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On Generalized q-sakaguchi Type Functions

N. Shilpa[1,*] and S. Latha[2]

1 PG Department of Mathematics, JSS College of Arts Commerce and Science, Ooty Road, Mysuru, Karnataka, India.

2 Department of Mathematics, Yuvaraja's College, University of Mysore, Mysuru, Karnataka, India.

Abstract: The aim of the present paper is to study new subclasses of functions defined by using generalized sakaguchi type functions and the concept of q derivative. The results investigated in this paper include coefficient inequalities, distortion inequalities, coefficient estimates etc.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the familiar class of functions $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$. A function $f(z) \in \mathcal{A}$ in (1) is said to be in the generalized sakaguchi class $\mathcal{S}(\alpha, s, t)$ defined by frasin [1] if it satisfies

$$\Re\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \alpha, \tag{2}$$

for some $\alpha(0 \le \alpha < 1), |t| \le 1, s \ne t$ and for all $z \in \mathcal{U}$. In 1910, Jackson [4, 5] presented a precise definition of q-difference operator and developed q-calculus in a systematic way.

$$\mathcal{D}_q f(z) = \frac{f(z)-f(qz)}{z(1-q)}, \quad (z \ne 0, 0 < q < 1), \quad (\mathcal{D}_q f(0) = f'(0)). \tag{3}$$

Equivalently (3), may be written as

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad z \ne 0,$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

Note that as $q \to 1$, $[n]_q \to n$. Using the concept of q-derivative we introduce new subclasses of functions associated with generalized sakaguchi type functions as follows.

* *E-mail: drshilpamaths@gmail.com*

Definition 1.1. *Let $\mathcal{S}_q(\alpha, s, t)$ denote the family of functions $f \in \mathcal{A}$ which satisfies*

$$\Re\left\{\frac{(s-t)z\mathcal{D}_q f(z)}{f(sz)-f(tz)}\right\} > \alpha, \quad \text{for all } \ z \in \mathcal{U}. \tag{4}$$

For arbitrary fixed numbers q, α and t, $0 \leq q < 1$, $0 \leq \alpha < 1$, $-1 \leq t < 1$, $s \neq t$.

Remark 1.2. *As $q \to 1$, the class $\mathcal{S}_q(\alpha, s, t)$ reduces to the subclass $\mathcal{S}(\alpha, s, t)$ studied by Frasin [1], for $s = 1$, the class $\mathcal{S}_q(\alpha, s, t)$ reduces to the subclass $\mathcal{S}_q(\alpha, t)$ studied by Hamid et. al. [3], also $\mathcal{S}_1(\alpha, 1, t) = \mathcal{S}(\alpha, t)$ the class introduced and studied by Owa et al. [6], $\mathcal{S}_1(0, -1) = \mathcal{S}(0, -1)$ the class introduced and studied by Sakaguchi [8] and $\mathcal{S}_1(0, t) = \mathcal{S}(t)$ the class introduced Rønning [7].*

The subclasss $\mathcal{T}_q(\alpha, s, t)$ is defined by $f \in \mathcal{T}_q(\alpha, s, t)$ if $z\mathcal{D}_q f(z) \in \mathcal{S}_q(\alpha, s, t)$. The following lemma is necessary to prove our main results.

Lemma 1.3 ([2]). *Let $P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $\quad (z \in \mathcal{U})$, with the condition $\Re\{p(z)\} > 0$, then $|p_n| \leq 2$, $\quad (n \geq 1)$.*

2. Main Results

Theorem 2.1. *If the function $f \in \mathcal{A}$ and satisfies*

$$\sum_{n=2}^{\infty} \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\}\,|a_n| \leq 1-\alpha, \quad u_n(s,t) = \sum_{j=1}^{n} s^{n-j} t^{j-1}, \tag{5}$$

then $f \in \mathcal{S}_q(\alpha, s, t)$.

