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Deductions on Slowly Changing Functions Oriented Bounds for the Zeros of Entire Functions

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Abstract : The aim of this paper is to deduce the bounds for the moduli of zeros of entire functions in connection with slowly changing functions.

Keywords : Zeros of entire functions, Proper ring shaped region, slowly changing function.

1 Introduction, Definitions and Notations

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n; |a_n| \neq 0$$

be a polynomial of degree n. Datt and Govil [2]; Govil and Rahaman [4]; Marden[8]; Mohammad[9]; Chattopadhyay, Das, Jain and Konwer[1]; Joyal, Labelle and Rahaman[5]; Jain [6, 7]; Sun and Hsieh[12]; Zilovic, Roytman, Combettes and Swamy [14]; Das and Datta[3] etc. worked in the theory of the distribution of the zeros of polynomials and obtained some newly developed results.

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In this paper we intend to establish some sharper results concerning the theory of distribution of zeros of entire functions on the basis of slowly changing functions.
The following definitions are well known :

Definition 1.1. *The order ρ and lower order λ of an entire function f are defined as*

$$\rho = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

Let $L \equiv L(r)$ be a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \longrightarrow \infty$ for every positive constant a. Singh and Barker[10] defined it in the following way:

Definition 1.2. *[10] A positive continuous function $L(r)$ is called a slowly changing function if for ε (> 0),*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \quad \text{for } r > r(\varepsilon) \quad \text{and}$$

uniformly for $k(\geq 1)$.
If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL^{'}(r)}{L(r)} = 0.$$

Somasundaram and Thamizharasi [11] introduced the notions of L-order and L-lower order for entire functions defined in the open complex plane $\mathbb{C}$ as follows:

Definition 1.3. *[11] The L-order ρ^L and the L-lower order λ^L of an entire function f are defined as*

$$\rho^L = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]} \quad \text{and} \quad \lambda^L = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[rL(r)]}.$$

The more generalised concept for L-order and L-lower order are L^*-order and L^*-lower order respectively. Their definitions are as follows:

Definition 1.4. *The L^*-order ρ^{L^*} and the L^*-lower order λ^{L^*} of an entire function f are defined as*

$$\rho^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]} \quad \text{and} \quad \lambda^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log[re^{L(r)}]}.$$

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *If $f(z)$ is an entire function of L-order ρ^L, then for every $\varepsilon > 0$ the inequality*

$$N(r) \le [rL(r)]^{\rho^L+\varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \le [rL(r)]$.

Proof. Let us suppose that $f(0) = 1$. This supposition can be made without loss of generality because if $f(z)$ has a zero of order $'m'$ at the origin then we may consider $g(z) = c \cdot \frac{f(z)}{z^m}$ where c is so chosen that $g(0) = 1$. Since the function $g(z)$ and $f(z)$ have the same order therefore it will be unimportant for our investigations that the number of zeros of $g(z)$ and $f(z)$ differ by m.

We further assume that $f(z)$ has no zeros on $|z| = 2[rL(r)]$ and the zeros z_i's of $f(z)$ in $|z| < [rL(r)]$ are in non decreasing order of their moduli so that $|z_i| \le |z_{i+1}|$. Also let ρ^L supposed to be finite.

Now we shall make use of Jenson's formula as state below

$$\log|f(0)| = -\sum_{i=1}^{n} \log \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(R\, e^{i\phi}\right)\right| d\phi. \tag{2.1}$$

Let us replace R by $2r$ and n by $N(2r)$ in (2.1)

$$\therefore \log|f(0)| = -\sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\; e^{i\phi}\right)\right| d\phi.$$

Since $f(0) = 1, \therefore \log|f(0)| = \log 1 = 0$.

$$\therefore \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\; e^{i\phi}\right)\right| d\phi. \tag{2.2}$$

$$\text{L.H.S.} = \sum_{i=1}^{N(2r)} \log \frac{2r}{|z_i|} \ge \sum_{i=1}^{N(r)} \log \frac{2r}{|z_i|} \ge N(r)\log 2 \tag{2.3}$$

because for large values of r,

$$\log \frac{2r}{|z_i|} \ge \log 2.$$

$$\begin{aligned} \text{R.H.S} &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(2r\; e^{i\phi}\right)\right| d\phi \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \log M(2r)\, d\phi = \log M(2r). \end{aligned} \tag{2.4}$$

□

Again by definition of order ρ^L of $f(z)$ we have for every $\varepsilon > 0$ and as $L(2r) \sim L(r)$,

$$\begin{aligned} \log M(2r) &\le (2rL(2r))^{\rho^L+\varepsilon/2} \\ i.e., \log M(2r) &\le (2rL(r))^{\rho^{L+\varepsilon/2}}. \end{aligned} \tag{2.5}$$

Hence from (2.2) by the help of $(2.3), (2.4)$ and (2.5) we have

$$N(r)\log 2 \leq (2rL(r))^{\rho^L+\varepsilon/2}$$

$$N(r) \leq \frac{2^{\rho^L+\varepsilon/2}}{\log 2} \cdot \frac{(rL(r))^{\rho^L+\varepsilon}}{(rL(r))^{\varepsilon/2}} \leq (rL(r))^{\rho^L+\varepsilon}.$$

This proves the lemma.

