Int. J. Math. And Appl., 6(4)(2018), 117–132
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On the General Product-connectivity Index of Generalized xyz–point-line Transformation Graphs

B. Basavanagoud[1,*] and Chetana S. Gali[1]

1 Department of Mathematics, Karnatak University, Dharwad, Karnataka, India.

Abstract: The general product-connectivity index is a molecular descriptor defined as $R_\alpha(G) = \sum_{uv \in E(G)} (d_G(u)d_G(v))^\alpha$, where $d_G(u)$ denotes the degree of vertex u in a graph G and α is a real number. In this paper, we compute the expressions for general product-connectivity index of generalized xyz–point-line transformation graphs.

MSC: 05C07.

Keywords: Subdivision graph, Semitotal-point graph, Semitotal-line graph, General product-connectivity index, Generalized xyz–point-line transformation graphs.

 Accepted on: 07.10.2018

1. Introduction

Throughout this paper, we consider simple, finite and undirected graphs. Let G be a graph with the vertex set $V(G)$ consisting of n vertices and edge set $E(G)$ of m edges. The *degree of a vertex* $v \in V(G)$ is the number of edges incident with v in G and is denoted by $d_G(v)$. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The *degree of an edge* $e = uv$ in G is denoted by $d_G(e)$ and is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The *maximum* and *minimum* vertex degree of G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$ respectively. The vertices and edges of G are *members*. Here, we denote the adjacency (or incidence) of members by the symbol $\sim$ and nonadjacency (or nonincidence) by $\nsim$. As usual $\overline{G}$ is the *complement of a graph*, $L(G)$ is the *line graph* and $S(G)$ is the *subdivision graph*. For unexplained terminology, we refer [10, 14].

The *semitotal-point graph* $T_2(G)$ [18] is a graph whose vertex set is $V(G) \cup E(G)$, and where two vertices are adjacent if and only if they corresponds to (i) two adjacent vertices of G or (ii) one is a vertex of G and the other is an edge of G incident with it in G. The *semitotal-line graph* $T_1(G)$ [18] is a graph whose vertex set is $V(G) \cup E(G)$, and where two vertices are adjacent if and only if they corresponds to (i) two adjacent edges in G or (ii) one is a vertex of G and the other is an edge of G incident with it in G. The *partial complement of subdivision graph* $\overline{S}(G)$ [12] is a graph with the vertex set $V(G) \cup E(G)$ such that two vertices of $\overline{S}(G)$ are adjacent if and only if one corresponds to a vertex v of G and other to an edge e of G and v is not incident to e in G.

For a graph $G = (V(G), E(G))$, let G^0 be the graph with the vertex set $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$.

* *E-mail: b.basavanagoud@gmail.com*

Definition 1.1 ([8])**.** *Given a graph* $G = (V(G), E(G))$ *and three variables* $x, y, z \in \{0, 1, +, -\}$*, the* xyz−transformation $T^{xyz}(G)$ *of* G *is the graph with the vertex set* $V(T^{xyz}(G)) = V(G) \cup E(G)$ *and the edge set* $E(T^{xyz}(G)) = E(G^x) \cup E(L(G)^y) \cup E(W)$*, where* $W = S(G)$ *if* $z = +$*,* $W = \overline{S}(G)$ *if* $z = -$*,* W *is the graph with* $V(W) = V(G) \cup E(G)$ *and with no edges if* $z = 0$*, and* W *is the complete bipartite graph with parts* $V(G)$ *and* $E(G)$ *if* $z = 1$*.*

Since there are sixty four distinct 3-permutations of $\{1, 0, +, -\}$, there are sixty four different xyz–transformations of a given graph G. These *xyz−transformation graphs* are also called as *generalized xyz–point-line transformation graphs*[1]. In literature we found that $T^{00+}(G)$ is the subdivision graph, $T^{+++}(G)$ is the total graph, $T^{---}(G)$ is the complement of total graph, $T^{00-}(G)$ is the partial complement of subdivision graph, $T^{-++}(G)$ is the quasi-total graph [2] and $T^{1++}(G)$ is the quasivertex-total graph [15]. Note that if $x, y, z \in \{+, -\}$, then we obtain *total transformation graphs* [19]. The vertex v of $T^{xyz}(G)$ corresponding to a vertex v of G is referred to as *point-vertex* and vertex e of $T^{xyz}(G)$ corresponding to an edge e of G is referred to as *line-vertex*. In Figure 1, self-explanatory examples of some $T^{xyz}(G)$ graphs are depicted, dark circles represents the point-vertices and light circles represents the line-vertices of $T^{xyz}(G)$.