Proof. We show that if f satisfies (5) then

$$\left|\frac{(s-t)z\mathcal{D}_q f(z)}{f(sz)-f(tz)} - 1\right| < 1-\alpha.$$

Evidently, since

$$\begin{aligned}\frac{(s-t)z\mathcal{D}_q f(z)}{f(sz)-f(tz)} - 1 &= \frac{\sum_{n=2}^{\infty}([n]_q - u_n(s,t))a_n z^n}{z + \sum_{n=2}^{\infty} a_n u_n(s,t) z^n} \\ &= \frac{\sum_{n=2}^{\infty}([n]_q - u_n(s,t))a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n u_n(s,t) z^{n-1}}\end{aligned}$$

We have

$$\left|\frac{(s-t)z\mathcal{D}_q f(z)}{f(sz)-f(tz)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}|[n]_q - u_n(s,t)||a_n|}{1 - \sum_{n=2}^{\infty}|a_n||u_n(s,t)|}.$$

Therefore if f satisfies (5), then we have

$$\left|\frac{(s-t)z\mathcal{D}_q f(z)}{f(sz)-f(tz)} - 1\right| < 1-\alpha.$$

This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *If the function $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}[n]_q \left\{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\right\}|a_n| \leq 1-\alpha, \quad u_n(s,t) = \sum_{j=1}^{n} s^{n-j}t^{j-1}, \tag{6}$$

for $0 \leq \alpha < 1$, then $f \in \mathcal{T}_q(\alpha, s, t)$.

Proof. Since $f \in \mathcal{T}_q(\alpha, s, t)$ if and only if $z\mathcal{D}_q f(z) \in \mathcal{S}_q(\alpha, s, t)$, the result follows. □

We discuss the coefficient inequalities for function f in the subclasses $\mathcal{S}_q(\alpha, s, t)$ and $\mathcal{T}_q(\alpha, s, t)$.

Theorem 2.3. *If $f \in \mathcal{S}_q(\alpha, s, t)$, then*

$$|a_n| \leq \prod_{j=1}^{n-1} \frac{2(1-\alpha)|u_j(s,t)| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|}, \tag{7}$$

Proof. We define the function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha}\left(\frac{(s-t)z\mathcal{D}_q f(z)}{f(sz) - f(tz)} - \alpha\right) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

where $p(z)$ is Carathéodory function and $f(z) \in \mathcal{S}_q(\alpha, s, t)$. Since

$$(s-t)z\mathcal{D}_q f(z) = (f(sz) - f(tz))(\alpha + (1-\alpha)p(z)),$$

we have

$$\sum_{n=2}^{\infty}([n]_q - u_n(s,t))a_n z^n = \left(z + \sum_{n=2}^{\infty} a_n u_n(s,t) z^n\right)\left(1 + (1-\alpha)\sum_{n=1}^{\infty} p_n z^n\right)$$

where $u_n(s,t) = \sum\limits_{j=1}^{n} s^{n-j}t^{j-1}$. Equating coefficients of z^n on both sides we have

$$a_n = \frac{(1-\alpha)}{([n]_q - u_n(s,t))}\sum_{j=1}^{n-1} u_{n-j}(s,t)a_{n-j}p_j, \quad a_1 = 1.$$

By Lemma 1.3, we get

$$|a_n| \leq \frac{2(1-\alpha)}{|[n]_q - u_n(s,t)|}\sum_{j=1}^{n-1}|u_j(s,t)|a_j|, \quad a_1 = 1. \tag{8}$$

Now we prove that

$$\frac{(1-\alpha)}{|[n]_q - u_n(s,t)|}\sum_{j=1}^{n-1}|u_j(s,t)|a_j| \leq \prod_{j=1}^{n-1}\frac{2(1-\alpha)|u_j(s,t)| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|}. \tag{9}$$

We proof (9) by the induction method.

For $n = 2$, from (8), we have

$$|a_2| \leq \frac{2(1-\alpha)}{|[2]_q - u_2(s,t)|}.$$

(9) yields

$$|a_2| \leq \frac{2(1-\alpha)|u_1(s,t)| + |[1]_q - u_1(s,t)|}{|[2]_q - u_2(s,t)|} \leq \frac{2(1-\alpha)+1}{|[2]_q - u_2(s,t)|}.$$

For $n = 3$, from (8), we get

$$|a_3| \leq \frac{2(1-\alpha)}{|[3]_q - u_3(s,t)|}(1 + |u_2(s,t)||a_2|)$$

$$\leq \frac{2(1-\alpha)}{|[3]_q - u_3(s,t)|}\left(1 + |u_2(s,t)|\frac{2(1-\alpha)}{|[2]_q - u_2(s,t)|}\right).$$