In the line of Lemma 2.1, we may state the following lemma:

Lemma 2.2. *If $f(z)$ is an entire function of L^* -order ρ^{L^*}, then for every $\varepsilon > 0$ the inequality*

$$N(r) \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}$$

holds for all sufficiently large r where $N(r)$ is the number of zeros of $f(z)$ in $|z| \leq [re^{L(r)}]$.

Proof. With the initial assumptions as laid down in Lemma 1, let us suppose that $f(z)$ has no zeros on $|z| = 2[re^{L(r)}]$ and the zeros z_i's of $f(z)$ in $|z| < [re^{L(r)}]$ are in non decreasing order of their moduli so that $|z_i| \leq |z_{i+1}|$. Also let ρ^{L^*} supposed to be finite.

In view of (2.1),(2.2),(2.3) and (2.4), by definition of ρ^{L^*} and as $L(2r) \sim L(r)$, we get for every $\varepsilon > 0$ that

$$\log M(2r) \leq [2re^{L(2r)}]^{\rho^{L^*}+\varepsilon/2}$$
$$i.e., \log M(2r) \leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}. \tag{2.6}$$

Hence by the help of $(2.3), (2.4)$ and (2.6) we obtain from (2.2) that

$$N(r)\log 2 \leq [2re^{L(r)}]^{\rho^{L^*}+\varepsilon/2}$$

$$N(r) \leq \frac{2^{\rho^{L^*}+\varepsilon/2}}{\log 2} \cdot \frac{[re^{L(r)}]^{\rho^{L^*}+\varepsilon}}{[rL(r)]^{\varepsilon/2}} \leq [re^{L(r)}]^{\rho^{L^*}+\varepsilon}.$$

Thus the lemma is established. □

3 Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let $P(z)$ be an entire function defined as*

$$P(z) = a_0 + a_1 z + \ldots\ldots + a_n z^n + \ldots\ldots$$

with L-order ρ^L. Also for all sufficiently large r in the disc $|z| \leq [rL(r)]$, $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 and t_0' are the positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - \ldots\ldots - |a_0| = 0$$

and

$$h(t) \equiv |a_0|\, t^{N(r)} - |a_1|\, t^{N(r)-1} - .... - \left|a_{N(r)}\right| = 0$$

respectively in $|z| \leq [rL(r)]$ *and* $N(r)$ *denotes the number of zeros of* $P(z)$ *in* $|z| \leq [rL(r)]$ *for sufficiently large* r*.*

Proof. Since $P(z)$ is an entire function of L-order ρ^L, then from Lemma 2.1 we have for sufficiently large r in the disc $|z| \leq [rL(r)]$,

$$N(r) \leq (rL(r))^{\rho^L+\epsilon} \text{ for } \epsilon > 0.$$

Also $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Further $a_n \to 0$ as $n > N(r)$.
Hence we have

$$\begin{aligned} P(z) &= a_0 + a_1 z + ... + a_n z^n + ... \\ &\approx a_0 + a_1 z + ... + a_{N(r)} z^{N(r)}. \end{aligned}$$

Therefore

$$\begin{aligned} |P(z)| &\approx \left|a_0 + a_1 z + ... + a_{N(r)} z^{N(r)}\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_{N(r)-1}\right| |z|^{N(r)-1} .... - |a_0| \end{aligned} \tag{3.1}$$

in the disc $|z| \leq [rL(r)]$ for sufficiently large r. In fact (3.1) can be deduced in the following way

$$\begin{aligned} \left|a_0 + ... + a_{N(r)-1} z^{N(r)-1}\right| &\leq |a_0| + ... + \left|a_{N(r)-1}\right| |z|^{N(r)-1} \\ \text{i.e.,} \quad - |a_0| ..... - \left|a_{N(r)-1}\right| |z|^{N(r)-1} &\leq -\left|a_0 + ... + a_{N(r)-1} z^{N(r)-1}\right|. \end{aligned}$$

Hence

$$\begin{aligned} &\left|a_{N(r)} z^{N(r)} + a_{N(r)-1} z^{N(r)-1} + ... + a_0\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_{N(r)-1} z^{N(r)-1} + ... + a_0\right| \\ &\geq \left|a_{N(r)}\right| |z|^{N(r)} - \left|a_{N(r)-1}\right| |z|^{N(r)-1} - ..... - |a_0|. \end{aligned}$$