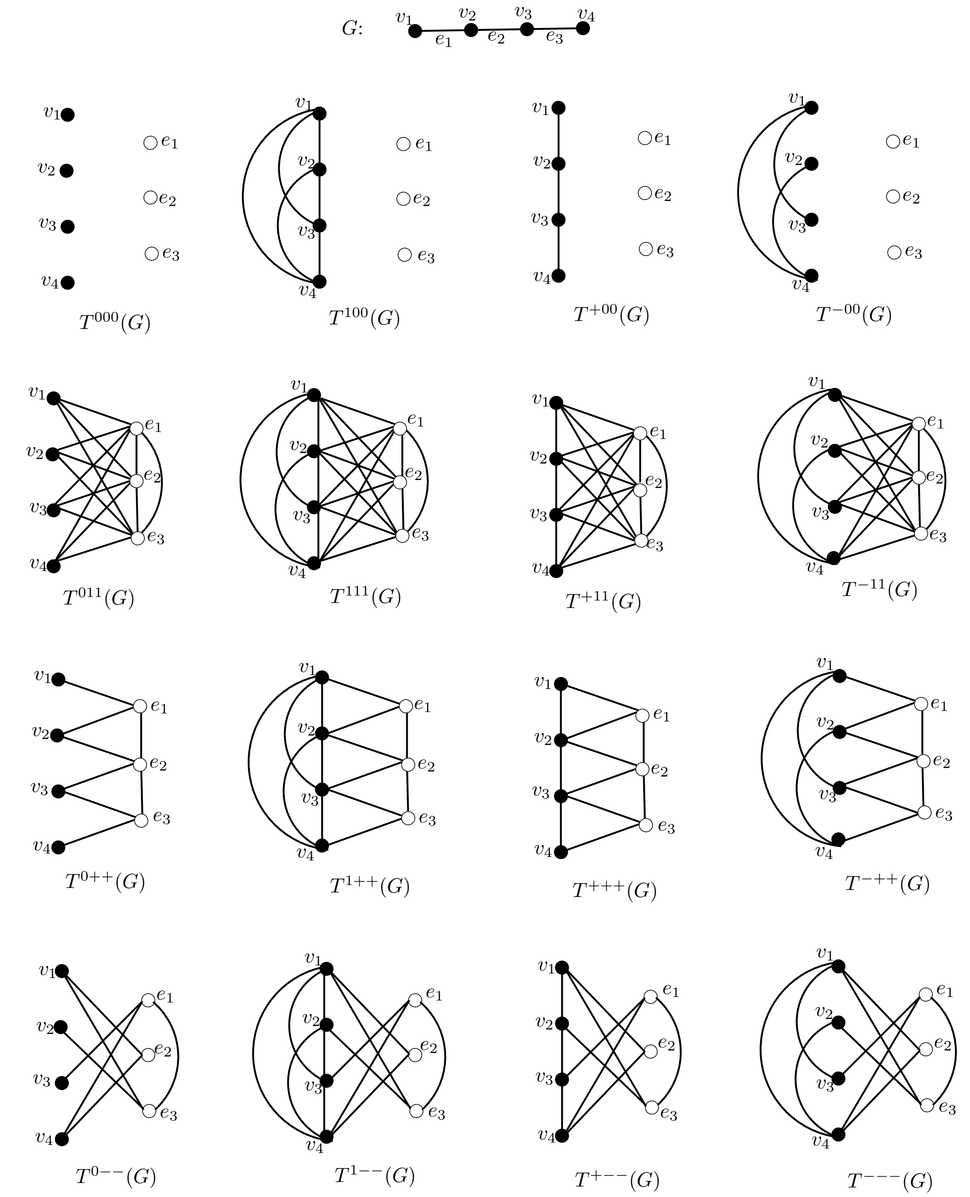

Figure 1. **Some generalized xyz–point-line transformation graphs**

Theorem 1.2 ([4]). *Let G be an $(n,\, m)$ graph and let v be the point-vertex of $T^{xyz}(G)$ corresponding to a vertex v of G. Then*

