi invertible elements of $\mathcal{A}$. If $\mathcal{A}$ is unital, $\mathcal{A}^{-1}$ is the set of all invertible elements of $\mathcal{A}$. Further, $\sigma_{\mathcal{A}}(a)$ and $r_{\mathcal{A}}(a)$ denote the spectrum and the spectral radius of a in $\mathcal{A}$. Then we have the following.

Proposition 2.1. *Let $(a, x) \in \mathcal{A} \times_c \mathcal{I}$. Then*

* *E-mail: hvdedania@yahoo.com*

(1). $(a,x) \in (\mathcal{A} \times_c \mathcal{I})^{-1}$ *iff* $a+x, a-x \in \mathcal{A}^{-1}$*;*

(2). $(a,x) \in (\mathcal{A} \times_c \mathcal{I})_{-1}$ *iff* $a+x, a-x \in \mathcal{A}_{-1}$*;*

(3). $\sigma_{\mathcal{A}\times_c\mathcal{I}}((a,x)) = \sigma_{\mathcal{A}}(a+x) \cup \sigma_{\mathcal{A}}(a-x)$*;*

(4). $r_{\mathcal{A}\times_c\mathcal{I}}((a,x)) = \max\{r_{\mathcal{A}}(a+x), r_{\mathcal{A}}(a-x)\}$*.*

Proposition 2.2. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{I}$ be closed in $\mathcal{A}$. Then $\mathcal{A} \times_c \mathcal{I}$ has a left approximate identity iff $\mathcal{A}$ has a left approximate identity. (Similar results are true for right, bounded left, bounded right approximate identity.)*

Proof. Let $\mathcal{A} \times_c \mathcal{I}$ has a left approximate identity $((e_\alpha, x_\alpha))_{\alpha\in\Lambda}$ and $a \in \mathcal{A}$. Then

$$\|e_\alpha a - a\| \leq \|e_\alpha a - a\| + \|x_\alpha a\| = \|(e_\alpha, x_\alpha)(a,0) - (a,0)\|_1$$

converges to 0 as $\alpha \to \infty$. Thus (e_α) is a left approximate identity for $\mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ has a left approximate identity (e_α). Then,

$$\|(e_\alpha, 0)(a,x) - (a,x)\|_1 \;=\; \|(e_\alpha a, e_\alpha x) - (a,x)\|_1 \;=\; \|(e_\alpha a - a) + (e_\alpha x - x)\|_1$$

converges to 0 as $\alpha \to \infty$ for every $(a,x) \in \mathcal{A} \times_c \mathcal{I}$. Thus $(e_\alpha, 0)$ is a left approximate identity for $\mathcal{A} \times_c \mathcal{I}$. Therefore $\mathcal{A} \times_c \mathcal{I}$ has a left approximate identity. The proof for the bounded approximate identity follows from the fact that a sequence $((e_\alpha, x_\alpha))$ in $\mathcal{A} \times_c \mathcal{I}$ is bounded then the sequence (e_α) is bounded in $\mathcal{A}$ and if a sequence (e_α) is bounded in $\mathcal{A}$, then the sequence $((e_\alpha, 0))$ is bounded in $\mathcal{A} \times_c \mathcal{I}$. □

Remark 2.3. *Let $\|\cdot\|$ be a norm on an algebra $\mathcal{A}$ and $\mathcal{I}$ be an ideal of $\mathcal{A}$. Let $\|(a,x)\|_\infty = \max\{\|a\|, \|x\|\}$ $((a,x) \in \mathcal{A} \times_c \mathcal{I})$. Then $\|\cdot\|_\infty$ is a linear norm but it may not be an algebra norm on $\mathcal{A} \times_c \mathcal{I}$.*

Definition 2.4. *Let $\mathcal{A}$ be an algebra. Then*

(1). An algebra norm $||\cdot||$ on $\mathcal{A}$ is a uniform norm *if $||a^2|| = ||a||^2$ $(a \in \mathcal{A})$.*

(2). $\mathcal{A}$ is a uniform *algebra if it admits a complete uniform norm.*

(3). An algebra norm $||\cdot||$ on a $*$-algebra $\mathcal{A}$ is a C^*-norm *if $||a^*a|| = ||a||^2 (a \in \mathcal{A})$.*

Lemma 2.5. *Let $\mathcal{I}$ be an ideal in a normed algebra $(\mathcal{A}, \|\cdot\|)$ and $(a,x) \in \mathcal{A} \times_c \mathcal{I}$. Define $|(a,x)| := \max\{\|a+x\|, \|a-x\|\}$. Then*

(1). $|\cdot|$ is an algebra norm on $\mathcal{A} \times_c \mathcal{I}$;

(2). $|\cdot|$ is a uniform norm on $\mathcal{A} \times_c \mathcal{I}$ iff $\|\cdot\|$ is a uniform norm on $\mathcal{A}$;

(3). Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{I}$ be a $*$-ideal in $\mathcal{A}$. Then $|\cdot|$ is a C^*-norm on $\mathcal{A} \times_c \mathcal{I}$ iff $\|\cdot\|$ is a C^*-norm on $\mathcal{A}$.