Now let us write

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - ..... - |a_0|. \tag{3.2}$$

Since (3.2) has one change of sign, by Descartes' rule of sign, the maximum number of positive root of (3.2) is one. Moreover

$$g(0) = -|a_0| < 0$$

and $g(\infty)$ is a positive quantity.
Clearly $t > t_0$ implies $g(t) > 0$.
If not, let for some $t_1 > t_0$, $g(t_1) < 0$.
Then $g(t) = 0$ has another positive root in (t_1, ∞) which gives a contradiction. Hence $g(t) > 0$ for $t > t_0$.
Therefore $|P(z)| > 0$ for $|z| > t_0$. So $P(z)$ does not vanish in $|z| > t_0$ and therefore all the zeros of $P(z)$ lie in $|z| \leq t_0$ where t_0 is the positive root of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - ..... - |a_0| = 0.$$

Now we give the proof of the other part of the theorem.
Let us consider

$$Q(z) = z^{N(r)} P\left(\frac{1}{z}\right) \tag{3.3}$$

for sufficiently large r in the disc $|z| \leq [rL(r)]$. Now

$$\begin{aligned} Q(z) &= z^{N(r)} P\left(\frac{1}{z}\right) \\ &\approx z^{N(r)}\left[a_0 + \frac{a_1}{z} + ... + a_{N(r)}\frac{1}{z^{N(r)}}\right] \\ &= a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}. \end{aligned} \tag{3.4}$$

Again we have

$$\left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \leq |a_1|\,|z|^{N(r)-1} + |a_2|\,|z|^{N(r)-2} + ... + \left|a_{N(r)}\right|$$

i.e.,

$$-\,|a_1|\,|z|^{N(r)-1} - ..... - \left|a_{N(r)}\right| \leq -\left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right|.$$

So we get that

$$\begin{aligned} \left|a_0 z^{N(r)} + ... + a_{N(r)}\right| &\geq |a_0|\,|z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \\ &\geq |a_0|\,|z|^{N(r)} - |a_1|\,|z|^{N(r)-1} - ..... - \left|a_{N(r)}\right|. \end{aligned} \tag{3.5}$$

Let us consider the equation

$$h(t) \equiv |a_0|\,|t|^{N(r)} - |a_1|\,|t|^{N(r)-1} - ..... - \left|a_{N(r)}\right| = 0. \tag{3.6}$$

Since (3.6) has one change of sign, by Descartes' rule of sign the maximum number of positive root of (3.6) is one. Moreover

$$h(0) = -\left|a_{N(r)}\right| < 0$$

and $h(\infty)$ is a positive quantity. So $h(t)$ has exactly one positive root. □

Let $t_0^{'}$ be the positive root of $h(t) = 0$. Clearly for $t > t_0^{'}$ we get $h(t) > 0$.
If not, let $t_1^{'} > t_0^{'}$. Then $h(t_1^{'}) < 0$. Hence $h(t) = 0$ has another positive root in $(t_1^{'}, \infty)$ which gives a contradiction.
Therefore $h(t) > 0$ for $t > t_0^{'}$ and $|Q(z)| > 0$ for $|z| > t_0^{'}$.
So $Q(z)$ does not vanish in $|z| > t_0^{'}$ and therefore all the zeros of $Q(z)$ lie in $|z| \leq t_0^{'}$. Let $z = z_0$ be any zero of $P(z) = 0$. Clearly $z_0 \neq 0$ as $|a_0| \neq 0$.

Putting $z = \frac{1}{z_0}$ in $Q(z)$ we get that

$$Q\left(\frac{1}{z_0}\right) = \left(\frac{1}{z_0}\right)^{N(r)} .P(z_{0)} = \left(\frac{1}{z_0}\right)^{N(r)} .0 = 0.$$

So $\frac{1}{z_0}$ is a zero of $Q(z)$. Therefore$\left|\frac{1}{z_0}\right| \leq t_0^{'}$ i.e, $|z_0| \geq \frac{1}{t_0^{'}}$. Since z_0 is any arbitrary zero of $P(z)$, all the zeros of $P(z)$ lie in $|z| \geq \frac{1}{t_0^{'}}$.

Hence all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0^{'}} \leq |z| \leq t_0$$

where t_0 and $t_0^{'}$ are the positive roots of

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - ..... - |a_0| = 0$$

and

$$h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - ..... \left|a_{N(r)}\right| = 0$$

respectively for sufficiently large r in the disc $|z| \leq [rL(r)]$.

This proves the theorem.