(1). $$d_{T^{xy0}(G)}(v) = \begin{cases} 0 & if\ x=0, y\in\{0,1,+,-\}. \\ n-1 & if\ x=1, y\in\{0,1,+,-\}. \\ d_G(v) & if\ x=+, y\in\{0,1,+,-\}. \\ n-1-d_G(v) & if\ x=-, y\in\{0,1,+,-\}. \end{cases}$$

(2). $$d_{T^{xy1}(G)}(v) = \begin{cases} m & if\ x=0, y\in\{0,1,+,-\}. \\ n+m-1 & if\ x=1, y\in\{0,1,+,-\}. \\ m+d_G(v) & if\ x=+, y\in\{0,1,+,-\}. \\ n+m-1-d_G(v) & if\ x=-, y\in\{0,1,+,-\}. \end{cases}$$

(3). $$d_{T^{xy+}(G)}(v) = \begin{cases} d_G(v) & if\ x=0, y\in\{0,1,+,-\}. \\ n-1+d_G(v) & if\ x=1, y\in\{0,1,+,-\}. \\ 2d_G(v) & if\ x=+, y\in\{0,1,+,-\}. \\ n-1 & if\ x=-, y\in\{0,1,+,-\}. \end{cases}$$

(4). $$d_{T^{xy-}(G)}(v) = \begin{cases} m-d_G(v) & if\ x=0, y\in\{0,1,+,-\}. \\ n+m-1-d_G(v) & if\ x=1, y\in\{0,1,+,-\}. \\ m & if\ x=+, y\in\{0,1,+,-\}. \\ n+m-1-2d_G(v) & if\ x=-, y\in\{0,1,+,-\}. \end{cases}$$

Theorem 1.3 ([4]). *Let G be an $(n,\, m)$ graph and let e be the line-vertex of $T^{xyz}(G)$ corresponding to an edge $e=uv$ of G. Then*

(1). $$d_{T^{xy0}(G)}(e) = \begin{cases} 0 & if\ y=0, x\in\{0,1,+,-\}. \\ m-1 & if\ y=1, x\in\{0,1,+,-\}. \\ d_G(u)+d_G(v)-2 & if\ y=+, x\in\{0,1,+,-\}. \\ m+1-d_G(u)-d_G(v) & if\ y=-, x\in\{0,1,+,-\}. \end{cases}$$

(2). $$d_{T^{xy1}(G)}(e) = \begin{cases} n & if\ y=0, x\in\{0,1,+,-\}. \\ n+m-1 & if\ y=1, x\in\{0,1,+,-\}. \\ n-2+d_G(u)+d_G(v) & if\ y=+, x\in\{0,1,+,-\}. \\ n+m+1-d_G(u)-d_G(v) & if\ y=-, x\in\{0,1,+,-\}. \end{cases}$$

(3). $$d_{T^{xy+}(G)}(e) = \begin{cases} 2 & if\ y=0, x\in\{0,1,+,-\}. \\ m+1 & if\ y=1, x\in\{0,1,+,-\}. \\ d_G(u)+d_G(v) & if\ y=+, x\in\{0,1,+,-\}. \\ m+3-d_G(u)-d_G(v) & if\ y=-, x\in\{0,1,+,-\}. \end{cases}$$

(4). $$d_{T^{xy-}(G)}(e) = \begin{cases} n-2 & if\ y=0, x\in\{0,1,+,-\}. \\ n+m-3 & if\ y=1, x\in\{0,1,+,-\}. \\ n-4+d_G(u)+d_G(v) & if\ y=+, x\in\{0,1,+,-\}. \\ n+m-1-d_G(u)-d_G(v) & if\ y=-, x\in\{0,1,+,-\}. \end{cases}$$

Topological indices are numerical parameters of a graph which are invariant under graph isomorphisms. The significance of topological indices is associated with quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR) [13].

