International Journal of Mathematics And Its Applications
Vol.2 No.1 (2014), pp.49-55.
ISSN: 2347-1557(online)

Study of the Orthogonal Property of Recurrence Relations and the Differential Equation in Generalized Rodrigue's Formula

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Abstract : Let $L_1 = L_1(I)$ be the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ be the gamma function.

Keywords : Lebesgue integrable functions, Rodrigue's formula, recurrence relations.

1 Introduction

The object of this paper is to derive some properties of generalized Laguerre functions and recurrences relations and also proved that the set of generalized functions. Laguerre polynomials is continuous. The Rodrigue's formula of generalized Laguerre polynomials is also derived. Lastly we have proved that the set of functions are orthogonal in $L_2(0, \infty)$. The fractional derivative D^α of order $\alpha \in (0, 1)$ of the absolutely continuous function f(x) is given by (see [1-7])

$$D_a^\alpha\, f(x) = I_a^{1-\alpha}\, Df(x), \quad D = \frac{d}{dx}$$

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and the fractional derivation D^α of order $\alpha \in (n-1,n)$ of the function $f(x)$ is given by

$$D_a^\alpha\, f(x) = I_a^{m-\alpha}\, D^n f(x), \quad D = \frac{d}{dx}$$

Now we have the following two definitions.

Definition 1.1. *Let $\alpha, \gamma \in (n-1,n)$, $n = 1,2,\ldots$ and $\beta, \alpha \in \Re$. We define function $Y_\alpha^\beta\,(\gamma, a; x)$ by*

$$Y_\alpha^\beta\,(\gamma, a; x) = D^\alpha x^{\gamma+\beta} e^{-ax}, \gamma + \beta > 0 \tag{1.1}$$

and the function $Y_{-\alpha}^\beta\,(-\gamma, a; x)$ by

$$Y_{-\alpha}^\beta\,(\gamma, a; x) = I^\alpha x^{-\gamma+\beta} e^{-ax}, \;\; -\gamma + \beta > -1 \tag{1.2}$$

Definition 1.2. *Let $\alpha, \gamma \in (n-1,n)$, $n = 1,2,\ldots$ and $\beta, \alpha \in \Re$. We define the generalized Rodrigue's formula by the two functions*

$$L_\alpha^\beta\,(\gamma, a; x) = \frac{e^{ax} x^{-\beta}}{\Gamma(1+\alpha)} Y_\alpha^\beta(\gamma, a; x), \quad \gamma + \beta > 0 \tag{1.3}$$

$$L_{-\alpha}^\beta\,(\gamma, a; x) = \frac{e^{ax} x^{-\beta}}{\Gamma(1+\alpha)} Y_{-\alpha}^\beta(-\gamma, a; x), \quad -\gamma + \beta > 0 \tag{1.4}$$

Here we study some of properties of the functions $L_\alpha^\beta\,(\gamma, a; x)$, and $Y_\alpha^\beta\,(\gamma, a; x)$, some recurrence relations and prove that the set of functions $\{L_\alpha^\beta\,(\gamma, a; x),\;\; \alpha \in \Re\}$ is continuous as a function of $\alpha \in \Re$. The continuation of the function $\{L_\alpha^\beta\,(\gamma, a; x)\}$, as $\alpha, \gamma \to n$ and $a = 1$ to the Rodrigue's formula of the Laguerre polynomials $L_n^\beta\,(x)$ are proved. Also we prove that the set of functions $\{L_{m/n}^\beta\,(\gamma, a; x),\; m, n = 1, 2, ...\}$ are orthogonal in $L_2(0, \infty)$.

Theorem 1.3. *Let $a; \gamma\; \in (n-1)$, $n = 1,2,\ldots$. If $\beta, \alpha \in \Re$, then*

$$(\alpha+1) L_{1+\alpha}^\beta = D L_\alpha^\beta(\gamma, a; x) + \left(\frac{\beta}{x} - a\right) L_\alpha^\beta(\gamma, a; x) \tag{1.5}$$

$$(\alpha+1) L_{1+\alpha}^\beta = D L_\alpha^\beta(\gamma, a; x) = \frac{\gamma+\beta}{x} L_\alpha^\beta(\gamma, a; x) - a L_\alpha^\beta(\gamma, a; x) \tag{1.6}$$

