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Spectral Properties of M-class A_k^* Operator

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Abstract: The Banach algebra on a non-zero complex Hilbert space H of all bounded linear operators are denoted by $B(H)$. An operator T is defined as an element in $B(H)$. If T belongs to $B(H)$, then T^* means the adjoint of T in $B(H)$. An operator T is called class $A(k)$ if $|T|^2 \leq \left(T^* |T|^{2k} T\right)^{\frac{1}{k+1}}$ for $k > 0$. An operator T is called class A_k if $|T|^2 \leq \left(\left|T^{k+1}\right|^{\frac{2}{k+1}}\right)$ for some positive integer k. S. Panayappan [11] introduced class A_k^* operator as "an operator T is called class A_k^* if $\left|T^k\right|^{\frac{2}{k}} \geq |T^*|^2$ where k is a positive integer" and studied Weyl and Weyl type theorems for the operator [9]. In this paper we introduced extended class A_k^* operator and studied some of its spectral properties. We also show that extended class A_k^* operators are closed under tensor product.

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Keywords: Class A_k^*, Quasi Class A_k^*, Weyl's theorem.

1. Introduction

An operator T is defined in $B(H)$ is an element in $B(H)$. Weyl and Weyl type theorems where studied for the following class of operators. Furuta et al introduced class $A(k)$, $k > 0$ as a class of operators including p-hyponormal and log-hyponormal operators and studied Weyl type theorems. L.A.Coburn studied Weyl's theorem for non normal operators [3] then M. Berkani studied generalized Weyl's theorem for hyponormal operators [1, 2]. Panayappan extended this concept and introduced class A_k operators and verified Weyl's theorem [11]. In 2016, D. Senthil Kumar studied aluthge transformation for N-class A_k operators [10]. In 2013, Panayappan et al introduced a new class of operators in a different manner called class A_k^* operator, quasi class A_k^* operators and studied Weyl and Weyl type theorems and also proved tensor product of two quasi class A_k^* operators is closed [9]. An operator T is called class A_k^* if $\left|T^k\right|^{\frac{2}{k}} \geq |T^*|^2$ where k is a positive integer.
If $k = 1$ then class A_k^* operator coincides with hyponormal operator [9]. In this paper, we extended class A_k^* operator as a new class of operator named M-class A_k^* operators and studied some of its spectral properties.

Definition 1.1. *An operator $T \in B(H)$ is said to be M-Class A_k^* operator if there exists positive real numbers M, k such that $|T*|^2 \leq M\left(\left|T^k\right|^{\frac{2}{k}}\right)$.*

Proposition 1.2. *If $M = 1$, then M-Class A_k^* operator coincides with class A_k^* operator. If $M = 1$ and $k = 1$, then M-Class A_k^* operator coincides with hyponormal operator. Hence, Hyponormal operator $\Rightarrow$ class A_k^* operator $\Rightarrow$ M-Class A_k^* operator.*

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2. Spectral Properties of M-Class A_k^* Operators

In this section, first we proved using matrix representation that the restriction of M-Class A_k^* operators to an invariant subspace is also M-Class A_k^*, and if T is M-Class A_k^* operator, then Weyl's theorem hold for T, T^* and $f(T)$ for $f \in H(\sigma(T))$ and if T^* has SVEP, then a-Weyl's theorem hold for T, T^* and $f(T)$ for $f \in H(\sigma(T))$.

Theorem 2.1. *If T is M-Class A_k^* operator for positive real numbers M and k, then $T|_N$ is also M-Class A_k^* operator where N is an invariant subspace of T.*