In the line of Theorem 1, we may state the following theorem in view of Lemma 2 :

Theorem 3.2. *Let $P(z)$ be an entire function defined as*

$$P(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}. Also for all sufficiently large r in the disc $|z| \leq [re^{L(r)}]$, $a_0 \neq 0$ and $a_{N(r)} \neq 0$. Also $a_n \to 0$ as $n > N(r)$. Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{t_0'} \leq |z| \leq t_0$$

where t_0 and t_0' are the positive roots of the equations

$$g(t) \equiv \left|a_{N(r)}\right| t^{N(r)} - \left|a_{N(r)-1}\right| t^{N(r)-1} - .... - |a_0| = 0$$

and

$$h(t) \equiv |a_0| t^{N(r)} - |a_1| t^{N(r)-1} - .... - \left|a_{N(r)}\right| = 0$$

respectively in $|z| \leq [re^{L(r)}]$ and $N(r)$ denotes the number of zeros of $P(z)$ in $|z| \leq [re^{L(r)}]$ for sufficiently large r.

The proof is omitted.

Theorem 3.3. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

with L-order ρ^L. Also for all sufficiently large r in the disc $|z| \leq [rL(r)]$, $a_{N(r)} \neq 0$ and $a_0 \neq 0$. Further $a_n \to 0$ as $n > N(r)$.
Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1+M'} < |z| < 1 + M$$

$$\textit{where } M = \max_{0 \leq k \leq N(r)-1} \left|\frac{a_k}{a_{N(r)}}\right| \textit{ and } M' = \max_{0 \leq k \leq N(r)-1} \left|\frac{a_k}{a_0}\right|.$$

Proof. Since $P(z)$ is an entire function of L-order ρ^L, then by Lemma 1 for sufficiently large values of r in $|z| \leq [rL(r)]$ we have $N(r) \leq [rL(r)]^{\rho^L+\epsilon}$ for $\epsilon > 0$. Also $a_0 \neq 0$, $a_{N(r)} \neq 0$, and $a_n \to 0$ as $n > N(r)$. Hence we may write

$$\begin{aligned} P(z) &= a_0 + a_1 z + ... + a_n z^n + ... \\ &\approx a_0 + a_1 z + ... + a_{N(r)} z^{N(r)}. \end{aligned}$$

Now

$$\begin{aligned} &\left| a_0 + a_1 z + ... + \ a_{N(r)-1} z^{N(r)-1} \right| \\ &\leq |a_0| + ... + \ \left| a_{N(r)-1} \right| |z|^{N(r)-1} \\ &= \left| a_{N(r)} \right| \left\{ \frac{|a_0|}{\left| a_{N(r)} \right|} + ... + \ \frac{\left| a_{N(r)-1} \right|}{\left| a_{N(r)} \right|} |z|^{N(r)-1} \right\} \end{aligned}$$

i.e.,

$$\begin{aligned} &\left| a_0 + a_1 z + ... + \ a_{N(r)-1} z^{N(r)-1} \right| \\ &\leq \left| a_{N(r)} \right| M \left(|z|^{N(r)-1} + |z|^{N(r)-2} + ... + 1 \right) \\ &= \left| a_{N(r)} \right| M \, |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\} \end{aligned}$$

where $|z| \neq 0$. Therefore when $|z| \neq 0$,

$$\begin{aligned} &- \left| a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1} \right| \\ &\geq - \left| a_{N(r)} \right| M \, |z|^{N(r)} \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\}. \end{aligned}$$

So for $|z| \neq 0$

$$\begin{aligned} &\left| a_{N(r)} \right| |z|^{N(r)} - \left| a_0 + a_1 z + ... + a_{N(r)-1} z^{N(r)-1} \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_{N(r)} \right| |z|^{N(r)} M \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\}. \end{aligned} \tag{3.7}$$

Now

$$\begin{aligned} |P(z)| &\approx \left| a_0 + a_1 z + ... + \ a_{N(r)} z^{N(r)} \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_0 + a_1 z + ... + \ a_{N(r)-1} z^{N(r)-1} \right|. \end{aligned}$$

Using (3.7) we have

$$\begin{aligned} |P(z)| &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_{N(r)} \right| |z|^{N(r)} M \left\{ \frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right\} \\ &= \left| a_{N(r)} \right| |z|^{N(r)} \left\{ 1 - M \left(\frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} \right) \right\} \ \text{ for } \ |z| \neq 0. \end{aligned}$$

i.e., when $|z| \neq 0$

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left\{1 - M\left(\frac{1}{|z|} + \frac{1}{|z|^2} + ... + \frac{1}{|z|^{N(r)}} + ...\right)\right\}.$$

