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On Edge Product Cordial Labeling of Some Product Related Graphs

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Abstract : An edge product cordial labeling is a variant of product cordial labeling. We have explored this concept in the context of different graph products.

Keywords : Graph Product, Product Cordial Labeling, Edge Product Cordial Labeling.

1 Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$ with order p and size q. For any graph theoretic notations and terminology, we follow West [9].

A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

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For a comprehensive bibliography of papers on the concept of graph labeling, readers are advised to refer Gallian [2]. The present paper is focused on edge product cordial labeling of graphs.

In 1987, Cahit [1] introduced cordial labeling as a weaker version of graceful labeling and harmonious labeling. In 2004, Sundaram *et al.* [4] have introduced the product cordial labeling in which the absolute difference of vertex labels in cordial labeling is replaced by the product of the vertex labels.

Vaidya and Barasara [5] have introduced the edge analogue of product cordial labeling and named it as edge product cordial labeling which is defined as follows.

For a graph $G = (V(G), E(G))$, an edge labeling function $f : E(G) \rightarrow \{0, 1\}$ induces a vertex labeling function $f^* : V(G) \rightarrow \{0, 1\}$ defined as $f^*(v) = \prod f(e_i)$ for $\{e_i \in E(G)/e_i$ is incident to $v_i\}$. Then f is called *edge product cordial labeling* of graph G if absolute difference of number of vertices with label 1 and label 0 is atmost 1 and absolute difference of number of edges with label 1 and label 0 is atmost 1. A graph G is called *edge product cordial* if it admits an edge product cordial labeling.

In [5, 6, 8], Vaidya and Barasara have proved several results on this newly defined concept while the edge product cordial labeling in the context of various graph operations is discussed by Vaidya and Barasara [7].

For any graph G, denote the number of vertices having label 1 as $v_f(1)$, the number of vertices having label 0 as $v_f(0)$, the number of edges having label 1 as $e_f(1)$ and the number of edges having label 0 as $e_f(0)$.

The product of two graphs is one of the important graph operations and mainly four different kinds of graph products are familiar. A detailed study on product graphs can be found in Hammack *et al.* [3].

The *cartesian product* $G \square H$ of two graphs $G = (V(G), E(G))$ and $E = (V(H), E(H))$ is a graph with the vertex set $V(G \square H) = V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (i) $u = u'$ and $vv' \in E(H)$, or (ii) $v = v'$ and $uu' \in E(G)$.

The *direct product (tensor product)* $G \times H$ of two graphs $G = (V(G), E(G))$ and $E = (V(H), E(H))$ is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

The *lexicographic product (graph composition)* $G[H]$ (also $G \circ H$) of two graphs $G = (V(G), E(G))$ and $E = (V(H), E(H))$ is a graph with the vertex set $V(G[H]) = V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if (i) $uu' \in E(G)$, or (ii) $u = u'$ and $vv' \in E(H)$.

The graph product is called non-trivial, if both G and H have at least two vertices.

The present work is aimed to study edge product cordial labeling in the context of cartesian product, direct product and lexicographic product.

2 Main Results

Theorem 2.1. *The graph $P_m \Box P_n$ is not an edge product cordial graph.*

Proof. For the graph $P_m \Box P_n$, $|V(P_m \Box P_n)| = mn$ and $|E(P_m \Box P_n)| = 2mn - m - n$. We will consider following two cases.

Case 1: When n is even.

Subcase 1: When m is odd.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{2mn - m - n - 1}{2}$ edges out of $2mn - m - n$ edges. The edges with label 0 will give rise at least $\frac{mn + m - 1}{2}$ vertices with label 0 and at most $\frac{mn - m + 1}{2}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = m - 1 > 1$. Consequently the graph is not edge product cordial.

Subcase 2: When m is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{2mn - m - n}{2}$ edges out of $2mn - m - n$ edges. The edges with label 0 will give rise at least $\frac{mn + m}{2}$ vertices with label 0 and at most $\frac{mn - m}{2}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = m > 1$. Consequently the graph is not edge product cordial.

Case 2: When n is odd.

Subcase 1: When $m \equiv 0 \pmod 4$

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{2mn - m - n - 1}{2}$ edges out of $2mn - m - n$ edges. The edges with label 0 will give rise at least $\frac{2mn + m}{4}$ vertices with label 0 and at most $\frac{2mn - m}{4}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = \frac{m}{2} > 1$. Consequently the graph is not edge product cordial.

