

Solution of BSM Differential Equation with Time Dependent Parameters for Standard Powered Option

S. J. Ghevariya*

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar-388120, Gujarat, India.

Abstract: This paper deals to the solution of Black-Scholes-Merton (BSM) differential equation with time dependent parameters for standard powered option which is a generalization of well known plain vanilla option.

MSC: 91B25, 91G20.

Keywords: BSM differential equation; BSM model; plain vanilla option, standard powered option.

© JS Publication.

1. Introduction

Pricing of options become highly demanded problem in the financial markets. There are various options pricing models which are being used to derive the fair market value of an option. Among these, the Black-Scholes-Merton (BSM) model is the most widely used. The very first European BSM model derived for plain vanilla option [10]. After that many BSM models have been derived for different payoff functions [7]. There are various methods like analytical method, Binomial method, Mellin transform, Projected Differential Transform method (PDTM) etc, to solve BSM differential equation for different payoff functions. The closed form solutions of BSM differential equation for different payoff functions have been derived. But the financial market point of view, it is demanded to derive solution of BSM differential equation with time dependent parameters. Further, the BSM differential equation with time dependent parameters has been derived [8]. Also, the solution of that equation for plain vanilla call option has been derived [9]. In this paper, the solution of BSM differential equation with time dependent parameters for standard powered call option has been derived which is a generalization of plain vanilla call option. The solution involves the technique of change of variables so that one can transform BSM differential equation with time dependent parameters into constant parameters directly. We derive the BSM model with time dependent coefficients is a product of option price with constant parameters, the ratio of strike prices in the original problem and in the transformed problem, a discount factor involving parameterized time variable. In section-2, the derivation of BSM model with time dependent parameters for standard powered call option has been discussed.

2. Solution of BSM Differential Equation

In this section, we derive the solution of BSM differential equation with time dependent parameters for standard powered call option, $\max\{S^p - K, 0\}$, where $p \in \mathbb{R}^+$, S is the value of underlying asset at expiration time T , and K is the striking

* E-mail: sjghevaria@gmail.com

price of the option.

The BSM differential equation on non dividend paying asset with time dependent parameters is given by

$$\frac{\partial F_C}{\partial t} + \frac{1}{2}(\sigma(t))^2 S^2 \frac{\partial^2 F_C}{\partial S^2} + r(t)S \frac{\partial F_C}{\partial S} - r(t)F_C = 0, \quad (1)$$

where $F_C(S, t)$ is the value of call option at time t which expires at T , $\sigma(t)$ and $r(t)$ are volatility and risk free interest rate at time t , respectively, S is the value of underlying asset at time t . Further, the payoff for standard powered call option is given by

$$F_C(S, T) = \max\{S^p - K, 0\}. \quad (2)$$

Now, the BSM differential equation with constant parameters is given by

$$\frac{\partial \bar{F}_C}{\partial \bar{t}} + \frac{1}{2}\sigma^2 \bar{S}^2 \frac{\partial^2 \bar{F}_C}{\partial \bar{S}^2} + r\bar{S} \frac{\partial \bar{F}_C}{\partial \bar{S}} - r\bar{F}_C = 0, \quad (3)$$

where $\bar{F}_C(\bar{S}, \bar{t})$ is the value of call option having asset price $\bar{S}$ at time $\bar{t} < \bar{T}$, σ and r are constant volatility and constant risk free interest rate. Further, the payoff for standard powered call option of the equation (3) is given by

$$\bar{F}_C(\bar{S}, \bar{T}) = \max\{\bar{S}^p - \bar{K}\}, \quad (4)$$

where $\bar{K}$ is the strike price of the option. We transform the equations (1) and (2) into equations (3) and (4) such that $\bar{t} = \bar{T}$ when $t = T$. Taking transformations

$$F_C(S, t) = \eta(t)\bar{F}_C(\bar{S}, \bar{t}), \quad \bar{S} = \lambda(t)S \quad \text{and} \quad \bar{t} = \rho(t). \quad (5)$$

Thus equation (1) reduces to

$$\frac{\partial \bar{F}_C}{\partial \bar{t}} + \frac{(\sigma(t))^2(\lambda(t))^2 S^2}{2\rho'(t)} \frac{\partial^2 \bar{F}_C}{\partial \bar{S}^2} + \frac{[r(t)\lambda(t) + \lambda'(t)]S}{\rho'(t)} \frac{\partial \bar{F}_C}{\partial \bar{S}} + \frac{\eta'(t) - r(t)\eta(t)}{\eta(t)\rho'(t)} \bar{F}_C = 0. \quad (6)$$

From equations (3) and (6), we get

$$\frac{(\sigma(t))^2(\lambda(t))^2 S^2}{2\rho'(t)} = \frac{1}{2}\sigma^2(\lambda(t))^2 \bar{S}^2, \quad \frac{[r(t)\lambda(t) + \lambda'(t)]S}{\rho'(t)} = r\lambda(t)S, \quad \text{and} \quad \frac{\eta'(t) - r(t)\eta(t)}{\eta(t)\rho'(t)} = -r. \quad (7)$$