$$a L_{\alpha-1}^\beta(\gamma, a; x) = (\lambda+\beta) L_{1-\alpha}^\beta\,(\gamma-1, a; x) - a L_\alpha^\beta(\gamma, a; x) \tag{1.7}$$

$$(1-\alpha) L_{1-\alpha}^\beta(\gamma, a; x) = \frac{\gamma+\beta}{x} L_{-\alpha}^{\beta-1}(\gamma, a; x) - a L_{-\alpha}^\beta(\gamma, a; x) \tag{1.8}$$

$$(1-\alpha) L_{1-\alpha}^\beta\,(\gamma, a; x) = (\gamma+\beta) L_{-\alpha}^\beta(\gamma-1, a; x) - a L_{-\alpha}^\beta(\gamma, a; x) \tag{1.9}$$

$$(1-\alpha) L_{1-\alpha}^\beta\,(\gamma, a; x) = D L_{-\alpha}^\beta(\gamma, a; x) + \left(\frac{\beta}{x} - a\right) L_{-\alpha}^\beta(\gamma, a; x) \tag{1.10}$$

$$L_\alpha^\beta\,(1+\gamma, a; x) = x L_\alpha^{\beta+1}(\gamma, a; x) = \frac{1}{x} L_\alpha^{\beta-1}(2+\gamma, a; x) \tag{1.11}$$

$$\alpha L_\alpha^\beta\,(1+\gamma, a; x) = (\gamma+\beta+1) L_{\alpha-1}^\beta(\gamma, a; x) - a L_{\alpha-1}^\beta\,(1+\gamma, a; x) \tag{1.12}$$

$$L_\alpha^\beta\,(1-\gamma, a; x) = x L_\alpha^{\beta+1}\,(-\gamma, a; x) = \frac{1}{x} L_\alpha^{\beta-1}\,(2-\gamma, a; x) \tag{1.13}$$

$$\alpha L_\alpha^\beta\,(1+\gamma, a; x) = (1-\gamma+\beta) L_{\alpha-1}^\beta(-\gamma, a; x) - a L_{\alpha-1}^\beta\,(1-\gamma, a; x) \tag{1.14}$$

Proof. Differentiation $L_{\alpha}^{\beta}\,(\gamma,a;x)$ gives

$$DL_{\alpha}^{\beta}\,(\gamma,a;x)=(\alpha+1)L_{\alpha+1}^{\beta}\,(\gamma,a;x)+aL_{\alpha}^{\beta}\,(\gamma,a;x)-\frac{\beta}{x}L_{\alpha}^{\beta}(\gamma,a;x) \tag{1.15}$$

Then

$$(\alpha+1)L_{1+\alpha}^{\beta}\,(\gamma,a;x)+DL_{\alpha}^{\beta}\,(\gamma,a;x)+\left(\frac{\beta}{x}-a\right)L_{\alpha}^{\beta}(\gamma,a;x) \tag{1.16}$$

From the properties of the fractional calculus and the definition $L_{\alpha}^{\beta}\,(\gamma,a;x)$ we get

$$\begin{aligned}L_{\alpha+1}^{\beta}\,(\gamma,a;x)&=\frac{e^{ax}x^{-\beta}}{\Gamma(2+\alpha)}D^{\alpha+1}x^{\gamma+\beta}e^{-ax}\\&=\frac{e^{ax}x^{-\beta}}{\Gamma(2+\alpha)}D^{\alpha}\left[(\gamma+\beta)\,x^{\gamma+\beta-1}e^{-ax}-ax^{\gamma+\beta}e^{ax}\right]\\&=\frac{1}{\alpha+1}\left[\frac{\gamma+\beta}{x}L_{\alpha}^{\beta-1}(\gamma,a;x)-aL_{\alpha}^{\beta}(\gamma,a;x)\right]\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}L_{\alpha}^{\beta}\,(\gamma,a;x)&=\frac{e^{ax}x^{-\beta}}{\Gamma(\alpha+1)}D^{\alpha}x^{\gamma+\beta}e^{-ax}\\&=\frac{e^{ax}x^{-\beta}}{\Gamma(\alpha+1)}D^{\alpha-1}\left[(\gamma+\beta)\,x^{\gamma+\beta-1}e^{-ax}-ax^{\gamma+\beta}e^{-ax}\right]\\&=\frac{1}{\alpha}\left[(\gamma+\beta)L_{\alpha-1}^{\beta}(\gamma-1,a;x)-aL_{\alpha-1}^{\beta}(\gamma,a;x)\right]\end{aligned} \tag{1.18}$$