Therefore

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left\{1 - M\sum_{j=1}^{\infty} \frac{1}{|z|^j}\right\} \text{ for } |z| \neq 0.$$

Now the geometric series $\sum\limits_{j=1}^{\infty} \frac{1}{|z|^j}$ is convergent when $\frac{1}{|z|} < 1$ i.e., when $|z| > 1$ and is equal to

$$\frac{1}{|z|}\frac{1}{1 - \frac{1}{|z|}} = \frac{1}{|z| - 1}.$$

On $|z| > 1$ we can write

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left(1 - \frac{M}{|z| - 1}\right).$$

Now on $|z| > 1$,

$$|P(z)| > 0 \text{ if } \left|a_{N(r)}\right| |z|^{N(r)} \left(1 - \frac{M}{|z| - 1}\right) \geq 0$$

$$\begin{aligned} &\text{i.e, if} \quad 1 - \frac{M}{|z| - 1} \geq 0 \\ &\text{i.e, if} \quad |z| - 1 \geq M \\ &\text{i.e, if} \quad |z| \geq M + 1. \end{aligned}$$

Therefore

$$|z| \geq M + 1 > 1 \text{ as } M > 0.$$

Hence

$$|P(z)| > 0 \text{ if } |z| \geq M + 1.$$

Therefore all the zeros of $P(z)$ lie in $|z| < M + 1$.
Secondly, we give the proof of the lower bound. Let us consider

$$Q(z) = z^{N(r)} P\left(\frac{1}{z}\right).$$

Therefore

$$\begin{aligned} Q(z) &= |z|^{N(r)} \left\{a_0 + \frac{a_1}{|z|} + ... + \frac{a_{N(r)}}{|z|^{N(r)}}\right\} \\ &= a_0 |z|^{N(r)} + a_1 |z|^{N(r)-1} + ... + a_{N(r)}. \end{aligned}$$

Now

$$\begin{aligned}
\left|a_1\left|z\right|^{N(r)-1}+......+a_{N(r)}\right| &\le \left|a_1\right|\left|z\right|^{N(r)-1}+......+\left|a_{N(r)}\right|\\
&= \left|a_0\right|\left(\frac{\left|a_1\right|}{\left|a_0\right|}\left|z\right|^{N(r)-1}+......+\frac{\left|a_{N(r)}\right|}{\left|a_0\right|}\right)\\
&\le \left|a_0\right| M'\left(\left|z\right|^{N(r)-1}+......+1\right)\\
&= \left|a_0\right| M'\left|z\right|^{N(r)}\left(\frac{1}{\left|z\right|}+......+\frac{1}{\left|z\right|^{N(r)}}\right).
\end{aligned}$$

Therefore

$$-\left|a_1\left|z\right|^{N(r)-1}+......+a_{N(r)}\right| \ge -\left|a_0\right| M'\left|z\right|^{N(r)}\left(\frac{1}{\left|z\right|}+......+\frac{1}{\left|z\right|^{N(r)}}\right).$$

So

$$\begin{aligned}
\left|Q(z)\right| &\ge \left|a_0\right|\left|z\right|^{N(r)}-\left|a_1 z^{N(r)-1}+....+a_{N(r)}\right|\\
&\ge \left|a_0\right|\left|z\right|^{N(r)}-\left|a_0\right| M'\left|z\right|^{N(r)}\left(\frac{1}{\left|z\right|}+......+\frac{1}{\left|z\right|^{N(r)}}\right)\\
&= \left|a_0\right|\left|z\right|^{N(r)}\left\{1-M'\left(\frac{1}{\left|z\right|}+......+\frac{1}{\left|z\right|^{N(r)}}\right)\right\}\\
&> \left|a_0\right|\left|z\right|^{N(r)}\left\{1-M'\left(\frac{1}{\left|z\right|}+......+\frac{1}{\left|z\right|^{N(r)}}+.....\right)\right\}.
\end{aligned}$$

Hence using above we get that

$$\left|Q(z)\right| > \left|a_0\right|\left|z\right|^{N(r)}\left\{1-M'\sum_{j=1}^{\infty}\frac{1}{\left|z\right|^{j}}\right\}.$$

Now the geometric series $\sum\limits_{j=1}^{\infty}\frac{1}{|z|^j}$ is convergent when $\frac{1}{|z|}<1$ i.e., $|z|>1$ and is equal to

$$\frac{1}{|z|}\frac{1}{1-\frac{1}{|z|}}=\frac{1}{|z|-1}.$$

On $|z|>1$ we may write

$$\left|Q(z)\right| > \left|a_0\right|\left|z\right|^{N(r)}\left(1-\frac{M'}{|z|-1}\right).$$

Now for $|z|>1$,

$$\begin{aligned}
\left|Q(z)\right|>0 \text{ if } &\quad \left|a_0\right|\left|z\right|^{N(r)}\left(1-\frac{M'}{|z|-1}\right)\ge 0\\
\text{i.e., if} &\quad 1-\frac{M'}{|z|-1}\ge 0\\
\text{i.e., if} &\quad |z|\ge 1+M'.
\end{aligned}$$

Therefore $|z|\ge 1+M'>1$ as $M'>0$.