The *first and second Zagreb indices* are defined in [9] as

$$M_1 = M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \text{ and } M_2 = M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) \text{ respectively.}$$

The *Randić index* (or *product-connectivity index*) of a graph G is defined as [17]

$$R(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)]^{-\frac{1}{2}}.$$

It has been extended to the *general product-connectivity index* defined as [7]

$$R_\alpha(G) = \sum_{uv \in E(G)} [d_G(u)\, d_G(v)]^\alpha, \text{ where } \alpha \text{ is any real number.}$$

The *first general Zagreb index* of a graph G is defined as [16]

$$M_1^\alpha(G) = \sum_{u \in V(G)} [d_G(u)]^\alpha, \text{ where } \alpha \text{ is any real number.}$$

The first and second Zagreb indices of semitotal-point graph, semitotal-line graph and total transformation graphs can be found in [3, 4, 11]. The expression for $R_\alpha(T_1(G))$, $R_\alpha(T_2(G))$ and bounds for general product-connectivity index of total transformation graphs obtained in [6]. The expression for first and second Zagreb indices and general sum-connectivity index of generalized xyz–point-line transformation graphs found in [4] and [5] respectively. In this paper, we compute expressions for general product-connectivity index of some generalized xyz–point-line transformation graphs.

2. General Product Connectivity Index of $T^{xyz}(G)$ Graphs

Theorem 2.1. *Let G be a graph of order n and size m. Then*

(1). $R_\alpha(T^{000}(G)) = 0$

(2). $R_\alpha(T^{100}(G)) = \binom{n}{2}(n-1)^{2\alpha}$

(3). $R_\alpha(T^{+00}(G)) = R_\alpha(G)$

(4). $R_\alpha(T^{-00}(G)) = \sum_{u \nsim v;\ u,v \in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha$

(5). $R_\alpha(T^{010}(G)) = \binom{m}{2}(m-1)^{2\alpha}$

(6). $R_\alpha(T^{110}(G)) = \binom{n}{2}(n-1)^{2\alpha} + \binom{m}{2}(m-1)^{2\alpha}$

(7). $R_\alpha(T^{+10}(G)) = R_\alpha(G) + \binom{m}{2}(m-1)^{2\alpha}$

(8). $R_\alpha(T^{-10}(G)) = \sum_{u \nsim v;\ u,v \in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha + \binom{m}{2}(m-1)^{2\alpha}$

(9). $R_\alpha(T^{001}(G)) = (nm)^{\alpha+1}$

(10). $R_\alpha(T^{101}(G)) = \binom{n}{2}(n+m-1)^{2\alpha} + m \cdot n^{\alpha+1}(n+m-1)^\alpha$

(11). $R_\alpha(T^{011}(G)) = \binom{m}{2}(n+m-1)^{2\alpha} + n \cdot m^{\alpha+1}(n+m-1)^\alpha$

(12). $R_\alpha(T^{111}(G)) = (n+m-1)^{2\alpha}\left[\binom{n}{2} + \binom{m}{2} + nm\right]$

Proof.

(1). $R_\alpha(T^{000}(G)) = \sum_{xy \in E(T^{000}(G))} [d_{T^{000}(G)}(x)\, d_{T^{000}(G)}(y)]^\alpha = \sum_{xy \in E(T^{000}(G))} [0 \cdot 0]^\alpha = 0.$

(2). $R_\alpha(T^{100}(G)) = \sum_{uv \in E(T^{100}(G)) \cap E(K_n)} [d_{T^{100}(G)}(u)\, d_{T^{100}(G)}(v)]^\alpha = \sum_{uv \in E(K_n)} [(n-1)(n-1)]^\alpha = \binom{n}{2}(n-1)^{2\alpha} = R_\alpha(K_n).$

(3). $R_\alpha(T^{+00}(G)) = \sum_{uv \in E(T^{+00}(G)) \cap E(G)} [d_{T^{+00}(G)}(u)\, d_{T^{+00}(G)}(v)]^\alpha = \sum_{uv \in E(G)} [d_G(u)\, d_G(v)]^\alpha = R_\alpha(G).$

(4).
$$R_\alpha(T^{-00}(G)) = \sum_{uv \in E(T^{-00}(G)) \cap E(\overline{G})} [d_{T^{-00}(G)}(u)\, d_{T^{-00}(G)}(v)]^\alpha = \sum_{uv \in E(\overline{G})} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha$$
$$= \sum_{u \nsim v;\ u,v \in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha = R_\alpha(\overline{G}).$$

(5). $R_\alpha(T^{010}(G)) = \sum_{e_ie_j \in E(K_m)} [(m-1)(m-1)]^\alpha = (m-1)^{2\alpha}\binom{m}{2} = R_\alpha(K_m).$

(6). Since $T^{110}(G) = K_n \cup K_m$, we have $R_\alpha(T^{110}(G)) = R_\alpha(K_n) + R_\alpha(K_m)$.