Corollary 2.6. *Let $\mathcal{I}$ be a closed ideal in a Banach algebra $\mathcal{A}$. Then $\mathcal{A} \times_c \mathcal{I}$ is a uniform algebra iff $\mathcal{A}$ is a uniform algebra.*

Proof. Let $\mathcal{I}$ be a closed ideal in a Banach algebra $\mathcal{A}$. Since $\mathcal{A} \cong \mathcal{A} \times \{0\}$ is a closed subalgebra of $\mathcal{A} \times_c \mathcal{I}$, $\mathcal{A}$ is a uniform algebra whenever $\mathcal{A} \times_c \mathcal{I}$ is a uniform algebra. Conversely, let $\|\cdot\|$ be the complete uniform norm on $\mathcal{A}$. Then, by Lemma 2.5(2), $|\cdot|$ is a uniform norm on $\mathcal{A} \times_c \mathcal{I}$. Next, let $((a_n, x_n))$ be a Cauchy sequence in $(\mathcal{A} \times_c \mathcal{I}, |\cdot|)$. Then, for each $n \in \mathbb{N}$,

$$\|a_n\| \leq \frac{1}{2}\{\|a_n + x_n\| + \|a_n - x_n\|\} \;\leq\; \max\{\|a_n + x_n\|, \|a_n - x_n\|\} \;=\; |(a_n, x_n)|.$$

This implies that (a_n) is a Cauchy sequence in $(\mathcal{A}, \|\cdot\|)$. Since $\|\cdot\|$ is a complete norm on $\mathcal{A}$, the sequence (a_n) converges to some $a \in \mathcal{A}$. By the similar argument, it follows that the sequence (x_n) converges to some $x \in \mathcal{I}$. Hence the sequence $((a_n, x_n))$ converges to $(a, x) \in \mathcal{A} \times_c \mathcal{I}$. Thus $|\cdot|$ is a complete uniform norm on $\mathcal{A} \times_c \mathcal{I}$. $\square$

3. Gel'fand Space and Shilov Boundary

Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. In this section, we calculate the Gel'fand space $\Delta(\mathcal{A} \times_c \mathcal{I})$. Note that the Gel'fand space of $\mathcal{A} \times_c \mathcal{I}$ is very much different from the Gel'fand space of $\mathcal{A} \times \mathcal{B}$ (see [4]).

Notations 3.1. *Let $\varphi \in \Delta(\mathcal{I})$ and $u \in \mathcal{I}$ such that $\varphi(u) = 1$. Define $\varphi^+, \varphi^- : \mathcal{A} \times_c \mathcal{I} \longrightarrow \mathbb{C}$ as $\varphi^+((a, x)) := \varphi(au) + \varphi(x)$ and $\varphi^-((a, x)) := \varphi(au) - \varphi(x)$ $((a, x) \in \mathcal{A} \times_c \mathcal{I})$. We note that φ^+, φ^- are independent of u. Let $F \subset \Delta(\mathcal{A})$. Define $F^+ := \{\varphi^+ : \varphi \in F\}$ and $F^- := \{\varphi^- : \varphi \in F\}$.*

Lemma 3.2. *Let $F \subset \Delta(\mathcal{A})$ and $G \subset \Delta(\mathcal{I})$. Then*

(1). $F^+, F^- \subset \Delta(\mathcal{A} \times_c \mathcal{I})$;

(2). $G^+, G^- \subset \Delta(\mathcal{A} \times_c \mathcal{I})$;

(3). $G^+ \cap G^- = F^+ \cap G^- = F^- \cap G^+ = \emptyset$.

Proof. **(1)** Let $\varphi \in F$. Choose $u \in \mathcal{A}$ such that $\varphi(u) = 1$. Then $\varphi^+((u, 0)) = 1 \neq 0$. It is clear that φ^+ is linear. We show that φ^+ is multiplicative. Also, $\varphi(au) = \varphi(a)$. So we have $\varphi^+((a, x)) = \varphi(a) + \varphi(x)$. Let $(a, x), (b, y) \in \mathcal{A} \times_c \mathcal{I}$. Then

$$\begin{aligned}
\varphi^+((a,x)(b,y)) &= \varphi^+(ab+xy, ay+xb) = \varphi(ab+xy) + \varphi(ay+xb) \\
&= \varphi(a)\varphi(b) + \varphi(x)\varphi(y) + \varphi(a)\varphi(y) + \varphi(x)\varphi(b) \\
&= (\varphi(a)+\varphi(x))(\varphi(b)+\varphi(y)) = \varphi^+((a,x))\,\varphi^+((b,y)).
\end{aligned}$$

Thus $\varphi^+ \in \Delta(\mathcal{A} \times_c \mathcal{I})$. Hence, $F^+ \subset \Delta(\mathcal{A} \times_c \mathcal{I})$. By similar arguments, it follows that $F^- \subset \Delta(\mathcal{A} \times_c \mathcal{I})$.
(2) Let $\varphi \in G$. Let $u \in \mathcal{I}$ be such that $\varphi(u) = 1$. Then it is clear that φ^- is a nonzero linear function on $\mathcal{A} \times_c \mathcal{I}$. To show that φ^- is multiplicative, let $(a, x), (b, y) \in \mathcal{A} \times_c \mathcal{I}$. Then

$$\begin{aligned}
\varphi^-((a,x)(b,y)) &= \varphi^-((ab+xy, ay+xb)) = \varphi((ab+xy)u) - \varphi(ay+xb) \\
&= \varphi(au)\varphi(bu) + \varphi(x)\varphi(y) - \varphi(au)\varphi(y) - \varphi(x)\varphi(bu) \\
&= (\varphi(au)-\varphi(x))(\varphi(bu)-\varphi(y)) = \varphi^-((a,x))\varphi^-((b,y)).
\end{aligned}$$