Also

$$\begin{aligned}L_{1-\alpha}^{\beta}\,(\gamma,a;x)&=\frac{e^{ax}x^{-\beta}}{\Gamma(2-\alpha)}D^{1-\alpha}x^{\gamma+\beta}e^{-ax}\\&=\frac{e^{ax}x^{-\beta}}{\Gamma(2-\alpha)}I^{\alpha}\left[(\gamma+\beta)\,x^{\gamma+\beta-1}e^{-ax}-ax^{\gamma+\beta}e^{-ax}\right]\\&=\frac{1}{1-\alpha}\left[\frac{\gamma+\beta}{x}L_{-\alpha}^{\beta-1}(\gamma,a;x)-aL_{-\alpha}^{\beta}(\gamma,a;x)\right]\\&=\frac{1}{1-\alpha}\left[(\gamma+\beta)L_{-\alpha}^{\beta}(\gamma-1,a;x)-aL_{-\alpha}^{\beta}(\gamma,a;x)\right]\end{aligned} \tag{1.19}$$

where

$$L_{-\alpha}^{\beta}\,(\gamma-1,a;x)=\frac{1}{x}L_{-\alpha}^{\beta-1}(\gamma,a;x)$$

Since

$$DL_{-\alpha}^{\beta}\,(\gamma,a;x)=(1-\alpha)L_{1-\alpha}^{\beta}(\gamma,a;x)+aL_{-\alpha}^{\beta}\,(\gamma,a;x)-\frac{\beta}{x}L_{-\alpha}^{\beta}(\gamma,a;x) \tag{1.20}$$

Then

$$(1-\alpha)L_{1-\alpha}^{\beta}(\gamma,a;x)=DL_{-\alpha}^{\beta}\,(\gamma,a;x)=+\left(\frac{\beta}{x}-a\right)L_{-\alpha}^{\beta}(\gamma,a;x) \tag{1.21}$$

From the definition of $L_{\alpha}^{\beta}(\gamma,a;x)$, we get

$$\begin{aligned}L_{\alpha}^{\beta}(1+\gamma,a;x)&=\frac{e^{ax}x^{-\beta}}{\Gamma(1+\alpha)}D^{\alpha}x^{1+\gamma+\beta}e^{-ax}\\&=\frac{xe^{ax}x^{-(\beta+1)}}{\Gamma(1+\alpha)}D^{\alpha}x^{1+\gamma+\beta}e^{-ax}\end{aligned} \tag{1.22}$$

$$= \frac{e^{ax}x^{-(\beta-1)}}{\Gamma(1+\alpha)} D^\alpha x^{2+\gamma+\beta-1} e^{-ax}$$
$$= x\Gamma_\alpha^\beta(\gamma, a; x)$$
$$= \frac{1}{x} L_\alpha^{\beta-1}(2+\gamma, a; x)$$

Also, we have

$$\begin{aligned} L_\alpha^\beta\,(1+\gamma, a; x) &= \frac{e^{ax}x^{-\beta}}{\Gamma(1+\alpha)} D^\alpha x^{1+\gamma+\beta} e^{-ax} \\ &= \frac{xe^{ax}x^{-\beta}}{\Gamma(1+\alpha)} D^{\alpha-1}[(\gamma+\beta+1)x^{\gamma+\beta}e^{-ax} - ax^{1+\gamma+\beta}e^{-ax}] \\ &= \frac{1}{\alpha}[(\gamma+\beta+1)L_{\alpha-1}^\beta(\gamma, a; x) - aL_{\alpha-1}^\beta(1+\gamma, a; x)] \end{aligned} \tag{1.23}$$

By the same way, we can easily prove the last two relations. From the properties of the fractional calculus and the definition of $Y_\alpha^\beta(\gamma, a; x)$, we can easily prove the following lemma. □