(7). Since $T^{+10}(G) = G \cup K_m$, we have $R_\alpha(T^{+10})(G) = R_\alpha(G) + R_\alpha(K_m)$.

(8). Since $T^{-10}(G) = \overline{G} \cup K_m$, we have $R_\alpha(T^{-10}(G)) = R_\alpha(\overline{G}) + R_\alpha(K_m)$.

(9). $R_\alpha(T^{001}(G)) = \sum_{ue \in E(T^{001}(G)) \cap E(K_{n,m})} [d_{T^{001}(G)}(u)\, d_{T^{001}(G)}(e)]^\alpha = \sum_{ue \in E(K_{n,m})} [m \cdot n]^\alpha = (mn)^{\alpha+1}.$

(10).
$$R_\alpha(T^{101}(G)) = \sum_{uv \in E(T^{101}(G)) \cap E(K_n)} [d_{T^{101}(G)}(u)\, d_{T^{101}(G)}(v)]^\alpha + \sum_{ue \in E(T^{101}(G)) \cap E(K_{n,m})} [d_{T^{101}(G)}(u)\, d_{T^{101}(G)}(e)]^\alpha$$
$$= \sum_{uv \in E(K_n)} [(n+m-1)(n+m-1)]^\alpha + \sum_{ue \in E(K_{n,m})} [(n+m-1)n]^\alpha$$
$$R_\alpha(T^{101}(G)) = \binom{n}{2}(n+m-1)^{2\alpha} + m \cdot n^{\alpha+1}(n+m-1)^\alpha.$$

(11).
$$R_\alpha(T^{011}(G)) = \sum_{e_ie_j \in E(K_m)} [(n+m-1)(n+m-1)]^\alpha + \sum_{ue \in E(K_{n,m})} [m(n+m-1)]^\alpha$$
$$= \binom{m}{2}(n+m-1)^{2\alpha} + n \cdot m^{\alpha+1}(n+m-1)^\alpha.$$

(12).
$$R_\alpha(T^{111}(G)) = \sum_{uv \in E(K_n)} [(n+m-1)(n+m-1)]^\alpha + \sum_{e_ie_j \in E(K_m)} [(n+m-1)(n+m-1)]^\alpha$$
$$+ \sum_{ue \in E(K_{n,m})} [(n+m-1)(n+m-1)]^\alpha.$$

□

Theorem 2.2. *Let G be an (n, m) graph and $\alpha < 0$. Then*

(1). $4^\alpha(\delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] \geq R_\alpha(T^{0+0}(G)) \geq 4^\alpha(\Delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$

(2). $R_\alpha(T^{1+0}(G)) \leq (n-1)^{2\alpha}\binom{n}{2} + 4^\alpha(\delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$

$R_\alpha(T^{1+0}(G)) \geq (n-1)^{2\alpha}\binom{n}{2} + 4^\alpha(\Delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$

(3). $R_\alpha(T^{++0}(G)) \leq R_\alpha(G) + 4^\alpha(\delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$

$R_\alpha(T^{++0}(G)) \geq R_\alpha(G) + 4^\alpha(\Delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$

(4). $R_\alpha(T^{-+0}(G)) \leq \sum\limits_{u\nsim v;\, u,v\in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha + 4^\alpha(\delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$

$R_\alpha(T^{-+0}(G)) \geq \sum\limits_{u\nsim v;\, u,v\in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha + 4^\alpha(\Delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right].$

Proof.