Hence $|Q(z)|>0$ for $|z|\ge 1+M'$. □

So all the zeros of $Q(z)$ lie in $|z| < 1+ M'$.
Let $z = z_0$ be any zero of $P(z)$. Therefore $P(z_0) = 0$. Clearly $z_0 \neq 0$ as $|a_0| \neq 0$. Putting $z = \frac{1}{z_0}$ in $|Q(z)|$ we have

$$\left|Q\left(\frac{1}{z_0}\right)\right| = \left(\frac{1}{z_0}\right)^n .P(z_0) = \left(\frac{1}{z_0}\right)^n .0 = 0$$

Therefore $z = \frac{1}{z_0}$ is a root of $Q(z)$. So

$$\left|\frac{1}{z_0}\right| < 1 + M',$$

which implies that

$$|z_0| > \left|\frac{1}{1+M'}\right|.$$

As z_0 is an arbitrary root of $P(z) = 0$, all the zeros of $P(z)$ lie in $|z| > \left|\frac{1}{1+M'}\right|$.
Hence all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1+M'} < |z| < 1 + M.$$

This proves the theorem.

In the line of Theorem 3.3, we may state the following theorem in view of Lemma 2.2 :

Theorem 3.4. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}. Also for all sufficiently large r in the disc $|z| \leq [re^{L(r)}]$, $a_{N(r)} \neq 0$ and $a_0 \neq 0$. Further $a_n \to 0$ as $n > N(r)$.
Then all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{1+M'} < |z| < 1 + M$$

$$\text{where } M = \max_{0 \leq k \leq N(r)-1} \left|\frac{a_k}{a_{N(r)}}\right| \text{ and } M' = \max_{0 \leq k \leq N(r)-1} \left|\frac{a_k}{a_0}\right|.$$

The proof is omitted.

Theorem 3.5. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L-order ρ^L. Also for sufficiently large r in the disc $|z| \leq [rL(r)], a_{N(r)} \neq 0, a_0 \neq 0$ and $a_n \to 0$ as $n > N(r)$. For any p, q with $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, all the zeros of $P(z)$ lie in the annular region

$$\frac{1}{\left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}} < |z| < \left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{p}\right]^{\frac{q}{p}}.$$

Proof. Given that $a_0 \neq 0$, $a_{N(r)} \neq 0$ and $a_n \to 0$ as $n > N(r)$. Therefore for sufficiently large r in the disc $|z| \leq [rL(r)]$ the existence of $N(r)$ implies that

$$\begin{aligned} P(z) &= a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ... \\ &\approx a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)}. \end{aligned}$$

Now

$$\begin{aligned} &\left| a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1} \right| \\ &\leq |a_0| + |a_1| \, |z| + ... \left| a_{N(r)-1} \right| |z|^{N(r)-1} \\ &= \left| a_{N(r)} \right| \left\{ \frac{|a_0|}{\left| a_{N(r)} \right|} + ... + \frac{\left| a_{N(r)-1} \right|}{\left| a_{N(r)} \right|} |z|^{N(r)-1} \right\} \\ &= \left| a_{N(r)} \right| \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right| |z|^j . \end{aligned} \tag{3.8}$$

Therefore using (3.8) we get that

$$\begin{aligned} |P(z)| &\approx \left| a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)} z^{N(r)} \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_0 + a_1 z + a_2 z^2 + ... + a_{N(r)-1} z^{N(r)-1} \right| \\ &\geq \left| a_{N(r)} \right| |z|^{N(r)} - \left| a_{N(r)} \right| \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right| |z|^j \\ \text{i.e.,} \quad |P(z)| &\geq \left| a_{N(r)} \right| \left\{ |z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right| |z|^j \right\}. \end{aligned}$$

By Holder's inequality we have

$$\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right| |z|^j \leq \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(|z|^j \right)^q \right)^{\frac{1}{q}} . \tag{3.9}$$