Lemma 1.4. *Let $\alpha, \alpha_1, \gamma \in (n-1, n)$, $n = 1, 2, \ldots$. If $\beta, \alpha \in \Re$, then*

$$D^{\alpha_1} L_\alpha^\beta(\gamma, a; x) = L_{\alpha_1+\alpha}^\beta(\gamma, a; x) = D^\alpha L_{\alpha_1}^\beta(\gamma, a; x) \tag{1.24}$$
$$I^{\alpha_1} Y_{-\alpha}^\beta(-\gamma, a; x) = Y_{-(\alpha_1+\alpha)}^\beta(-\gamma, a; x) = I^\alpha Y_{-\alpha_1}^\beta(-\gamma, a; x) \tag{1.25}$$
$$D^{\alpha_1} L_{-\alpha}^\beta(-\gamma, a; x) = I^\alpha Y_{\alpha_1}^\beta(-\gamma, a; x) \tag{1.26}$$
$$\left(a\gamma + \frac{(\alpha-)\beta}{x}\; L_\alpha^\beta\,(\gamma, a; x)\right) = 0$$

Theorem 1.5. *Let $\alpha, \gamma \in (n-1, n)$, $n = 1, 2, \ldots$ and $\beta, \alpha \in \Re$; $a > 0$. Then*

$$\begin{aligned} Y_\alpha^\beta\,(\gamma, a; x) &= (\gamma+\beta)\, Y_{\alpha-1}^{\beta-1}\,(\gamma, a; x) - \; aY_{\alpha-1}^\beta\,(\gamma, a; x) \\ &= (\gamma+\beta)\, Y_{\alpha-1}^\beta\,(\gamma-1, a; x) - \; aY_{\alpha-1}^\beta\,(\gamma, a; x) \end{aligned} \tag{1.27}$$
$$\begin{aligned} DY_\alpha^\beta\,(\gamma, a; x) &= Y_{\alpha+1}^\beta\,(\gamma, a; x) \\ &= (\gamma+\beta)\, Y_\alpha^{\beta-1}\,(\gamma, a; x) - aY_\alpha^{\beta-1}\,(\gamma, a; x) \\ &= (\gamma+\beta)\, Y_{\alpha-1}^\beta\,(\gamma-1, a; x) - aY_\alpha^\beta\,(\gamma, a; x) \end{aligned} \tag{1.28}$$
$$\begin{aligned} D^n Y_\alpha^\beta\,(\gamma, a; x) &= Y_{\alpha+n}^\beta\,(\gamma, a; x) \\ &= \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} (-a)^{n-k} \frac{\Gamma(1+\gamma+\beta)}{\Gamma(1+\gamma+\beta-k)} Y_\alpha^{\beta-k}\,(\gamma, a; x) \end{aligned} \tag{1.29}$$
$$xDY_{\alpha-1}^\beta\,(\gamma-1, a; x) = -Y_\alpha^\beta\,(\gamma, a; x) = -\alpha Y_{\alpha-1}^\beta\,(\gamma-1, a; x) \tag{1.30}$$

Proof. From the definition $Y_\alpha^\beta\,(\gamma, a; x)$, we have

$$\begin{aligned} Y_\alpha^\beta\,(\gamma, a; x) &= D^{\alpha-1}\{(\gamma+\beta)x^{\gamma+\beta-1}e^{-ax} - ax^{\gamma+\beta}e^{-ax}\} \\ &= (\gamma+\beta)D^{\alpha-1}x^{(\gamma-1)+\beta}e^{-ax} - aD^{\alpha-1}x^{\gamma+\beta}e^{-ax} \\ &= (\gamma+\beta)Y_{\alpha-1}^\beta(\gamma, a; x) - aY_{\alpha-1}^\beta\,(\gamma, a; x) \end{aligned}$$