(1).
$$\begin{aligned} R_\alpha(T^{0+0}(G)) &= \sum_{xy\in E(T^{0+0}(G))} [d_{T^{0+0}(G)}(x)\, d_{T^{0+0}(G)}(y)]^\alpha \\ &= \sum_{e_ie_j\in E(T^{0+0}(G))\cap E(L(G))} [d_{T^{0+0}(G)}(e_i)\, d_{T^{0+0}(G)}(e_j)]^\alpha \\ &= \sum_{e_ie_j\in E(L(G)), e_i=uv, e_j=vw} [(d_G(u)+d_G(v)-2)(d_G(v)+d_G(w)-2)]^\alpha . \end{aligned}$$

Note that $d_G(u) \leq \Delta$ and $d_G(u) \geq \delta$ for any vertex $u \in V(G)$. The equalities hold if and only if G is a regular graph.

$$R_\alpha(T^{0+0}(G)) \geq 4^\alpha(\Delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] \ as\ \alpha < 0$$

Similarly, we can show that $R_\alpha(T^{0+0}(G)) \leq 4^\alpha(\delta-1)^{2\alpha}\left[\frac{1}{2}M_1 - m\right]$. Also note that $R_\alpha(T^{0+0}(G)) = R_\alpha(L(G))$ but $T^{0+0}(G) \ncong L(G)$.

(2). Since $T^{1+0}(G) \cong K_n \cup L(G)$, we have $R_\alpha(T^{1+0}(G)) = R_\alpha(K_n) + R_\alpha(L(G)) = \binom{n}{2}(n-1)^{2\alpha} + R_\alpha(T^{0+0}(G))$.

(3). Since $T^{++0}(G) \cong G \cup L(G)$, we have $R_\alpha(T^{++0}(G)) = R_\alpha(G) + R_\alpha(L(G)) = R_\alpha(G) + R_\alpha(T^{0+0}(G))$.

(4). Since $T^{-+0}(G) \cong \overline{G} \cup L(G)$, we have $R_\alpha(T^{-+0}(G)) = R_\alpha(\overline{G}) + R_\alpha(L(G)) = R_\alpha(\overline{G}) + R_\alpha(T^{0+0}(G))$. □

Theorem 2.3. *Let G be an $(n,\, m)$ graph and $\alpha < 0$. Then*

(1). $R_\alpha(T^{0-0}(G)) \leq (m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$

$R_\alpha(T^{0-0}(G)) \geq (m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$

(2). $R_\alpha(T^{1-0}(G)) \leq (n-1)^{2\alpha}\binom{n}{2} + (m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2} + \frac{1}{2}M_1\right]$

$R_\alpha(T^{1-0}(G)) \geq (n-1)^{2\alpha}\binom{n}{2} + (m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$

(3). $R_\alpha(T^{+-0}(G)) \leq R_\alpha(G) + (m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$

$R_\alpha(T^{+-0}(G)) \geq R_\alpha(G) + (m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$

(4). $R_\alpha(T^{--0}(G)) \leq \sum\limits_{u\nsim v;\, u,v\in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha + (m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$

$R_\alpha(T^{--0}(G)) \geq \sum\limits_{u\nsim v;\, u,v\in V(G)} ([n-1-d_G(u)][n-1-d_G(v)])^\alpha + (m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right].$

Proof.

(1). $R_\alpha(T^{0-0}(G)) = \sum_{e_ie_j \in E(T^{0-0}(G)) \cap E(\overline{L(G)})} [d_{T^{0-0}(G)}(e_i)\ d_{T^{0-0}(G)}(e_j)]^\alpha$

$= \sum_{e_i \nsim e_j, e_i=uv, e_j=wx} [(m+1-d_G(u)-d_G(v))(m+1-d_G(w)-d_G(x))]^\alpha$

Note that $d_G(u) \leq \Delta$ and $d_G(u) \geq \delta$ for any vertex $u \in V(G)$. The equalities hold if and only if G is a regular graph.

$$R_\alpha(T^{0-0}(G)) \geq (m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] \ as\ \alpha < 0$$

Similarly, we can compute $R_\alpha(T^{0-0}(G)) \leq (m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$. Also note that $R_\alpha(T^{0-0}(G)) = R_\alpha(\overline{L(G)})$ but $T^{0-0}(G) \ncong \overline{L(G)}$.