Subcase 2: When $m \equiv 1 \pmod 4$

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{2mn - m - n}{2}$ edges out of $2mn - m - n$ edges. The edges with label 0 will give rise at least $\frac{2mn + m + 1}{4}$ vertices with label 0 and at most $\frac{2mn - m - 1}{4}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = \frac{m + 1}{2} > 1$. Consequently the graph is not edge product cordial.

Subcase 3: When $m \equiv 2 \pmod 4$

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{2mn - m - n - 1}{2}$ edges out of $2mn - m - n$ edges. The edges with label 0 will give rise at least $\frac{2mn + m + 2}{4}$ vertices with label 0 and at most $\frac{2mn - m - 2}{4}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = \frac{m + 2}{2} > 1$. Consequently the graph is not edge product cordial.

Subcase 4: When $m \equiv 3 \pmod 4$

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\dfrac{2mn-m-n}{2}$ edges out of $2mn-m-n$ edges. The edges with label 0 will give rise at least $\dfrac{2mn+m+3}{4}$ vertices with label 0 and at most $\dfrac{2mn-m-3}{4}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0)-v_f(1)| = \dfrac{m+3}{2} > 1$. Consequently the graph is not edge product cordial.

Hence, $P_m \Box P_n$ is not an edge product cordial graph. □

Theorem 2.2. *The graph $C_m \Box C_n$ is not an edge product cordial graph.*

Proof. For the graph $C_m \Box C_n$, $|V(C_m \Box C_n)| = mn$ and $|E(P_m \Box P_n)| = 2mn$. Without loss of generality we assume that $m \leq n$. We will consider following two cases.

Case 1: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to mn edges out of $2mn$ edges. The edges with label 0 will give rise at least $\dfrac{mn}{2}+m$ vertices with label 0 and at most $\dfrac{mn}{2}-m$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0)-v_f(1)| = 2m > 1$. Consequently the graph is not edge product cordial.

Case 2: When n is odd.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to mn edges out of $2mn$ edges. The edges with label 0 will give rise at least $\dfrac{mn+m}{2}$ vertices with label 0 and at most $\dfrac{mn-m}{2}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0)-v_f(1)| = m > 1$. Consequently the graph is not edge product cordial.

Hence, $C_m \Box C_n$ is not an edge product cordial graph. □

Theorem 2.3. *The graph $P_m \times P_n$ is an edge product cordial graph.*

Proof. For the graph $P_m \times P_n$, $|V(P_m \times P_n)| = mn$ and $|E(P_m \times P_n)| = 2(m-1)(n-1)$. We will consider following two cases.

Case 1: When m or n is even.

The graph $P_m \times P_n$ is a disconnected graph with two components. Both the components are of the order $\dfrac{mn}{2}$ and the size $(m-1)(n-1)$. Assign label 1 to all the edges of one component and label 0 to remaining edges. As a result of this procedure we have the following:

$$e_f(1) = e_f(0) = (m-1)(n-1),$$
$$v_f(1) = v_f(0) = \frac{mn}{2}.$$

Case 2: When m and n are odd.

The graph $P_m \times P_n$ is a disconnected graph with two components. First component is of order $\left\lceil \dfrac{mn}{2} \right\rceil$ and second component is of the order $\left\lfloor \dfrac{mn}{2} \right\rfloor$ while the size of both the component is $(m-1)(n-1)$.

Assign label 1 to all the edges of component having order $\left\lceil \frac{mn}{2} \right\rceil$ and label 0 to remaining edges. As a result of this procedure we have the following:

$$e_f(1) = e_f(0) = (m-1)(n-1),$$
$$v_f(1) - 1 = v_f(0) = \frac{mn-1}{2}.$$

Thus in both the cases we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| = 0$.

Hence, $P_m \times P_n$ is an edge product cordial graph. □

Example 2.4. *The graph $P_4 \times P_7$ and its edge product cordial labeling is shown in figure 1.*

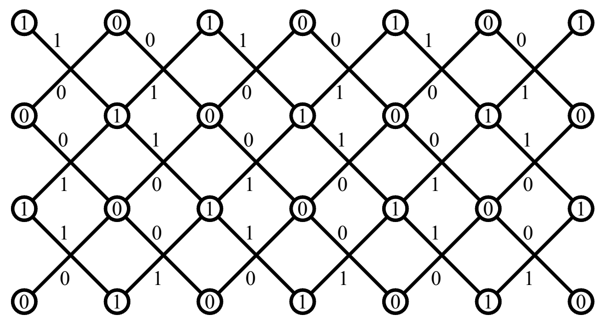

figure 1

Theorem 2.5. *The graph $C_m \times C_n$ is an edge product cordial graph for m and n even.*