Also

$$\begin{aligned} Y_{\alpha}^{\beta}\ (\gamma,a;x) &= D^{\alpha}\{(\gamma+\beta)x^{\gamma+\beta-1}e^{-ax} - ax^{\gamma+\beta}e^{-ax}\} \\ &= (\gamma+\beta)\ Y_{\alpha}^{\beta-1}\ (\gamma,a;x) - aY_{\alpha-1}^{\beta}\ (\gamma,a;x) \\ &= (\gamma+\beta)DY_{\alpha-1}^{\beta}(\gamma-1,a;x) - aY_{\alpha-1}^{\beta}\ (\gamma,a;x) \end{aligned}$$

where

$$DY_{\alpha-1}^{\beta-1}(\gamma,a;x) = Y_{\alpha}^{\beta-1}(\gamma,a;x)$$

and also

$$\begin{aligned} D^nY_{\alpha}^{\beta}\ (\gamma,a;x) &= D^{\alpha}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\ (D^kx^{\gamma+\beta})(D^{n-k}e^{-ax}) \\ &= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\ (-a)^{n-k}\ \frac{\Gamma(1+\gamma+\beta)}{\Gamma(1+\gamma+\beta-k)}D^{\alpha}x^{\gamma+\beta-k}\ e^{-ax} \\ &= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\ (-a)^{n-k}\ \frac{\Gamma(1+\gamma+\beta)}{\Gamma(1+\gamma+\beta-k)}Y_{\alpha}^{\beta-k}\ (\gamma,a;x) \end{aligned}$$

From the convergence of the power series expansion of $e^{-ax}x^{\gamma+\beta}$ and the properties of the fractional derivative we obtain

$$Y_{\alpha}^{\beta}\ (\gamma,a;x) = \sum_{m=0}^{\infty}\frac{(-a)^m}{m\ !}\ \frac{\Gamma(1+m+\gamma+\beta)}{\Gamma(1+m+\gamma-\alpha+\beta)}x^{m+\gamma-\alpha+\beta} \tag{1.31}$$

from which we can prove (by direct substitution) the last result. □

Theorem 1.6. *Let $\alpha,\gamma \in (n-1,n)$, $n = 1,2,\ldots$ and $\beta,\alpha \in \Re;\ a > 0$. Then*

$$xDL_{\alpha}^{\beta}(\gamma,a;x) = (\gamma+\beta)L_{\alpha}^{\beta-1}(\gamma,a;x) + (2ax-\beta)L_{\alpha}^{\beta}(\gamma,a;x) \tag{1.32}$$

$$\alpha xDL_{\alpha}^{\beta}(\gamma,a;x) + \alpha\beta L_{\alpha}^{\beta}(\gamma,a;x) = (\gamma+\beta)\times[(\beta-ax)\ L_{\alpha-1}^{\beta}(\gamma-1,a;x) + xDL_{\alpha-1}^{\beta}(\gamma-1,a;x)] \tag{1.33}$$

$$L_{\alpha+n}^{\beta}(\gamma,a;x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\sum_{k=0}^{n}k!\ \begin{pmatrix} n \\ k \end{pmatrix}\ x^{-k}L_k^{\beta-k}[D^kL_{\alpha}^{\beta}(\gamma,a;x)] \tag{1.34}$$

Where $L_k^{\beta-k}$ is the well known Laguerre polynomials.

$$xDL_{\alpha-1}^{\beta}(\gamma-1,a;x) - \alpha L_{\alpha}^{\beta}(\gamma,a;x) = (ax-\beta-\alpha)\ L_{\alpha-1}^{\beta}(\gamma-1,a;x) \tag{1.35}$$

Proof. From the definition $L_{\alpha}^{\beta}(\gamma,a;x)$ and Equations (1.32)-(1.35) respectively, we have the result. □

Theorem 1.7. *The function $Y_{\alpha}^{\beta}(\gamma,a;x)$ is particular solution of the differential equation.*

$$xD^2Y_{\alpha}^{\beta}(\gamma,a;x) + (1+\alpha-\gamma-\beta+ax)DY_{\alpha}^{\beta}(\gamma,a;x) + (1+\alpha)\ aY_{\alpha}^{\beta}(\gamma,a;x) = 0 \tag{1.36}$$

Proof. Multiplying (1.33) by x and from (1.35) into it we have

$$xDY_{\alpha}^{\beta}(\gamma,a;x) = -axY_{\alpha}^{\beta}(\gamma,a;x) + (\gamma+\beta)Y_{\alpha}^{\beta}(\gamma,a;x) - \alpha(\gamma+\beta)Y_{\alpha-1}^{\beta}(\gamma-1,a;x) \tag{1.37}$$