In view of (3.9) we obtain that

$$\begin{aligned} |P(z)| &\geq \left| a_{N(r)} \right| \left\{ |z|^{N(r)} - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(|z|^j \right)^q \right)^{\frac{1}{q}} \right\} \\ &= \left| a_{N(r)} \right| \left\{ |z|^{N(r)} - |z|^{N(r)} \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \frac{|z|^{jq}}{|z|^{N(r)q}} \right)^{\frac{1}{q}} \right\} \\ &= \left| a_{N(r)} \right| |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(\frac{1}{|z|^q} \right)^{N(r)-j} \right)^{\frac{1}{q}} \right\} \\ &= \left| a_{N(r)} \right| |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{N(r)} \left(\frac{1}{|z|^q} \right)^{j} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Now the geometric series $\sum_{j=1}^{N(r)} \left(\frac{1}{|z|^q}\right)^j$ is convergent for

$$\frac{1}{|z|^q} < 1$$

$$\text{i.e., for } |z|^q > 1$$
$$\text{i.e., for } |z| > 1$$

and is convergent to

$$\frac{1}{|z|^q} . \frac{1}{1 - \frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.$$

So

$$\left(\sum_{j=1}^{N(r)} \left(\frac{1}{|z|^q}\right)^j \right)^{\frac{1}{q}} \text{ converges to } \left(\frac{1}{|z|^q - 1}\right)^{\frac{1}{q}} \text{ for } |z| > 1.$$

Therefore on $|z| > 1$

$$|P(z)| > \left|a_{N(r)}\right| |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\}.$$

Now if $|P(z)| > 0$ then we have

$$\left|a_{N(r)}\right| |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\} \geq 0$$

$$\text{i.e,} \qquad 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \geq 0$$

$$\text{i.e,} \qquad (|z|^q - 1)^{\frac{1}{q}} \geq \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{1}{p}}$$

$$\text{i.e.,} \qquad |z|^q - 1 \geq \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}}$$

$$\text{i.e.,} \qquad |z| \geq \left[1 + \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}.$$

Clearly

$$\left[1 + \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_j}{a_{N(r)}} \right|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} > 1.$$

Therefore $|P(z)| > 0$ for

$$|z| \geq \left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Therefore all the zeros of $P(z)$ lie in

$$|z| < \left[1 + \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}. \quad (3.10)$$

For the lower bound let us take $Q(z) = z^{N(r)} P(\frac{1}{z})$. Therefore

$$\begin{aligned} Q(z) &= z^{N(r)} P\left(\frac{1}{z}\right) \\ &\approx z^{N(r)} \left\{a_0 + \frac{a_1}{z} + ... + \frac{a_{N(r)}}{z^{N(r)}}\right\} \\ &= a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}. \end{aligned}$$

Therefore

$$|Q(z)| \approx \left|a_0 z^{N(r)} + a_1 z^{N(r)-1} + ... + a_{N(r)}\right|.$$

Now

$$\begin{aligned} \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| &\leq |a_1| |z|^{N(r)-1} + ... + \left|a_{N(r)}\right| \\ &= |a_0| \left\{\frac{|a_1|}{|a_0|} |z|^{N(r)-1} + ... + \frac{\left|a_{N(r)}\right|}{|a_0|}\right\} \\ &= |a_0| \sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j. \end{aligned} \quad (3.11)$$

Therefore using (3.11) we get that

$$\begin{aligned} |Q(z)| &\geq |a_0| |z|^{N(r)} - \left|a_1 z^{N(r)-1} + ... + a_{N(r)}\right| \\ &\geq |a_0| |z|^{N(r)} - |a_0| \sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j \\ &= |a_0| \left\{|z|^{N(r)} - \sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j\right\}. \end{aligned}$$

Now by Holder's inequality we have

$$\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right| |z|^j \leq \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(|z|^j\right)^q\right)^{\frac{1}{q}}. \quad (3.12)$$

Using (3.12) we obtain from above that

$$|Q(z)| \geq |a_0| \left\{|z|^{N(r)} - \left(\sum_{j=0}^{N(r)-1} \left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(|z|^j\right)^q\right)^{\frac{1}{q}}\right\}$$

$$= |a_0| \left\{ |z|^{N(r)} - |z|^{N(r)} \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \frac{\left(|z|^j\right)^q}{|z|^{N(r)q}} \right)^{\frac{1}{q}} \right\}$$

$$= |a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \frac{\left(|z|^j\right)^q}{|z|^{N(r)q}} \right)^{\frac{1}{q}} \right\}$$

$$= |a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \frac{1}{|z|^{q(N(r)-j)}} \right)^{\frac{1}{q}} \right\}$$

$$= |a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)-1} \left(\frac{1}{|z|^q} \right)^{(N(r)-j)} \right)^{\frac{1}{q}} \right\}$$

$$> |a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=0}^{N(r)} \left(\frac{1}{|z|^q} \right)^{j} \right)^{\frac{1}{q}} \right\}.$$

Therefore

$$|Q(z)| \geq |a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^{N(r)} \left(\frac{1}{|z|^q} \right)^{j} \right)^{\frac{1}{q}} \right\}.$$