Thus $\varphi^- \in \Delta(\mathcal{A} \times_c \mathcal{I})$. Hence, $G^- \subset \Delta(\mathcal{A} \times_c \mathcal{I})$. By similar arguments, it follows that $G^+ \subset \Delta(\mathcal{A} \times_c \mathcal{I})$.
(3) Suppose that $\widetilde{\eta} \in F^+ \cap G^-$. Then there exist $\varphi \in F, \psi \in G$ such that $\varphi^+ = \widetilde{\eta} = \psi^-$ on $\mathcal{A} \times_c \mathcal{I}$. Then, $2\varphi(x) = \varphi^+((x, x)) = \psi^-((x, x)) = 0$ $(x \in \mathcal{I})$. Thus $\varphi \equiv 0$ on $\mathcal{I}$. Therefore, $\psi(x) = \psi^-((x, 0)) = \varphi^+((x, 0)) = \varphi(x) = 0$ $(x \in \mathcal{I})$. Thus $\psi \equiv 0$ on $\mathcal{I}$, a contradiction. Hence $F^+ \cap G^- = \emptyset$. By similar arguments, it follows that $G^+ \cap G^- = F^- \cap G^+ = \emptyset$. $\square$

Theorem 3.3. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. Then $\Delta(\mathcal{A} \times_c \mathcal{I}) \cong \Delta^+(\mathcal{A}) \biguplus \Delta^-(\mathcal{I})$.*

Proof. It follows from Lemma 3.2 that $\Delta^+(\mathcal{A}) \biguplus \Delta^-(\mathcal{I}) \subset \Delta(\mathcal{A} \times_c \mathcal{I})$.
For the reverse inclusion, let $\widetilde{\eta} \in \Delta(\mathcal{A} \times_c \mathcal{I})$. Define $\varphi(a) = \widetilde{\eta}((a, 0))$ on $\mathcal{A}$ and $\psi(x) = \widetilde{\eta}((0, x))$ on $\mathcal{I}$. Then $\widetilde{\eta}((a, x)) = \varphi(a) + \psi(x)$ $((a, x) \in \mathcal{A} \times_c \mathcal{I})$. Also, if $\varphi \equiv 0$ on $\mathcal{A}$, then $\psi(x)^2 = \widetilde{\eta}((0, x))^2 = \widetilde{\eta}((0, x)^2) = \widetilde{\eta}((x, 0))^2 = \varphi(x)^2 = 0$ $(x \in \mathcal{I})$.

Hence $\widetilde{\eta} \equiv 0$ on $\mathcal{A} \times_c \mathcal{I}$. This is not possible. Therefore, there exists $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. Also, $\varphi(ab) = \widetilde{\eta}((ab, 0)) = \widetilde{\eta}((a, 0))\widetilde{\eta}((b, 0)) = \varphi(a)\varphi(b)$ $(a, b \in \mathcal{A})$. Hence $\varphi \in \Delta(\mathcal{A})$. Now, there are two cases.

Case -(i): $\widetilde{\eta} = 0$ on $\{0\} \times \mathcal{I}$. So that $\psi = 0$ on $\mathcal{I}$. Therefore, for every $x \in \mathcal{I}$,

$$\varphi(x)^2 = \widetilde{\eta}((x, 0))^2 = \widetilde{\eta}((x, 0)^2) = \widetilde{\eta}((0, x)^2) = \widetilde{\eta}((0, x))^2 = 0.$$

So $\varphi(x) = 0$ $(x \in \mathcal{I})$. Hence, $\varphi = \psi$ on $\mathcal{I}$. Also, for $(a, x) \in \mathcal{A} \times_c \mathcal{I}$,

$$\widetilde{\eta}((a, x)) = \widetilde{\eta}((a, 0)) + \widetilde{\eta}((0, x)) = \varphi(a) + \psi(x) = \varphi(a) + \varphi(x) = \varphi^+((a, x)).$$

Thus we get $\widetilde{\eta} = \varphi^+ \in \Delta^+(\mathcal{A})$.

Case-(ii): $\widetilde{\eta} \neq 0$ on $\{0\} \times \mathcal{I}$. So that $\psi \neq 0$ on $\mathcal{I}$. Since ψ is linear, there exists $y \in \mathcal{I}$ such that $\psi(y) = 1$. Then, for each $x \in \mathcal{I}$,

$$\begin{aligned} \varphi(x) &= \varphi(x)\psi(y) = \widetilde{\eta}((x, 0))\widetilde{\eta}((0, y)) = \widetilde{\eta}((x, 0)(0, y)) \\ &= \widetilde{\eta}((0, xy)) = \widetilde{\eta}((y, 0)(0, x)) = \varphi(y)\psi(x) \end{aligned} \tag{1}$$

Now, $\varphi(y)^2 = \widetilde{\eta}((y, 0))^2 = \widetilde{\eta}((y, 0)^2) = \widetilde{\eta}((0, y)^2) = \psi(y)^2 = 1$ implies that $\varphi(y) = \pm 1$. If $\varphi(y) = 1$, then from equation (1), $\varphi(x) = \psi(x) (x \in \mathcal{I})$. So that

$$\begin{aligned} \widetilde{\eta}((a, x)) &= \varphi(a) + \psi(x) = \varphi(a)\varphi(y) + \varphi(x) \\ &= \varphi(ay) + \varphi(x) = \varphi^+((a, x)) \quad ((a, x) \in \mathcal{A} \times_c \mathcal{I}). \end{aligned}$$

Thus $\widetilde{\eta} = \varphi^+ \in \Delta^+(\mathcal{A})$. If $\varphi(y) = -1$, then from equation (1), we get $\varphi(x) = -\psi(x)$ $(x \in \mathcal{I})$ and $\varphi(u) = 1$, where $u = -y$. So that

$$\begin{aligned} \widetilde{\eta}((a, x)) &= \varphi(a) + \psi(x) = \varphi(a)\varphi(u) - \varphi(x) \\ &= \varphi(au) - \varphi(x) = \varphi^-((a, x)) \quad ((a, x) \in \mathcal{A} \times_c \mathcal{I}). \end{aligned}$$