Differentiating this equation and from (1.33) into it, we obtain the result. □

Theorem 1.8. *The function $L^{\beta}_{\alpha}(\gamma, a; x)$ is particular solution of the differential equation*

$$xD^2Y^{\beta}_{\alpha}(\gamma, a; x) + (1+\alpha+\beta-\gamma-ax)DY^{\beta}_{\alpha}(\gamma, a; x) + \left(ay\frac{(\alpha-\gamma)\beta}{x}L^{\beta}_{\alpha}(\gamma, a; x)\right) = 0 \tag{1.38}$$

Proof. From (1.1) into (1.14), we obtain the result. □

2 Orthogonality Property

Theorem 2.1. *For any real numbers $\alpha_1 \neq \alpha_2, \neq \gamma_1 \neq \gamma_2$, we have*

$$\int_0^\infty e^{-ax}x^{\beta}L^{\beta}_{\alpha_1}(\gamma_1, a; x)\ \ L^{\beta}_{\alpha_2}(\gamma_2, a; x)\ dx = 0$$

for $\alpha_1 = \gamma_1$ $\alpha_2 = \gamma_2$.

Proof. Let $u^{\beta}_{\alpha}(\gamma, a; x) = e^{-ax/2}x^{\beta/2}L^{\beta}_{\alpha}(\gamma, a; x)$, $\alpha \in \Re^+$, then by the direct calculation we can prove that

$$xD^2u^{\beta}_{\alpha}(\gamma, a; x) + (1+\alpha-\gamma)\ Du^{\beta}_{\alpha}(\gamma, a; x) + \left[(1+\alpha+\gamma+\beta)\frac{a}{2} + (\alpha-\gamma)\frac{\beta}{2x} - \frac{\beta^2}{4x} - \frac{a^2}{4}x\right]u^{\beta}_{\alpha}(\gamma, a; x) = 0 \tag{2.1}$$

Then for any positive real number α_1, γ_1 and α_2, γ_2, we have

$$\begin{aligned} &xD^2u^{\beta}_{\alpha_1}(\gamma_1, a; x) + (1+\alpha_1-\gamma_1)\ Du^{\beta}_{\alpha_1}(\gamma_1, a; x) \\ &\qquad + \left[(1+\alpha_1+\gamma_1+\beta)\frac{a}{2} + (\alpha_1-\gamma_1)\frac{\beta}{2x} - \frac{\beta^2}{4x} - \frac{a^2}{4}x\right]u^{\beta}_{\alpha_1}(\gamma_1, a; x) = 0 \quad (2.2) \\ &xD^2u^{\beta}_{\alpha_2}(\gamma_2, a; x) + (1+\alpha_2-\gamma_2)\ Du^{\beta}_{\alpha_2}(\gamma_2, a; x) \\ &\qquad + \left[(1+\alpha_2+\gamma_2+\beta)\frac{a}{2} + (\alpha_2-\gamma_2)\frac{\beta}{2x} - \frac{\beta^2}{4x} - \frac{a^2}{4}x\right]u^{\beta}_{\alpha_2}(\gamma_2, a; x) = 0 \quad (2.3) \end{aligned}$$

By multiplying the first equation by $u^{\beta}_{\alpha_2}(\gamma_2, a; x)$ and the second equation by $u^{\beta}_{\alpha_1}(\gamma_1, a; x)$, subtracting the resulting equations and integrating from 0 to ∞ we obtain

$$\begin{aligned} \int_0^\infty u^{\beta}_{\alpha_1}(\gamma_1, a; x)\ u^{\beta}_{\alpha_2}(\gamma_2, a; x)\ dx &= (\gamma_1-\alpha_1)\ \int_0^\infty u^{\beta}_{\alpha_2}(\gamma_2, a; x)\ Du^{\beta}_{\alpha_1}(\gamma_1, a; x)\ dx \\ &\quad + (\alpha_2-\gamma_2)\int_0^\infty u^{\beta}_{\alpha_1}(\gamma_1, a; x)\ Du^{\beta}_{\alpha_2}(\gamma_2, a; x)\ dx \end{aligned} \tag{2.4}$$

□

3 Conclusion

We see that our assumption are true and exits the result for $\alpha_1 = \gamma_1$ and $\alpha_2 = \gamma_2$, we have the result.

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