International Journal of Mathematics And Its Applications
Vol.1 No.1 (2013), pp.47-52.
ISSN: 2347-1557(online)

Asymptotically Lacunary Statistically Equivalent Sequences of Interval Numbers

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Abstract : In this article we present the following definition of asymptotic equivalence which is natural combination of the definition for asymptotically equivalent and lacunary statistical convergence of interval numbers. Let $\theta = (k_r)$ be a lacunary sequence, then the two sequnces $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k)$ of interval numbers are said to be asymptotically lacunary statistically equivalent to multiple $\overline{1} = [1, 1]$ provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{1} \right) \geq \varepsilon \right\} \right| = 0.$$

Keywords : Asymptotically equivalent, lacunary sequence, interval numbers.

1 Introduction

The idea of statistical convergence for single sequences was introduced by Fast [7] in 1951. Schoenberg [15] studied statistical convergence as a summability method and listed some elementary properties of statistical convergence. Both of these authors noted that if a bounded sequence is statistically convergent,

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AMS Subject Classification: 40A05, 40C05, 46D25.

then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to statistically sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

Chiao in [1] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz in [16] introduced the concepts of bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric spaces. Later, Esi [4] introduced and studied strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence of interval numbers. Recently, Esi [5] introduced and studied asymptotically $\lambda-$statistical equivalent sequences of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analytic properties.We denote the set of all real valued closed intervals by $\mathrm{I}\mathbb{R}$. Any element of $\mathrm{I}\mathbb{R}$ is called closed interval and denoted by $\overline{x}$;that is $\overline{x} = \{x \in \mathbb{R} : \ a \leq x \leq b\}$. An interval number $\overline{x}$ is a closed subset of real numbers and the set of all interval numbers is denoted by $\mathrm{I}\mathbb{R}$[1]. Let x_l and x_r be first and last points of an interval number $\overline{x}$. For $\overline{x}_1, \overline{x}_2 \in \mathrm{I}\mathbb{R}$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{1_l}=x_{2_l}$, $x_{1_r}=x_{2_r}$. $\overline{x}_1 + \overline{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \leq x \leq x_{1_r} + x_{2_r}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \ \alpha x_{1_l} \leq x \leq \alpha x_{1_r}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in \mathbb{R} : \ \alpha x_{1_r} \leq x \leq \alpha x_{1_l}\}$, where $\alpha \in \mathbb{R}$.

Let us suppose that $A = \{x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}\}$. Then the multiplication of $\overline{x}_1$ and $\overline{x}_2$ defined by

$$\overline{x}_1.\overline{x}_2 = \{x \in \mathbb{R} : \ \min A \leq x \leq \max A\}.$$

Also $\frac{\overline{x}_1}{\overline{x}_2}$ can be defined as similar to multiplication if $0 \notin \overline{x}_2$.The set of all interval numbers $\mathrm{I}\mathbb{R}$ is a complete metric space defined by

$$d(\overline{x}_1, \overline{x}_2) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\} \ [11].$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x_k}$ is called k^{th} term of sequence $\overline{x} = (\overline{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebric properties of w^i can be found in [16].

Now we give the definition of convergence of interval numbers:

A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \geq k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$.Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$ [1].

By a lacunary sequence $\theta = (k_r)$, $r = 0, 1, 2, ...$,where $k_0 = 0$, we shall mean increasing sequence of non-negative integers $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The notion of lacunary sequences have been discussed in [6, 8, 14, 17, 6, 8, 14, 17] and many others.

Marouf [18] presented definition for asymptotically equivalent sequences as follows: Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\frac{x_k}{y_k} = 1, \quad \text{(denoted by } x \sim y)$$

In [19], Patterson extended those concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices. In [20], Patterson and Savas extended the definitions presented [19] to lacunary sequences. This paper extend the definitions of asymptotical equivalent sequences to interval sequences by using lacunary sequences.

2 Definitions and Notations

Definition 2.1. *Let $\theta = (k_r)$ be a lacunary sequence. Two non-negative sequences $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k)$ of interval numbers are said to be asymptotically lacunary statistical equivalent of multiple $\overline{L} = [L_l, L_r]$ provided that for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right| = 0, \quad \left(\text{denoted by } \overline{x} \overset{\overline{s}_\theta^{\overline{L}}}{\sim} \overline{y} \right)$$

and simply asymptotically lacunary statistical equivalent if $\overline{L} = \overline{1} = [1,1]$. Furthermore, let $\overline{s}_\theta^{\overline{L}}$ denotes the set of $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k)$ of interval numbers such that $\overline{x} \overset{\overline{s}_\theta^{\overline{L}}}{\sim} \overline{y}$.

Definition 2.2. *Let $\theta = (k_r)$ be a lacunary sequence. Two non-negative sequences $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k)$ of interval numbers are strong asymptotically lacunary statistical equivalent of multiple $\overline{L}$ provided that*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) = 0,$$

$\left(\text{denoted by } \overline{x} \overset{\overline{N}_\theta^{\overline{L}}}{\sim} \overline{y} \right)$ and simply strong asymptotically lacunary statistical equivalent if $\overline{L} = \overline{1} = [1,1]$. In addition, let $\overline{N}_\theta^{\overline{L}}$ denotes the set of $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k)$ of interval numbers such that $\overline{x} \overset{\overline{N}_\theta^{\overline{L}}}{\sim} \overline{y}$.