Proof. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of H onto N and $T|_N = T_1 = (PTP)|_N$ and $TP = PTP$. Since T is M-class A_k^* operator and P is a projection on N, $P\left(M\left|T^k\right|^{2/k} - |T^*|^2\right)P \geq 0$. By Hansen's Inequality [4, 10],

$$
\begin{aligned}
P\left(M\left|T^k\right|^{\frac{2}{k}}\right)P &\leq M\left(PT^{*k}T^kP\right)^{1/k} \\
&= \begin{pmatrix} M\left|T_1^k\right|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{k}} \\
&= \begin{pmatrix} M\left|T_1^k\right|^{\frac{2}{k}} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Hence,

$$
M\begin{pmatrix} \left|T_1^k\right|^{\frac{2}{k}} & 0 \\ 0 & 0 \end{pmatrix} \geq P(M\left|T^k\right|^{\frac{2}{k}})P \geq P\left|T^*\right|^2 P = \begin{pmatrix} |T_1^*|^2 + |T_2^*|^2 & 0 \\ 0 & 0 \end{pmatrix}
$$

Hence $M\left|T_1^k\right|^{\frac{2}{k}} - |T_1^*|^2 \geq |T_2^*|^2 \geq 0$. Hence $T|_N$ is M-Class A_k^* operator on an invariant subspace N of T. □

Theorem 2.2. *If T is M-Class A_k^* operator for positive real numbers M and k, $\lambda \in \sigma_P(T)$ where $\lambda \neq 0$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $Ker(T-\lambda) \oplus ran(T-\lambda)^*$, then T_3 is M-Class A_k^* operator and $T_2 = 0$.*

Proof. Let P be the orthogonal projection of H onto $Ker(T-\lambda)$. Since T is M-Class A_k^* operator, $M\left|T^K\right|^{2/k} - |T^*|^2 \geq 0$ this implies that $0 \leq P\left[M\left|T^K\right|^{2/k} - |T^*|^2\right]P$, where $P\left|T^*\right|^2 P = \begin{pmatrix} |\lambda|^2 + T_2T_2^* & 0 \\ 0 & 0 \end{pmatrix}$ and $P\left|T^K\right|^2 P = \begin{pmatrix} |\lambda|^{2k} & 0 \\ 0 & 0 \end{pmatrix}$. Therefore,

$$
\begin{aligned}
P\left[M|T^K|^{2/k}\right]P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} &\geq P\left|T^*\right|^2 P \\
&= \begin{pmatrix} |\lambda|^2 + T_2T_2^* & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Hence $T_2T_2^* = 0$ implies that $T_2 = 0$. Therefore,

$$
0 \leq M\left|T^K\right|^{2/k} - TT^* = \begin{pmatrix} 0 & 0 \\ 0 & M\left|T_3^k\right|^{2/k} - |T_3^*|^2 \end{pmatrix}.
$$

Hence T_3 is M-Class A_k^* operator. □

Theorem 2.3. *If T is M-Class A_k^* operator for positive real numbers M and k and $(T-\lambda)x = 0$ for all $\lambda \neq 0$ and $x \in H$ then $(T-\lambda)^* x = 0$.*

Proof. Using Schwarz's and Holder McCarthy inequalities,

$$\begin{aligned}
|\lambda|^2 \|x\|^2 &= \|Tx\|^2 \\
&= f(\sigma(T)) - \pi_{00}(T) \\
&= \langle T^*Tx, x\rangle \\
&= \langle (T^*T)x, x\rangle \\
&= \left\langle |T|^2 x, x\right\rangle \\
&\leq \left\langle M\left(\left|T^k\right|\right)^{2/k} x, x\right\rangle \\
&\leq \left\langle M\left(T^k x,\, T^k x\right)\right\rangle^{2/k} \|x\|^{2(1-2/k)} \\
&\leq \left\langle M\left(T^k x, T^k x\right)\right\rangle^{2/k} \|x\|^{2((k-2)/k)} \\
&= M\,|\lambda|^2 \|x\|^2 .
\end{aligned}$$

Hence $|\lambda|^2 \langle x, x\rangle = \langle T^*Tx, x\rangle = \left\langle M\left(\left|T^k\right|\right)^{2/k} x, x\right\rangle$. Since, $\left\langle M\left(\left|T^k\right|\right)^{2/k} x\right\rangle$ and x are linearly independent. Therefore, $M\left(\left|T^k\right|\right)^{2/k} x = |\lambda|^2 x$

$$\left\|(M\left|T^k\right|^{2/k} - |T^*|^2)^{1/2} x\right\|^2 = \left\langle \left|M(\left|T^k\right|)^{2/k} - (TT^*)\right| x, x\right\rangle = 0.$$