Thus, $\widetilde{\eta} = \varphi^- \in \Delta^-(\mathcal{I})$. Hence $\Delta(\mathcal{A} \times_c \mathcal{I}) \subset \Delta^+(\mathcal{A}) \biguplus \Delta^-(\mathcal{I})$. Thus $\Delta(\mathcal{A} \times_c \mathcal{I})$ and $\Delta^+(\mathcal{A}) \biguplus \Delta^-(\mathcal{I})$ are set theoretically same. By arguments as in [4, Theorem 2.2], we can show that they are homeomorphic. □

Theorem 3.4 ([6, Corollary 3.3.4])**.** *Let X be a locally compact Hausdorff space, and let $\mathcal{A}$ be a subalgebra of $C_0(X)$ which strongly separates the points of X. Then a point $x \in X$ belongs to the Shilov boundary of $\mathcal{A}$ if and only if given any open neighbourhood U of x, there exist $f \in \mathcal{A}$ such that $\|f|_{X \setminus U}\|_\infty < \|f|_U\|_\infty$.*

Theorem 3.5. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. Then $\partial(\mathcal{A} \times_c \mathcal{I}) = \partial^+(\mathcal{A}) \biguplus \partial^-(\mathcal{I})$.*

Proof. Let $\varphi_0 \in \partial\mathcal{A}$. Let $\widetilde{U}$ be a neighborhood of φ_0^+. Set $U = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{U}\} \cup \{\psi \in \Delta(\mathcal{I}) : \psi^- \in \widetilde{U}\}$. Then U is a neighborhood of φ_0. Therefore, by Theorem 3.4, there exists $a \in \mathcal{A}$ such that $\|\widehat{a}|_{\Delta(\mathcal{A}) \setminus U}\|_\infty < \|\widehat{a}|_U\|_\infty$. If $\psi^- \in \Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{U}$, then $\psi \in \Delta(\mathcal{A}) \setminus U$. If $\varphi^+ \in \Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{U}$, then $\varphi \in \Delta(\mathcal{A}) \setminus U$. This gives $\|(a, 0)^\wedge|_{\Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{U}}\|_\infty = \|\widehat{a}|_{\Delta(\mathcal{A}) \setminus U}\|_\infty$. Also $(a, 0)^\wedge(\varphi^+) = \widehat{a}(\varphi)$ for every $\varphi^+ \in \widetilde{U}$. Hence

$$\|(a, 0)^\wedge|_{\Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{U}}\|_\infty = \|\widehat{a}|_{\Delta(\mathcal{A}) \setminus U}\|_\infty < \|\widehat{a}|_U\|_\infty = \|(a, 0)^\wedge|_{\widetilde{U}}\|_\infty.$$

Therefore, by Theorem 3.4, $\varphi_0^+ \in \partial(\mathcal{A} \times_c \mathcal{I})$. Thus $\partial^+(\mathcal{A}) \subset \partial(\mathcal{A} \times_c \mathcal{I})$. Let $\psi_0 \in \partial\mathcal{I}$. Let $\widetilde{V}$ be a neighborhood of ψ_0^-. Set $V = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{V}\} \cup \{\psi \in \Delta(\mathcal{I}) : \psi^- \in \widetilde{V}\}$. Then V is a neighborhood of ψ_0. Therefore, by Theorem 3.4, there exists $x \in \mathcal{I}$ such that $\|\widehat{x}|_{\Delta(\mathcal{I})\setminus V}\|_\infty < \|\widehat{x}|_V\|_\infty$. If $\psi^- \in \Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{V}$, then $(x,-x)^\wedge(\psi^-) = 2\widehat{x}(\psi)$. If $\varphi^+ \in \Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{V}$, then $(x,-x)^\wedge(\varphi^+) = 0$. This gives $\|(x,-x)^\wedge|_{\Delta(\mathcal{A}\times_c\mathcal{I})\setminus\widetilde{V}}\|_\infty = 2\|\widehat{x}|_{\Delta(\mathcal{I})\setminus V}\|_\infty$. Hence

$$\|(x,-x)^\wedge|_{\Delta(\mathcal{A}\times_c\mathcal{I})\setminus\widetilde{V}}\|_\infty = 2\|\widehat{x}|_{\Delta(\mathcal{I})\setminus V}\|_\infty < 2\|\widehat{x}|_V\|_\infty = \|(x,-x)^\wedge|_{\widetilde{V}}\|_\infty.$$

Therefore, by Theorem 3.4, $\psi_0^- \in \partial(\mathcal{A} \times_c \mathcal{I})$. Thus $\partial^-(\mathcal{I}) \subset \partial(\mathcal{A} \times_c \mathcal{I})$.

For the reverse inclusion, let $\varphi_0^+ \in \partial(\mathcal{A} \times_c \mathcal{I})$. Let U be a neighborhood of $\varphi_0 \in \Delta(\mathcal{A})$. Then $\widetilde{U} = U^+$ is a neighborhood of φ_0^+ in $\Delta(\mathcal{A} \times_c \mathcal{I})$. Since $\varphi_0^+ \in \partial(\mathcal{A} \times_c \mathcal{I})$,by Theorem 3.4, there exists $(a,x) \in \mathcal{A} \times_c \mathcal{I}$ such that $\|(a,x)^\wedge|_{\Delta(\mathcal{A}\times_c\mathcal{I})\setminus\widetilde{U}}\|_\infty < \|(a,x)^\wedge|_{\widetilde{U}}\|_\infty$. This gives $\|(a+x)^\wedge|_{\Delta(\mathcal{A})\setminus U}\|_\infty < \|(a+x)^\wedge|_U\|_\infty$. Therefore $\varphi_0 \in \partial\mathcal{A}$.