(2). Since $T^{1-0}(G) \cong K_n \cup \overline{L(G)}$, we have $R_\alpha(T^{1-0}(G)) = R_\alpha(K_n) + R_\alpha(\overline{L(G)}) = \binom{n}{2}(n-1)^{2\alpha} + R_\alpha(T^{0-0}(G))$.

(3). Since $T^{+-0}(G) \cong G \cup \overline{L(G)}$, we have $R_\alpha(T^{+-0}(G)) = R_\alpha(G) + R_\alpha(\overline{L(G)}) = R_\alpha(G) + R_\alpha(T^{0-0}(G))$.

(4). Since $T^{--0}(G) \cong \overline{G} \cup \overline{L(G)}$, we have $R_\alpha(T^{--0}(G)) = R_\alpha(\overline{G}) + R_\alpha(\overline{L(G)}) = R_\alpha(\overline{G}) + R_\alpha(T^{0-0}(G))$. □

Theorem 2.4. *Let G be a graph of order n and size m. Then*

(1). $R_\alpha(T^{+01}(G)) = \sum_{uv \in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + n^\alpha m \sum_{u \in V(G)} [m+d_G(u)]^\alpha$

(2). $R_\alpha(T^{-01}(G)) = \sum_{u \nsim v;\ u,v \in V(G)} \{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha + n^\alpha m \sum_{u \in V(G)} [n+m-1-d_G(u)]^\alpha$

(3). $R_\alpha(T^{+11}(G)) = \sum_{uv \in E(G)} \{[m+d_G(u)][d_G(v)]\}^\alpha + (n+m-1)^\alpha m \sum_{u \in V(G)} [m+d_G(u)]^\alpha + \binom{m}{2}(n+m-1)^{2\alpha}$

(4). $R_\alpha(T^{-11}(G)) = \sum_{u \nsim v;\ u,v \in V(G)} \{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha + \binom{m}{2}(n+m-1)^{2\alpha}$
$+ (n+m-1)^\alpha m \sum_{u \in V(G)} [n+m-1-d_G(u)]^\alpha$.

Proof.

(1). $R_\alpha(T^{+01}(G)) = \sum_{uv \in E(T^{+01}(G)) \cap E(G)} [d_{T^{+01}(G)}(u) d_{T^{+01}(G)}(v)]^\alpha + \sum_{ue \in E(T^{+01}(G)) \cap E(K_{n,m})} [d_{T^{+01}(G)}(u) d_{T^{+01}(G)}(e)]^\alpha$

$= \sum_{uv \in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + \sum_{ue \in E(K_{n,m})} \{[m+d_G(u)]n\}^\alpha$

$= \sum_{uv \in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + n^\alpha \sum_{u \in V(G)} m[m+d_G(u)]^\alpha.$

(2). $R_\alpha(T^{-01}(G)) = \sum_{uv \in E(\overline{G})} \{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha + \sum_{ue \in E(K_{n,m})} [(n+m-1-d_G(u))n]^\alpha$

$= \sum_{u \nsim v;\ u,v \in V(G)} [\{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha + n^\alpha \sum_{u \in V(G)} m[n+m-1-d_G(u)]^\alpha.$

(3). $R_\alpha(T^{+11}(G)) = \sum_{uv \in E(G)} [(m+d_G(u))(m+d_G(v))]^\alpha + \sum_{ue \in E(K_{n,m})} \{[m+d_G(u)](n+m-1)\}^\alpha$

$+ \sum_{e_ie_j \in E(K_m)} [(n+m-1)(n+m-1)]^\alpha$

$= \sum_{uv \in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + (n+m-1)^\alpha \sum_{u \in V(G)} m[m+d_G(u)]^\alpha + \binom{m}{2}(n+m-1)^{2\alpha}.$

$$(4).\ R_\alpha(T^{-11}(G)) = \sum_{uv\in E(\overline{G})} [(n+m-1-d_G(u)) + (n+m-1-d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(n+m-1)(n+m-1)]^\alpha$$
$$+ \sum_{ue\in E(K_{n,m})} \{[n+m-1-d_G(u)](n+m-1)\}^\alpha$$
$$= \sum_{u\nsim v;\ u,v\in V(G)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + \binom{m}{2}(n+m-1)^{2\alpha}$$
$$+ (n+m-1)^\alpha \sum_{u\in V(G)} m[n+m-1-d_G(u)]^\alpha.$$