Therefore, $(TT^*)x = M\left(\left|T^k\right|\right)^{2/k} x = |\lambda|^2 x = 0 \Rightarrow (T-\lambda)^* x = 0.$ ☐

Corollary 2.4. *If T is M-Class A_k^* operator for positive real numbers M and k, $0 \neq \lambda \in \sigma_P(T)$ then T is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $Ker(T-\lambda) \oplus ran(T-\lambda)^*$, where T_3 is M-Class A_k^* and $Ker(T_3 - \lambda) = \{0\}$.*

An operator T is called normaloid if $r(T) = \|T\|$, where $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. An operator T is called hereditarily normaloid, if every part of it is normaloid. If iso $\sigma(T) \subseteq \pi(T)$ then an operator T is called polaroid where $\pi(T)$ is the set of poles of the resolvent of T and iso $\sigma(T)$ is the set of all isolated points of $\sigma(T)$. An operator T is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator T is said to be reguloid if for every isolated point λ of $\sigma(T)$, $\lambda I - T$ is relatively regular. An operator T is known as relatively regular if and only if ker T and $T(X)$ are complemented. Hence, we can say that **Polaroid⇒reguloid⇒isoloid**.

Theorem 2.5. *If T is M-Class A_k^* operator for positive real numbers M and k,then for $\lambda \in C$, if $\sigma(T) = \lambda$ then $T = \lambda$.*

Proof.

Case (A): Let $\lambda = 0$. It is obvious that, Hyponormal operator ⊂ k-paranormal ⊂ normaloid [11]. Therefore M-Class A_k^* operator is also normaloid. Therefore $T = 0$.

Case (B): Let $\lambda \neq 0$. Since T is M-Class A_k^* operator then T is invertible, so is also M-Class A_k^*. Hence it is also normaloid. We know that, if $\lambda \in T$ then $\frac{1}{\lambda} \in T^{-1}$. Hence $\|T\| \left\|T^{-1}\right\| = |\lambda| \left|\frac{1}{\lambda}\right| = 1 \Rightarrow T$ is covexoid (i.e) $w(T) = \{\lambda\} \Rightarrow T = \lambda$. ☐

Since class A_k^* operator are k^* paranormal, by [8] class A_k^* operators are normaloid by the inclusion property M- class A_k^* operators are also normaloid and by [7] we have the following results.

Theorem 2.6. *If T is M-Class A_k^* operator for positive real numbers M and k, then*

(1). T is Polaroid.

(2). T is isoloid.

(3). If $\lambda \in \sigma(T)$ is a isolated point then $E_\lambda H = Ker(T-\lambda)$ and hence λ is an eigen value of T.

(4). If $\lambda \neq 0$ be an isolated point in $\sigma(T)$, then E_λ is self adjoint and satisfies

$$E_\lambda H = Ker(T-\lambda) = Ker(T-\lambda)^*.$$

(5). T has SVEP, $P(\lambda I - T) \leq 1$ for every $\lambda \in C$ and T^* is reguloid.

(6). Weyl's theorem holds for T and T^*. In addition, T^* has SVEP, then a-Weyl's theorem holds for both T and T^* and for $f(T)$ for every $f \in H(\sigma(T))$.

Theorem 2.7. *If T is M-Class A_k^* operator for positive real numbers M and k then $(T-\lambda)$ has finite ascent for $\lambda \in C$.*