Let $\psi_0^- \in \partial(\mathcal{A} \times_c \mathcal{I})$. Let V be a neighborhood of $\psi_0 \in \Delta(\mathcal{I})$. Then V^- is a neighborhood of ψ_0^- in $\Delta(\mathcal{A} \times_c \mathcal{I})$. Since $\psi_0^- \in \partial(\mathcal{A} \times_c \mathcal{I})$,by Theorem 3.4, there exists $(a,x) \in \mathcal{A} \times_c \mathcal{I}$ such that $\|(a,x)^\wedge|_{\Delta(\mathcal{A}\times_c\mathcal{I})\setminus\widetilde{V}}\|_\infty < \|(a,x)^\wedge|_{\widetilde{V}}\|_\infty$. Hence $\|(a-x)^\wedge|_{\Delta(\mathcal{I})\setminus V}\|_\infty \leq \|(a-x)^\wedge|_V\|_\infty$. Therefore, by Theorem 3.4, $\psi_0 \in \partial\mathcal{I}$. Hence $\partial(\mathcal{A} \times_c \mathcal{I}) \subset \partial^+(\mathcal{A}) \uplus \partial^-(\mathcal{I})$. □

Remark 3.6. *Let $\varphi \in \Delta(\mathcal{I})$. Then there exists $u \in \mathcal{I}$ such that $\varphi(u) = 1$. Define Opt. Lett. $\varphi(a) := \varphi(au)$. Then Opt. Lett. $\varphi \in \Delta(\mathcal{A})$. Thus every $\varphi \in \Delta(\mathcal{I})$ can be extended to $\Delta(\mathcal{A})$. Therefore, $\Delta(\mathcal{I}) \subset \Delta(\mathcal{A})$. Also, it is clear that $\Delta(\mathcal{A}) = \Delta(\mathcal{I}) \cup \{\varphi \in \Delta(\mathcal{A}) : \mathcal{I} \subset \ker\varphi\}$. Hence $\Delta^+(\mathcal{A}) \cup \Delta^-(\mathcal{I}) = \Delta^+(\mathcal{A}) \cup \Delta^-(\mathcal{A})$ as sets.*

Theorem 3.7. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{I}$ be closed ideal in $\mathcal{A}$. Then $\mathcal{A} \times_c \mathcal{I}$ is semisimple if and only if $\mathcal{A}$ is semisimple.*

Proof. Suppose that $\mathcal{A} \times_c \mathcal{I}$ is semisimple. Let $a \in \mathcal{A}$ such that $\varphi(a) = 0$ $(\varphi \in \Delta(\mathcal{A}))$. Let $\psi \in \Delta(\mathcal{A})$ and $u \in \mathcal{A}$ such that $\psi(u) = 1$. Then $\psi^+((a,0)) = \psi(au) + \psi(0) = \psi(a)\psi(u) + 0 = 0$. Now let $\psi \in \Delta(\mathcal{I})$. Then, by Remark 3.6, *Opt. Lett.* $\psi \in \Delta(\mathcal{A})$. So, by the assumption, $\psi(av) =$ *Opt. Lett.* $\psi(a) = 0$. Hence $\psi^-((a,0)) = \psi(av) = 0$. Thus $\widetilde{\eta}((a,0)) = 0$ for all $\widetilde{\eta} \in \Delta^+(\mathcal{A}) \uplus \Delta^-(\mathcal{I})$. Since $\mathcal{A} \times_c \mathcal{I}$ is semisimple, $(a,0) = (0,0)$ gives $a = 0$. Thus $\mathcal{A}$ is semisimple.

Conversely, suppose that $\mathcal{A}$ is semisimple. Let $(a,x) \in \mathcal{A}\times_c\mathcal{I}$ be such that $\widetilde{\eta}((a,x)) = 0$ $(\widetilde{\eta} \in \Delta(\mathcal{A}\times_c\mathcal{I})$. Let $\varphi \in \Delta(\mathcal{A})$. Then $\varphi^+, \varphi^- \in \Delta(\mathcal{A} \times_c \mathcal{I})$. So that $\varphi^+((a,x)) = \varphi^-((a,x)) = 0$. Then $\varphi(a) + \varphi(x) = \varphi(a) - \varphi(x) = 0$. Hence $\varphi(a) = \varphi(x) = 0$. Since $\varphi \in \Delta(\mathcal{A})$ is arbitrary and $\mathcal{A}$ is semisimple, we get $a = x = 0$. Hence $\mathcal{A} \times_c \mathcal{I}$ is semisimple. □

4. Uniqueness and Separation Properties

We start with the following lemma which will be used in the proofs of main results.

Lemma 4.1. *Let $\mathcal{A}$ be a semisimple, commutative Banach algebra and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. Let $\widetilde{F} \subset \Delta(\mathcal{A}\times_c\mathcal{I})$. Define $F_\mathcal{A} = \{\varphi \in \Delta(\mathcal{A}) : \varphi^+ \in \widetilde{F}$ or $\varphi^- \in \widetilde{F}\}$. Then*

(1). $F_\mathcal{A}^+ \cup F_\mathcal{A}^- = \widetilde{F}$;

(2). If $\widetilde{F}$ is closed, then $F_\mathcal{A}$ is closed in $\Delta(\mathcal{A})$;

(3). If $\widetilde{F}$ is a set of uniqueness for $\mathcal{A} \times_c \mathcal{I}$, then so is $F_\mathcal{A}$ for $\mathcal{A}$.