Also from (7), we derive

$$\begin{aligned} |a_3| &\leq \left(\frac{2(1-\alpha)+1}{|[2]_q - u_2(s,t)|}\right)\left(\frac{2(1-\alpha)|u_2(s,t)| + |[2]_q - u_2(s,t)|}{|[3]_q - u_3(s,t)|}\right) \\ &\leq \left(\frac{2(1-\alpha)+1}{|[3]_q - u_3(s,t)|}\right)\left(\frac{2(1-\alpha)|u_2(s,t)|}{|[2]_q - u_2(s,t)|} + 1\right). \end{aligned}$$

Let the hypothesis be true for $n = m$. From (8), we have

$$|a_m| \leq \frac{2(1-\alpha)}{|[m]_q - u_m(s,t)|}\sum_{j=1}^{m-1}|u_j(s,t)|a_j|, a_1 = 1.$$

From (8), we have

$$|a_m| \leq \prod_{j=1}^{m-1}\frac{2(1-\alpha)|u_j(s,t)| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|}.$$

By the induction hypothesis , we have

$$\frac{2(1-\alpha)}{|[m]_q - u_m(s,t)|}\sum_{j=1}^{m-1}|u_j(s,t)|a_j| \leq \prod_{j=1}^{m-1}\frac{2(1-\alpha)|u_j(s,t)| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|}.$$

Multiplying both sides by

$$\frac{2(1-\alpha)|u_m(s,t)| + |[m]_q - u_m(s,t)|}{|[m+1]_q - u_{m+1}(s,t)|},$$

we have

$$\begin{aligned} \prod_{j=1}^{m}\frac{2(1-\alpha)|u_j(s,t)| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|} &\geq \frac{2(1-\alpha)}{|[m]_q - u_m(s,t)|}\,\frac{2(1-\alpha)|u_m(s,t)| + |[m]_q - u_m(s,t)|}{|[m+1]_q - u_{m+1}(s,t)|}\sum_{j=1}^{m-1}|u_j(s,t)|a_j| \\ &= \frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}(s,t)|}\left\{\frac{2(1-\alpha)|u_m(s,t)|}{|[m]_q - u_m(s,t)|}\sum_{j=1}^{m-1}|u_j(s,t)|a_j| + \sum_{j=1}^{m-1}|u_j(s,t)|a_j|\right\} \\ &\geq \frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}(s,t)|}\left\{|u_m(s,t)||a_m| + \sum_{j=1}^{m-1}|u_j(s,t)|a_j|\right\} \\ &\geq \frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}(s,t)|}\sum_{j=1}^{m}|u_j(s,t)|a_j| \end{aligned}$$

Hence

$$\frac{2(1-\alpha)}{|[m+1]_q - u_{m+1}(s,t)|}\sum_{j=1}^{m}|u_j(s,t)|a_j| \leq \prod_{j=1}^{m}\frac{2(1-\alpha)|u_j| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|}.$$

which shows that the inequality (9) is true for $n = m+1$, and the result is true. □

Theorem 2.4. *If $f \in \mathcal{T}_q(\alpha, s, t)$, then*

$$|a_n| \leq \frac{1}{[n]_q}\prod_{j=1}^{n-1}\frac{2(1-\alpha)|u_j(s,t)| + |[j]_q - u_j(s,t)|}{|[j+1]_q - u_{j+1}(s,t)|}, \quad \textit{for} \;\; n \geq 2. \tag{10}$$

We now define $\mathcal{S}_{0,q}(\alpha, s, t)$ and $\mathcal{T}_{0,q}(\alpha, s, t)$ as follows.

$$\mathcal{S}_{0,q}(\alpha, s, t) = \{f(z) \in \mathcal{A} : \; f(z) \text{ satisfies } (5)\}$$

$$\mathcal{T}_{0,q}(\alpha, s, t) = \{f(z) \in \mathcal{A} : \; f(z) \text{ satisfies } (6)\}.$$

For functions f in the classes $\mathcal{S}_{0,q}(\alpha, s, t)$ and $\mathcal{T}_{0,q}(\alpha, s, t)$ we derive the following results.