Now the geometric series $\sum_{j=1}^{\infty} \left(\frac{1}{|z|^q} \right)^j$ is convergent for

$$\frac{1}{|z|^q} < 1$$

$$\text{i.e., for} \qquad |z|^q > 1.$$

Therefore for $|z| > 1$ and the series is convergent to

$$\frac{1}{|z|^q} \frac{1}{1 - \frac{1}{|z|^q}} = \frac{1}{|z|^q - 1}.$$

So

$$\left(\sum_{j=1}^{\infty} \left(\frac{1}{|z|^q} \right)^j \right)^{\frac{1}{q}} \text{ is convergent to } \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \text{ for } |z| > 1.$$

Therefore on $|z| > 1$,

$$|Q(z)| > |a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\}.$$

Now if $|Q(z)| > 0$ then

$$|a_0| \, |z|^{N(r)} \left\{ 1 - \left(\sum_{j=0}^{N(r)-1} \left| \frac{a_{N(r)-j}}{a_0} \right|^p \right)^{\frac{1}{p}} \left(\frac{1}{|z|^q - 1} \right)^{\frac{1}{q}} \right\} \geq 0$$

$$\text{i.e,}\quad 1-\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}}\geq 0$$

$$\text{i.e.,}\quad 1\geq\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}\left(\frac{1}{|z|^q-1}\right)^{\frac{1}{q}}$$

$$\text{i.e.,}\quad (|z|^q-1)^{\frac{1}{q}}\geq\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{1}{p}}$$

$$\text{i.e.,}\quad |z|^q-1\geq\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}$$

$$\text{i.e.,}\quad |z|\geq\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Clearly

$$\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}>1.$$

Therefore $|Q(z)|>0$ if

$$|z|\geq\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Therefore all the zeros $Q(z)$ lie in

$$|z|<\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

Let $z=z_0$ be any other zero of $P(z)$. Therefore $P(z_0)=0$. Clearly $z_0\neq 0$ as $a_0\neq 0$. Putting $z=\frac{1}{z_0}$ in $Q(z)$ we have

$$Q(z_0)=\left(\frac{1}{z_0}\right)^{N(r)}.P(z_0)=\left(\frac{1}{z_0}\right)^{N(r)}.0=0$$

Therefore $z=\frac{1}{z_0}$ is a zero of $Q(z)$. So

$$\left|\frac{1}{z_0}\right|<\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}$$

$$\text{i.e.,}\ |z_0|>\frac{1}{\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}}.$$

As z_0 is an arbitrary zero of $P(z)$ so all the zeros of $P(z)$ lie in

$$|z| > \frac{1}{\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}}. \tag{3.13}$$

Hence combining (3.10) and (3.13) we may say that all the zeros of $P(z)$ lie in the ring shaped region

$$\frac{1}{\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}} < |z| < \left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$

This proves the theorem. □

In the line of Theorem 3.5, we may state the following theorem in view of Lemma 2.2 :

Theorem 3.6. *Let $P(z)$ be an entire function defined by*

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

with L^*-order ρ^{L^*}. Also for sufficiently large r in the disc $|z| \le [re^{L(r)}], a_{N(r)} \neq 0, a_0 \neq 0$ and $a_n \to 0$ as $n > N(r)$. For any p, q with $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, all the zeros of $P(z)$ lie in the annular region

$$\frac{1}{\left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_{N(r)-j}}{a_0}\right|^p\right)^{\frac{q}{p}}\right]^{\frac{1}{q}}} < |z| < \left[1+\left(\sum_{j=0}^{N(r)-1}\left|\frac{a_j}{a_{N(r)}}\right|^p\right)^{p}\right]^{\frac{q}{p}}.$$

Corollary 3.7. *In particular if we take $p = 2, q = 2$ in Theorem 3.5 then we get that all the zeros of the polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ lie in the ring shaped region*

$$\frac{1}{\left[1+\left(\sum_{j=0}^{n}\left|\frac{a_n}{a_0}\right|^2\right)\right]^{\frac{1}{2}}} < |z| < \left[1+\left(\sum_{j=0}^{n}\left|\frac{a_j}{a_n}\right|^2\right)\right]^{\frac{1}{2}}.$$

Corollary 3.8. *In particular if we take $p = 2, q = 2$ in Theorem 3.6 then we get that all the zeros of the polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ lie in the ring shaped region*

$$\frac{1}{\left[1+\left(\sum_{j=0}^{n}\left|\frac{a_n}{a_0}\right|^2\right)\right]^{\frac{1}{2}}} < |z| < \left[1+\left(\sum_{j=0}^{n}\left|\frac{a_j}{a_n}\right|^2\right)\right]^{\frac{1}{2}}.$$

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