Proof. **(1)** This is trivial.

(2) Suppose that $\widetilde{F} \subset \Delta(\mathcal{A} \times_c \mathcal{I})$ is closed. Let $\varphi \in Opt.\ Lett.\ F_{\mathcal{A}}$. Then there exists a net (φ_α) in $F_{\mathcal{A}}$ such that $\varphi_\alpha \longrightarrow \varphi$. Then we get a subnet (φ_{α_i}) of (φ_α) such that either $\{\varphi^+_{\alpha_i}\} \subset \widetilde{F}$ or $\{\varphi^-_{\alpha_i}\} \subset \widetilde{F}$. Also, $\varphi^+_{\alpha_i} \longrightarrow \varphi^+$ and $\varphi^-_{\alpha_i} \longrightarrow \varphi^-$. Since $\widetilde{F}$ is closed, either $\varphi^+ \in \widetilde{F}$ or $\varphi^- \in \widetilde{F}$. So that $\varphi \in F_{\mathcal{A}}$. Thus $F_{\mathcal{A}}$ is closed in $\Delta(\mathcal{A})$.

(3) Suppose that $\widetilde{F}$ is a set of uniqueness for $\mathcal{A} \times_c \mathcal{I}$. Let $a \in \mathcal{A}$ such that $\widehat{a}|_{F_{\mathcal{A}}} = 0$. Then $(a,0)^\wedge(\varphi^+) = \varphi(a) = \widehat{a}(\varphi) = 0 = (a,0)^\wedge(\varphi^-)$ $(\varphi \in F_{\mathcal{A}})$. Thus $(a,0)^\wedge = 0$ on $\widetilde{F}$. This implies that $(a,0) = (0,0)$ as $\widetilde{F}$ is a set of uniqueness for $\mathcal{A} \times_c \mathcal{I}$. Thus $a = 0$. Hence $F_{\mathcal{A}}$ is a set of uniqueness for $\mathcal{A}$. □

Definition 4.2 ([1, 3])**.** *An algebra $\mathcal{A}$ has* unique uniform norm property (UUNP) *if $\mathcal{A}$ has exactly one uniform norm.*

Theorem 4.3. *Let $\mathcal{A}$ be a semisimple, commutative Banach algebra and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. Then $\mathcal{A} \times_c \mathcal{I}$ has UUNP if and only if $\mathcal{A}$ has UUNP.*

Proof. Let $\mathcal{A} \times_c \mathcal{I}$ have UUNP. Let $F \subset \Delta(\mathcal{A})$ be a closed set of uniqueness for $\mathcal{A}$. Then $F^+ \uplus F^-$ is a closed subset of $\Delta(\mathcal{A} \times_c \mathcal{I})$. Moreover, it is also a set of uniqueness for $\mathcal{A} \times_c \mathcal{I}$. Since $\mathcal{A} \times_c \mathcal{I}$ has UUNP, by [3, Theorem 2.3], $\partial^+(\mathcal{A}) \uplus \partial^-(\mathcal{I}) \subset F^+ \uplus F^-$. Since $\Delta^+(\mathcal{A})$ and $\Delta^-(\mathcal{I})$ are disjoint, $\partial^+(\mathcal{A}) \subset F^+$. So, $\partial\mathcal{A} \subset F$. Thus $\partial\mathcal{A}$ is the smallest closed set of uniqueness for $\mathcal{A}$. Hence, by [3, Theorem 2.3], $\mathcal{A}$ has UUNP.
Conversely, suppose that $\mathcal{A}$ has UUNP. Since $\mathcal{A}$ is semisimple, $\mathcal{A} \times_c \mathcal{I}$ is also semisimple by Theorem 3.7. Let $\widetilde{F} \subset \Delta(\mathcal{A} \times_c \mathcal{I})$ be a closed set of uniqueness for $\mathcal{A} \times_c \mathcal{I}$. Then, by Lemma 4.1, $F_{\mathcal{A}}$ is a closed set of uniqueness for $\mathcal{A}$ and $F^+_{\mathcal{A}} \uplus F^-_{\mathcal{A}} = \widetilde{F}$. Since $\mathcal{A}$ has UUNP, by [3, Theorem 2.3], $\partial\mathcal{A} \subset F_{\mathcal{A}}$. Hence $\partial^+(\mathcal{A}) \subset F^+_{\mathcal{A}}$. Also we may assume that $\mathcal{A}$ has identity due to [2, Theorem 3.1]. Then, by [6, Theorem 3.4.13], $\partial I \subset \partial A$. Therefore $\partial I \subset F_{\mathcal{A}}$. Which implies that Hence $\partial^-(I) \subset F^-_{\mathcal{A}}$. Hence

$$\partial(\mathcal{A} \times_c \mathcal{I}) = \partial^+(\mathcal{A}) \uplus \partial^-(\mathcal{I}) \subset F^+_{\mathcal{A}} \uplus F^-_{\mathcal{A}} = \widetilde{F}.$$

Thus $\partial(\mathcal{A} \times_c \mathcal{I})$ is the smallest closed set of uniqueness for $\mathcal{A} \times_c \mathcal{I}$. Hence, again by [3, Theorem 2.3], $\mathcal{A} \times_c \mathcal{I}$ has UUNP. □

Definition 4.4 ([1, 3])**.** *A $*$-algebra $\mathcal{A}$ has* unique C^*-norm property (UC*NP) *if $\mathcal{A}$ has exactly one C^* norm.*

Theorem 4.5. *Let $\mathcal{A}$ be a $*$-semisimple, Banach $*$-algebra and $\mathcal{I}$ be a closed $*$-ideal of $\mathcal{A}$. Then*

(1). If $\mathcal{A} \times_c \mathcal{I}$ has UC*NP, then $\mathcal{A}$ has UC*NP;

(2). Suppose that $\mathcal{A}$ is commutative. If $\mathcal{A}$ has UC*NP, then $\mathcal{A} \times_c \mathcal{I}$ has UC*NP.