Theorem 2.5. *If $f \in \mathcal{S}_{0,q}(\alpha, s, t)$, then*

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - A_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z|^n + A_j|z|^{j+1}, \tag{11}$$

where

$$A_j = \frac{1 - \alpha - \sum_{n=2}^{\infty} \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\} |a_n|}{j + 1 - \alpha|u_{j+1}(s,t)|} \quad (j \geq 2). \tag{12}$$

Proof. From the inequality (5), we know that

$$\sum_{n=j+1}^{\infty} \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\} |a_n| \leq 1 - \alpha - \sum_{n=2}^{j} \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\} |a_n|.$$

On the other hand

$$|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)| \geq [n]_q - \alpha|u_n(s,t)|,$$

and hence $[n]_q - \alpha|u_n(s,t)|$ is monotonically increasing with respect to n. Thus we deduce

$$(j + 1 - \alpha|u_{j+1}(s,t)|) \sum_{n=j+1}^{\infty} |a_n| \leq 1 - \alpha - \sum_{n=2}^{j} \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\} |a_n|.$$

which implies that

$$\sum_{n=j+1}^{\infty} |a_n| \leq A_j.$$

Therefore we have that

$$|f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z|^n + A_j|z|^{j+1}$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{j} |a_n||z|^n - A_j|z|^{j+1}.$$

This completes the proof of the theorem. □

Analogously we prove

Theorem 2.6. *If $f \in \mathcal{T}_{0,q}(\alpha, t)$ then*

$$|z| - \sum_{n=2}^{j} |a_n||z|^n - B_j|z|^{j+1} \leq |f(z)| \leq |z| + \sum_{n=2}^{j} |a_n||z|^n + B_j|z|^{j+1},$$

and

$$1 - \sum_{n=2}^{j} [n]_q|a_n||z|^{n-1} - C_j|z|^{j-1} \leq |f'(z)| \leq 1 + \sum_{n=2}^{j} [n]_q|a_n||z|^{n-1} + C_j|z|^{j-1}.$$

Where

$$B_j = \frac{1 - \alpha - \sum_{n=2}^{j} [n]_q \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\} |a_n|}{(j+1)\{j + 1 - \alpha|u_{j+1}(s,t)|\}}, \quad (j \geq 2).$$

and

$$C_j = \frac{1 - \alpha - \sum_{n=2}^{j} [n]_q \{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|\} |a_n|}{j + 1 - \alpha|u_{j+1}(s,t)|}, \quad (j \geq 2).$$

Remark 2.7. *By the definitions of the classes $\mathcal{S}_{0,q}(\alpha,s,t)$, and $\mathcal{T}_{0,q}(\alpha,s,t)$, evidently we have $\mathcal{S}_{0,q}(\alpha,s,t) \subset \mathcal{S}_{0,q}(\beta,s,t)$; $(0 \leq \beta \leq \alpha < 1)$, and $\mathcal{T}_{0,q}(\alpha,s,t) \subset \mathcal{T}_{0,q}(\beta,s,t)$; $(0 \leq \beta \leq \alpha < 1)$. Let us discuss the relation between $\mathcal{S}_{0,q}(\beta,s,t)$ and $\mathcal{T}_{0,q}(\alpha,s,t)$.*

Theorem 2.8. *If $f \in \mathcal{T}_{0,q}(\alpha,s,t)$, then $\in \mathcal{S}_{0,q}(\frac{1+\alpha}{2},s,t)$.*

Proof. Let $f(z) \in \mathcal{T}_{0,q}(\alpha,s,t)$. Then if $f(z)$ satisfies

$$\frac{|[n]_q - u_n(s,t)| + (1-\beta)|u_n(s,t)|}{1-\beta} \leq [n]_q \frac{|[n]_q - u_n(s,t)| + (1-\alpha)|u_n(s,t)|}{1-\alpha} \tag{13}$$

for all $n \geq 2$, then we have that $f(z) \in \mathcal{S}_{0,q}(\beta,s,t)$. From 13, we have

$$\beta \leq 1 - \frac{(1-\alpha)|[n]_q - u_n(s,t)|}{[n]_q|[n]_q - u_n(s,t)| + (1-\alpha)([n]_q - 1)|u_n(s,t)|}.$$

Furthermore, since for all $n \geq 2$

$$\frac{|[n]_q - u_n(s,t)|}{[n]_q|[n]_q - u_n(s,t)| + (1-\alpha)([n]_q - 1)|u_n(s,t)|} \leq \frac{1}{n} \leq \frac{1}{2},$$

we obtain

$$f(z) \in \mathcal{S}_{0,q}\left(\frac{1+\alpha}{2},s,t\right).$$

□

Remark 2.9. *As $q \to 1$, $s = 1$ we obtain the results in [6], and for $s = 1$ we obtain the results in [3].*

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