Further simplifying equation (7), we get

$$\rho(t) = -\frac{1}{\sigma^2} \int_t^T \sigma(s)ds + C_1, \quad \lambda(t) = C_2 e^{-\int_t^T (r\rho'(s) - r(s))ds}, \quad \text{and} \quad (8)$$

$$\eta(t) = C_3 e^{-\int_t^T (r(s) - r\rho'(s))ds}, \quad (9)$$

where $C_1, C_2 > 0$ and $C_3 > 0$ are constants of integration which are to be determined so that equations (2) and (4) satisfies when $t = T$. The equations (2), (4) and (5) yields

$$\eta(T)(\lambda(T))^p = 1, \quad (\lambda(T))^p = \frac{\bar{K}}{K}.$$

Substituting these values in equation (9), we get

$$C_1 = \bar{T}, \quad C_2 = \left(\frac{\bar{K}}{K}\right)^{\frac{1}{p}} \quad \text{and} \quad C_3 = \left(\frac{K}{\bar{K}}\right).$$

Hence, solution of BSM differential equation (1) with time dependent parameters is given by

$$F_C(S, t) = \left(\frac{K}{\bar{K}}\right) \left[e^{-\int_t^T \left(r(s) - \frac{r}{\sigma^2} (\sigma(s))^2\right) ds} \right] \bar{F}_C(\bar{S}, \bar{T}), \quad (10)$$

$$\bar{S} = \left(\frac{\bar{K}}{K}\right)^{\frac{1}{p}} e^{-\int_t^T \left(\frac{r}{\sigma^2} (\sigma(s))^2 - r(s)\right) ds}, \quad \text{and} \quad \bar{t} = -\frac{1}{\sigma^2} \int_t^T (\sigma(s))^2 ds + \bar{T},$$

where $\bar{F}_C(\bar{S}, \bar{T})$ is the closed form solution of equation (3) and (4), which is given by

$$\begin{aligned} \bar{F}_C(\bar{S}, \bar{T}) &= \bar{S}^p e^{(p-1)(r + \frac{\sigma^2}{2})(\bar{T} - \bar{t})} N(d_1) - \bar{K} e^{-r(\bar{T} - \bar{t})} N(d_2), \\ d_1 &= \frac{\ln(\frac{\bar{S}}{\bar{K}}) + (r + (2p-1)\frac{\sigma^2}{2})(\bar{T} - \bar{t})}{\sigma\sqrt{\bar{T} - \bar{t}}}, \quad d_2 = \frac{\ln(\frac{\bar{S}}{\bar{K}}) + (r - \frac{\sigma^2}{2})(\bar{T} - \bar{t})}{\sigma\sqrt{\bar{T} - \bar{t}}} \quad \text{and} \end{aligned}$$

$N(\cdot)$ is a cumulative distribution function of standard normal random variable.

3. Conclusion

The BSM model with time varying parameters for plain vanilla option become special case by taking $p = 1$ in the equation (10). The described method shows that the price of European call option on a non dividend paying asset is a product of solution of BSM equation with constant parameters, ratio of strike prices and discounted factor with parameterized time. This method is applicable for European put option also. Further, the solution of BSM differential equation with time dependent parameters can be derived for general payoff functions.

References

- [1] H. V. Dedania and S. J. Ghevariya, *Option Pricing Formula for Modified Log-payoff Function*, International Journal of Mathematics and Soft Computing, 3(2)(2013), 129-140.
- [2] H. V. Dedania and S. J. Ghevariya, *Option Pricing Formulas for Fractional Polynomial Payoff Function*, Inter. Jr. of Pure and Applied Mathematical Sciences, 6(1)(2013), 43-48.
- [3] H. V. Dedania and S. J. Ghevariya, *Graphical Interpretation of Various BSM Formulas*, Global Journal of Pure and Applied Mathematics, 13(9)(2017), 6107-6112.
- [4] S. J. Ghevariya, *BSM Option Pricing Formulas Through Probabilistic Approach*, Mathematics Today, 32(2017), 31-34.
- [5] S. J. Ghevariya, *BSM European Put Option Pricing Formula for ML-Payoff Function with Mellin Transform*, Int. Jr. of Mathematics and its Applications, 6(2-B)(2018), 33-36.
- [6] S. J. Ghevariya, *BSM Model for ML-Payoff Function through PDTM*, Asian-European Journal of Mathematics, To appear.
- [7] E. G. Haug, *The Complete Guide to Option Pricing Formulas*, McGraw-Hill, 2nd Ed., (2007).
- [8] R. Merton, *Theory of rational option pricing*, Bell Journal of Management Sciences, 4(1973), 141-183.
- [9] M. R. Rodrigo and R. S. Mamon, *An alternative approach to solving the Black-Scholes equation with time-varying parameters*, Applied Mathematics Letters, 19(2)(2006), 398-402.
- [10] P. Wilmott, S. Howison and J. Dewynne, *Mathematics of Financial Derivatives*, Cambridge University Press, (2002).
- [11] P. Wilmott, *Paul Wilmott on Quantitative Finance*, John Wiley & Sons, 2nd Ed., (2006).