Proof. **(1)** Suppose that $\mathcal{A} \times_c \mathcal{I}$ has UC*NP. Let $|\cdot|_{\mathcal{A}}$ be the largest C^*-norm on $\mathcal{A}$. Define $|(a,x)| = \max\{\,|a+x|_{\mathcal{A}},\ |a-x|_{\mathcal{A}}\,\}$ $((a,x) \in \mathcal{A} \times_c \mathcal{I})$. Then, by Lemma 2.5 (3), $|\cdot|$ is a C^*-norm on $\mathcal{A} \times_c \mathcal{I}$. Now, let $|||\cdot|||_{\mathcal{A}}$ be any C^*-norm on $\mathcal{A}$. Define $|||(a,x)||| = \max\{\,|||a+x|||_{\mathcal{A}},\ |||a-x|||_{\mathcal{A}}\,\}$ $((a,x) \in \mathcal{A} \times_c \mathcal{I})$. Then, by Lemma 2.5 (3), $|||\cdot|||$ is also a C^*-norm on $\mathcal{A} \times_c \mathcal{I}$. Hence, by the hypothesis, $|\cdot| = |||\cdot|||$ on $\mathcal{A} \times_c \mathcal{I}$. Now,

$$|||a|||_{\mathcal{A}} = \max\{|||a|||_{\mathcal{A}}, |||a|||_{\mathcal{A}}\} = |||(a,0)||| = |(a,0)| = |a|_{\mathcal{A}} \quad (a \in \mathcal{A}).$$

Thus $\mathcal{A}$ has UC*NP.

(2) Suppose that $\mathcal{A}$ is commutative and it has UC*NP. Since $\mathcal{I}$ is a closed $*$-ideal in $\mathcal{A}$, by Theorem 2.2 of [1], $\mathcal{I}$ has UC*NP. Let $\Delta^h(\mathcal{A})$ denote the Hermitian Gel'fand space of $\mathcal{A}$. Let $\widetilde{F}$ be a proper closed subset of $\Delta^h(\mathcal{A} \times_c \mathcal{I})$. Set $F_{\mathcal{A}} = \{\varphi \in \Delta^h(\mathcal{A}) : \varphi^+ \in \widetilde{F} \text{ or } \varphi^- \in \widetilde{F}\}$. Then $F_{\mathcal{A}}$ is a proper closed subset of $\Delta^h(\mathcal{A})$ such that $F^+_{\mathcal{A}} \uplus F^-_{\mathcal{A}} = \widetilde{F}$. Since $\mathcal{A}$ has UC*NP, by [1, Proposition 1.3], there exists a nonzero element $a \in \mathcal{A}$ such that $\widehat{a}|_{F_{\mathcal{A}}} = 0$. Then $(a,0)^\wedge|_{\widetilde{F}} = \widehat{a}|_{F_{\mathcal{A}}} = 0$. Therefore, by [1, Proposition 1.3], $\mathcal{A} \times_c \mathcal{I}$ has UC*NP. □

Definition 4.6. *A semisimple, commutative algebra $\mathcal{A}$ is* weakly regular *if for each proper closed set $F \subset \Delta(\mathcal{A})$, there exists $a \in \mathcal{A}$ such that $\widehat{a}|_F = 0$.*

Theorem 4.7. *$\mathcal{A} \times_c \mathcal{I}$ is weakly regular if and only if $\mathcal{A}$ is weakly regular.*

Proof. Let $\mathcal{A} \times_c \mathcal{I}$ be weakly regular. Let F be a proper closed subset of $\Delta(\mathcal{A})$. Then $F^+ \cup F^-$ is a proper closed subset of $\Delta(\mathcal{A} \times_c \mathcal{I})$. Hence, by the hypothesis, there exists a non-zero element $(a, x) \in \mathcal{A} \times_c \mathcal{I}$ such that $(a, x)^\wedge|_{F^+ \cup F^-} = 0$. This implies that $(a, x)^\wedge|_{F^+} = 0$ and $(a, x)^\wedge|_{F^-} = 0$. Then $(a + x)^\wedge(\varphi) = (a, x)^\wedge(\varphi^+) = 0$ and $(a - x)^\wedge(\varphi) = (a, x)^\wedge(\varphi^-) = 0$ $(\varphi \in F)$. Thus $(a + x)^\wedge|_F = (a - x)^\wedge|_F = 0$. Also, note that either $a + x \neq 0$ or $a - x \neq 0$ as (a, x) is non-zero. Hence $\mathcal{A}$ is weakly regular.