Proof. By Theorem 2.5, for $\lambda \neq 0$

$$Ker(T-\lambda) \subseteq Ker(T-\lambda)^*.$$

Hence if $x \in \ker(T-\lambda)^2$, then $(T-\lambda)^*(T-\lambda)x = 0$ for $\lambda \neq 0$. Hence $\|(T-\lambda)x\|^2 = 0$ implies $x \in \ker(T-\lambda)$. Hence $\ker(T-\lambda)^2 = \ker(T-\lambda)$. If $\lambda = 0$, it is sufficient to prove $\ker T^{2k} \subset \ker T^k$. Let $x \in \ker T^{2k}$ and $x \neq 0$. By holder MC Carthy inequality,

$$\begin{aligned}
0 = \left\|T^{2k}\right\|^2 &= \left\langle \left|T^{2k}\right|^2 x, x\right\rangle \\
&\geq \left\langle \left|T^{2k}\right|^{2/k} x, x\right\rangle^k \\
&\geq \left\langle \left|T^2\right|^2 x, x\right\rangle^k \|x\|^{2k} \\
&= \|Tx\|^{2/k} \|x\|^{2k}
\end{aligned}$$

Hence $x \in \ker T \subset \ker T^k \Rightarrow T$ has finite asent. □

Theorem 2.8. *If T is M-Class A_k^* operator for positive real numbers M and k then $f(w(T)) = w(f(T))\ \forall\ f \in (\sigma(T))$.*

Proof. If T is M-Class A_k^* operator for positive real numbers M and k then T is of finite Ascent (by Theorem 2.9) by [5], (Proposition 38.5) $ind(T-\lambda) \neq 0$ for all complex numbers λ. Therefore by Theorem 5 of [13] $f(w(T)) = w(f(T))\ \forall\ f \in (\sigma(T))$. □

Theorem 2.9. *If T is M-Class A_k^* operator for positive real numbers M and k, then Weyl's theorem holds for $f(T)$ for every $f \in (\sigma(T))$.*

Proof. By Theorem 2.7, T is isoloid and Weyl's theorem holds for T. By lemma of [6],

$$f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T)), \quad forevery\, f \in H(\sigma(T)).$$

By Theorem 2.8,

$$f(w(T)) = w(f(T))\ \forall\ f \in (\sigma(T)).$$

Hence, $\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T)) - \pi_{00}(T) = f(w(T)) = w(f(T))$. Hence, Weyl's theorem holds for $f(T)\ \ \forall f \in H(\sigma(T))$. □

3. Tensor Product of M-Class A_k^* Operators

In this section, we proved that M-Class A_k^* operators are closed under tensor product.

Theorem 3.1. *If $T \in B(H)$ and $S \in B(K)$ are non-zero operators, then $T \otimes S$ is M-Class A_k^* operator if and only if T and S are M-Class A_k^* operators. $|T^*|^2 \leq M\left|T^K\right|^{2/k}$.*

Proof. Assume that T and S are M-Class A_k^* operators. Then

$$\begin{aligned} M\left|(T\otimes S)^k\right|^{2/k} &= M|T^k|^{2/k} \otimes M|S^k|^{2/k} \\ &\geq |T^*|^2 \otimes |S^*|^2 \\ &= |T^* \otimes S^*|^2 \end{aligned}$$

Hence, $T \otimes S$ is M-Class A_k^* operator.

Conversely, assume that $T \otimes S$ is M-Class A_k^* operator. Without loss of generality, it is enough to show that T is M-Class A_k^* operator. Since $|T^* \otimes S^*|^2 \leq M\left|(T\otimes S)^k\right|^{2/k}$. We have $|T^*|^2 \otimes |S^*|^2 \leq M\left|T^k\right|^{2/k} \otimes M\left|S^k\right|^{2/k}$. Therefore,

$$\begin{aligned} \|T^*\|^2 &= \sup\left\{\left\langle |T^*|^2 x, x\right\rangle : x \in H \, and \, \|x\| = 1\right\} \\ &\leq \sup\left\{\left\langle M|T^k|^{2/k} x, x\right\rangle : x \in H \, and \, \|x\| = 1\right\} \\ &\leq M \sup\left\{\left\langle |T^k|^2 x, x\right\rangle^{1/k} : x \in H \, and \, \|x\| = 1\right\} \\ &\leq M\|T^k\|^{2/k} \\ &\leq M\|T^*\|^2 \end{aligned}$$

Similarly, $\|S^*\|^2 \leq M\|S^*\|^2$. Hence both T and S are M-Class A_k^* operators. □

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