Proof. For the graph $C_m \times C_n$, $|V(C_m \times C_n)| = mn$ and $|E(C_m \times C_n)| = 2mn$. For even m and n, the graph $C_m \times C_n$ is a disconnected graph with two components. Both the components are of the order $\frac{mn}{2}$ and the size mn. Assign label 1 to all the edges of one component and label 0 to remaining edges. As a result of this procedure we have the following:

$$e_f(1) = e_f(0) = mn,$$
$$v_f(1) = v_f(0) = \frac{mn}{2}.$$

Thus we have $|v_f(0) - v_f(1)| = 0$ and $|e_f(0) - e_f(1)| = 0$. Hence, $C_m \times C_n$ is an edge product cordial graph for m and n even. □

Example 2.6. *The graph $C_4 \times C_6$ and its edge product cordial labeling is shown in figure 2. Here grey edges are labeled with 1 and black edges are labeled with 0.*

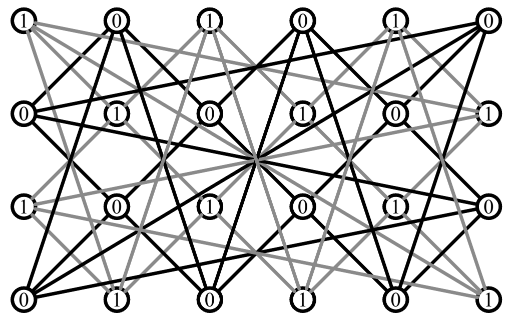

figure 2

Theorem 2.7. *The graph $C_m \times C_n$ is not an edge product cordial graph for m or n odd.*

Proof. For the graph $C_m \times C_n$, $|V(C_m \times C_n)| = mn$ and $|E(C_m \times C_n)| = 2mn$. Without loss of generality we assume that $m \leq n$. We will consider following three cases.

Case 1: When $m = 3$ and $n = 3$.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to 9 edges out of 18 edges. The edges with label 0 will give rise at least 6 vertices with label 0 and at most 3 vertices with label 1 out of total 9 vertices. Therefore $|v_f(0) - v_f(1)| = 3$. Consequently the graph is not edge product cordial.

Case 2: When m is odd or n is odd but not both.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to mn edges out of $2mn$ edges. The edges with label 0 will give rise at least $\dfrac{mn+4}{2}$ vertices with label 0 and at most $\dfrac{mn-4}{2}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = 4$. Consequently the graph is not edge product cordial.

Case 3: When both m and n are odd.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to mn edges out of $2mn$ edges. The edges with label 0 will give rise at least $\dfrac{mn+5}{2}$ vertices with label 0 and at most $\dfrac{mn-5}{2}$ vertices with label 1 out of total mn vertices. Therefore $|v_f(0) - v_f(1)| = 5$. Consequently the graph is not edge product cordial.

Hence, $C_m \times C_n$ is not an edge product cordial graph for m or n odd. □

Theorem 2.8. *The graph $P_n[P_2]$ is not an edge product cordial graph.*

Proof. For the graph $P_n[P_2]$, $|V(P_n[P_2])| = 2n$ and $|E(P_n[P_2])| = 5n - 4$. In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\left\lfloor \dfrac{5n-4}{2} \right\rfloor$ edges out of $5n - 4$ edges.

The edges with label 0 will give rise at least $n+1$ vertices with label 0 and at most $n-1$ vertices with label 1 out of total $2n$ vertices. Therefore $|v_f(0) - v_f(1)| = 2$. Consequently the graph is not edge product cordial. Hence, $P_m[P_2]$ is not an edge product cordial graph. □

Theorem 2.9. *The graph $C_n[P_2]$ is not an edge product cordial graph.*

Proof. For the graph $C_n[P_2]$, $|V(C_n[P_2])| = 2n$ and $|E(C_n[P_2])| = 5n$. In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\left\lfloor \frac{5n}{2} \right\rfloor$ edges out of $5n$ edges. The edges with label 0 will give rise at least $n+2$ vertices with label 0 and at most $n-2$ vertices with label 1 out of total $2n$ vertices.

Therefore $|v_f(0) - v_f(1)| = 4$. Consequently the graph is not edge product cordial. Hence, $C_m[P_2]$ is not an edge product cordial graph. □

3 Concluding Remarks

We have investigated edge product cordial labeling for the larger graph obtained by means of three graph products, namely cartesian product, direct product and lexicographic product.

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