3 Main Results

Theorem 3.1. *Let $\theta = (k_r)$ be a lacunary sequence. Then*

(a) If $\overline{x} \overset{\overline{N}_\theta^{\overline{L}}}{\sim} \overline{y}$ then $\overline{x} \overset{\overline{s}_\theta^{\overline{L}}}{\sim} \overline{y}$, where $0 \notin \overline{y} = (\overline{y}_k)$.

(b) If $\overline{x} = (\overline{x}_k)$ and $\overline{0} \neq \overline{y} = (\overline{y}_k) \in \overline{l}_\infty$ and $\overline{x} \overset{\overline{s}_\theta^{\overline{L}}}{\sim} \overline{y}$ then $\overline{x} \overset{\overline{N}_\theta^{\overline{L}}}{\sim} \overline{y}$.

(c) $\overline{s}_\theta^{\overline{L}} \cap \overline{l}_\infty = \overline{N}_\theta^{\overline{L}} \cap \overline{l}_\infty$, where $\overline{l}_\infty = \{ \overline{x} = (\overline{x}_k) : \ \sup_k \{ |x_{k_l}|, |x_{k_r}| \} < \infty \}$, which introduced in [16].

Proof.

(a) If $\varepsilon > 0$ and $\overline{x} \overset{\overline{N}_\theta^{\overline{L}}}{\sim} \overline{y}$, then

$$\begin{aligned} \sum_{k \in I_r} d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) &\geq \sum_{k \in I_r \ \& \ d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon} d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \\ &\geq \varepsilon . \left| \left\{ k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right| . \end{aligned}$$

Therefore $\overline{x} \overset{\overline{s}_\theta^L}{\sim} \overline{y}$.

(b) Suppose that $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k) \in \bar{l}_\infty$ and $\overline{x} \overset{\overline{s}_\theta^L}{\sim} \overline{y}$. Then we can assume that $d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \leq T$ for all k. Given $\varepsilon > 0$

$$\frac{1}{h_r}\sum_{k\in I_r} d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) = \frac{1}{h_r}\sum_{k\in I_r \ \& \ \overline{d}\left(\frac{X_k}{Y_k}, L\right)\geq\varepsilon} d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) + \frac{1}{h_r}\sum_{k\in I_r \ \& \ \overline{d}\left(\frac{X_k}{Y_k}, L\right)<\varepsilon} d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right)$$

$$\leq \frac{T}{h_r}\left|\left\{k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right| + \varepsilon$$

Therefore $\overline{x} \overset{\overline{N}_\theta^L(F)}{\sim} \overline{y}$.

(c) It follows from (a) and (b).

□

Theorem 3.2. *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$, then $\overline{x} \overset{\overline{s}^L}{\sim} \overline{y}$ implies $\overline{x} \overset{\overline{s}_\theta^L}{\sim} \overline{y}$, where $\overline{s}^L$ the set of $\overline{x} = (\overline{x}_k)$ and $0 \notin \overline{y} = (\overline{y}_k)$ of interval numbers such that*

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right| = 0.$$

Proof. Suppose that $\liminf_r q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $\overline{x} \overset{\overline{s}^L}{\sim} \overline{y}$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k \leq k_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right| \geq \frac{1}{k_r}\left|\left\{k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right|$$

$$\geq \frac{\delta}{1+\delta}.\frac{1}{h_r}\left|\left\{k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right|.$$

□

Theorem 3.3. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$, then $\overline{x} \overset{\overline{s}_\theta^L}{\sim} \overline{y}$ implies $\overline{x} \overset{\overline{s}^L}{\sim} \overline{y}$.*

Proof. If $\limsup_r q_r < \infty$,then there exists $B > 0$ such that $q_r < C$ for all $r \geq 1$. Let $\overline{x} \overset{\overline{s}_\theta^L}{\sim} \overline{y}$ and $\varepsilon > 0$.There exists $B > 0$ such that for every $j \geq B$

$$A_j = \frac{1}{h_j}\left|\left\{k \in I_j : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right| < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, 3, ...$ Now let n be any integer with $k_{r-1} < n < k_r$,where $r \geq B$. Then

$$\frac{1}{n}\left|\left\{k \leq n : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right| \leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L}\right) \geq \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1}} \left| \left\{ k \in I_1 : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right| + \frac{1}{k_{r-1}} \left| \left\{ k \in I_2 : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{1}{k_{r-1}} \left| \left\{ k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right|$$
$$= \frac{k_1}{k_{r-1}k_1} \left| \left\{ k \in I_1 : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right| + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)} \left| \left\{ k \in I_2 : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{k_B - k_{B-1}}{k_{r-1}(k_B - k_{B-1})} \left| \left\{ k \in I_B : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right|$$
$$+ ... + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})} \left| \left\{ k \in I_r : d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \geq \varepsilon \right\} \right|$$
$$= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + ... + \frac{k_B - k_{B-1}}{k_{r-1}} A_B + ... + \frac{k_r - k_{r-1}}{k_{r-1}} A_r$$
$$\leq \left\{ \sup_{j \geq 1} A_j \right\} \frac{k_B}{k_{r-1}} + \left\{ \sup_{j \succeq B} A_j \right\} \frac{k_r - k_B}{k_{r-1}}$$
$$\leq K . \frac{k_B}{k_{r-1}} + \varepsilon . C.$$

□

Theorem 3.4. *Let* $\theta = (k_r)$ *be a lacunary sequence with* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$*, then* $\overline{x} \overset{\overline{s^L}}{\sim} \overline{y} \Leftrightarrow \overline{x} \overset{\overline{s^L_\theta}}{\sim} \overline{y}$*, where* $0 \notin \overline{y} = (\overline{y}_k)$ *.*

Proof. The result follows from Theorem 3.2 and Theorem 3.3. □

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