Conversely, assume that $\mathcal{A}$ is weakly regular. Let $\widetilde{F}$ be a proper closed subset of $\Delta(\mathcal{A} \times_c \mathcal{I})$. Then, by Lemma 4.1, $F_{\mathcal{A}}$ is a proper closed subset of $\Delta(\mathcal{A})$ such that $F_{\mathcal{A}}^+ \cup F_{\mathcal{A}}^- = \widetilde{F}$. Therefore, by the hypothesis, there exists $a \in \mathcal{A}$ such that $\widehat{a}|_{F_{\mathcal{A}}} = 0$. Now let $\widetilde{\eta} \in \widetilde{F}$. Then either $\widetilde{\eta} = \varphi^+$ or $\widetilde{\eta} = \varphi^-$ for some $\varphi \in F_{\mathcal{A}}$. Suppose that $\widetilde{\eta} = \varphi^+$ for some $\varphi \in F_{\mathcal{A}}$. Then $(a, 0)^\wedge(\widetilde{\eta}) = (a, 0)^\wedge(\varphi^+) = \varphi(a) = \widehat{a}(\varphi) = 0$. Similarly, if $\widetilde{\eta} = \varphi^-$, then also $(a, 0)^\wedge(\widetilde{\eta}) = 0$. Thus, in each case, $(a, 0)^\wedge(\widetilde{\eta}) = 0$. Therefore $(a, 0)^\wedge|_{\widetilde{F}} = 0$. Hence $\mathcal{A} \times_c \mathcal{I}$ is weakly regular. $\square$

Definition 4.8. *$\mathcal{A}$ is* regular *if for every closed set $F \subset \Delta(\mathcal{A})$ and an element $\varphi \in \Delta(\mathcal{A}) \setminus F$, there exists an element $a \in \mathcal{A}$ such that $\widehat{a}(\varphi) = 1$ and $\widehat{a}|_F = 0$.*

Theorem 4.9. *$\mathcal{A} \times_c \mathcal{I}$ is regular if and only if $\mathcal{A}$ is regular.*

Proof. Let $\mathcal{A} \times_c \mathcal{I}$ be regular. Let F be a closed subset of $\Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{A}) \setminus F$. Then F^+ is a closed subset of $\Delta(\mathcal{A} \times_c \mathcal{I})$ and $\psi^+ \in \Delta(\mathcal{A} \times_c \mathcal{I}) \setminus (F^+)$. Hence, by the hypothesis, there exists $(a, x) \in \mathcal{A} \times_c \mathcal{I}$ such that $(a, x)^\wedge|_{F^+} = 0$ and $(a, x)^\wedge(\psi^+) = 1$. This implies that $(a + x)^\wedge(\varphi) = \varphi^+((a, x)) = (a, x)^\wedge(\varphi^+) = 0$ $(\varphi \in F)$ and $(a + x)^\wedge(\psi) = (a, x)^\wedge(\psi^+) = 1$. Thus $\mathcal{A}$ is regular.

Conversely, assume that $\mathcal{A}$ is regular. Since A is semisimple, $\mathcal{A} \times_c \mathcal{I}$ is also semisimple. Let $\widetilde{F}$ be a closed subsets of $\Delta(\mathcal{A} \times_c \mathcal{I})$ and $\widetilde{\psi} \in \Delta(\mathcal{A} \times_c \mathcal{I}) \setminus \widetilde{F}$. Then, by Lemma 4.1, $F_{\mathcal{A}}$ is a proper closed subset of $\Delta(\mathcal{A})$ such that $F_{\mathcal{A}}^+ \cup F_{\mathcal{A}}^- = \widetilde{F}$. Also, either $\widetilde{\psi} = \psi^+$ or $\widetilde{\psi} = \psi^-$ for some $\psi \in \Delta(\mathcal{A}) \setminus F_{\mathcal{A}}$, by Remark 3.6. Therefore, by the hypothesis, there exists $a \in \mathcal{A}$ such that $\widehat{a}|_{F_{\mathcal{A}}} = 0$ and $\widehat{a}(\psi) = 1$. Now let $\widetilde{\eta} \in \widetilde{F}$. Then either $\widetilde{\eta} = \varphi^+$ or $\widetilde{\eta} = \varphi^-$ for some $\varphi \in F_{\mathcal{A}}$. Suppose that $\widetilde{\eta} = \varphi^+$ for some $\varphi \in F_{\mathcal{A}}$. Then $(a, 0)^\wedge(\widetilde{\eta}) = (a, 0)^\wedge(\varphi^+) = \varphi(a) = \widehat{a}(\varphi) = 0$. Similarly, if $\widetilde{\eta} = \varphi^-$, then also $(a, 0)^\wedge(\widetilde{\eta}) = 0$. Thus, in each case, $(a, 0)^\wedge(\widetilde{\eta}) = 0$. Therefore $(a, 0)^\wedge|_{\widetilde{F}} = 0$. Also, $(a, 0)^\wedge(\widetilde{\psi}) = 1$. Hence $\mathcal{A} \times_c \mathcal{I}$ is regular. $\square$

References

[1] B. A. Barnes, *The properties of ∗-regularity and uniqueness of C^*-norm in a general ∗-algebra*, Trans. American Math. Soc., 279(2)(1983), 841-859.

[2] S. J. Bhatt and H. V. Dedania, *Uniqueness of uniform norm and adjoining identity in Banach algebras*, Proc. Indian Acad. Sci.(Math. Sci.), 105(4)(1995), 405-409.

[3] S. J. Bhatt and H. V. Dedania, *Banach algebras with unique uniform norm* , Proc. American Math. Soc., 124(2)(1996), 579-584.

[4] H. V. Dedania and H. J. Kanani, *Some Banach algebra properties in the cartesian product of Banach algebras*, Annals of Funct. Anal., 5(1)(2014), 51-55.

[5] H. J. Kanani, *Spectral and uniqueness properties in various Banach algebra products*, Sardar Patel University, (2016).

[6] E. Kaniuth, *A Course in Commutative Banach Algebras*, Springer Verlag, New York, (2009).