□

Theorem 2.5 ([6]). *Let G be a graph of order n and size m. Then*

(1). $R_\alpha(T^{00+}(G)) = R_\alpha(S(G)) = 2^\alpha\ M_1^{\alpha+1}$

(2). $R_\alpha(T^{00-}(G)) = R_\alpha(\overline{S}(G)) = (n-2)^\alpha \sum_{u\in V(G)} [m-d_G(u)]^{\alpha+1}$

Theorem 2.6. *Let G be a graph of order n and size m. Then*

(1). $R_\alpha(T^{10+}(G)) = \sum_{u,v\in V(G)} [(n-1+d_G(u))(n-1+d_G(v))]^\alpha + 2^\alpha \sum_{u\in V(G)} d_G(u)[n-1+d_G(u)]^\alpha$

(2). $R_\alpha(T^{-0+}(G)) = (n-1)^{2\alpha}\left[\binom{n}{2} - m\right] + 2^{\alpha+1}m(n-1)^\alpha$

(3). $R_\alpha(T^{01+}(G)) = (m+1)^{2\alpha}\binom{m}{2} + (m+1)^\alpha\ M_1^{\alpha+1}$

(4). $R_\alpha(T^{11+}(G)) = \sum_{u,v\in V(G)} [(n-1+d_G(u))(n-1+d_G(v))]^\alpha + (m+1)^{2\alpha}\binom{m}{2}$
$+ (m+1)^\alpha \sum_{u\in V(G)} d_G(u)[n-1+d_G(u)]^\alpha$

(5). $R_\alpha(T^{+1+}(G)) = 4^\alpha R_\alpha(G) + (m+1)^{2\alpha}\binom{m}{2} + [2(m+1)]^\alpha\ M_1^{\alpha+1}$

(6). $R_\alpha(T^{-1+}(G)) = (n-1)^{2\alpha}\left[\binom{n}{2} - m\right] + (m+1)^{2\alpha}\binom{m}{2} + 2m[(n-1)(m+1)]^\alpha.$

Proof.

$$(1).\ R_\alpha(T^{10+}(G)) = \sum_{uv\in E(K_n)} [(n-1+d_G(u))(n-1+d_G(v)]^\alpha + \sum_{ue\in E(S(G))} \{[n-1+d_G(u)]\cdot 2\}^\alpha$$
$$= \sum_{u,v\in V(G)} [(n-1+d_G(u))(n-1+d_G(v)]^\alpha + 2^\alpha \sum_{u\in V(G)} d_G(u)[n-1+d_G(u)]^\alpha.$$

$$(2).\ R_\alpha(T^{-0+}(G)) = \sum_{uv\in E(\overline{G})} [(n-1)(n-1)]^\alpha + \sum_{ue\in E(S(G))} [(n-1)\cdot 2]^\alpha$$
$$= (n-1)^{2\alpha}\left[\binom{n}{2} - m\right] + 2^{\alpha+1}m(n-1)^\alpha.$$

$$(3).\ R_\alpha(T^{01+}(G)) = \sum_{e_ie_j\in E(K_m)} [(m+1)(m+1)]^\alpha + \sum_{ue\in E(S(G))} [d_G(u)(m+1)]^\alpha$$
$$= (m+1)^{2\alpha}\binom{m}{2} + (m+1)^\alpha \sum_{u\in V(G)} d_G(u)[d_G(u)]^\alpha.$$

(4). $$R_\alpha(T^{11+}(G)) = \sum_{uv\in E(K_n)} [(n-1+d_G(u))(n-1+d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(m+1)(m+1)]^\alpha + \sum_{ue\in E(S(G))} [n-1+d_G(u)(m+1)]^\alpha$$
$$= \sum_{u,v\in V(G)} [(n-1+d_G(u))(n-1+d_G(v))]^\alpha + (m+1)^{2\alpha}\binom{m}{2} + (m+1)^\alpha \sum_{u\in V(G)} d_G(u)[n-1+d_G(u)]^\alpha.$$

(5). $$R_\alpha(T^{+1+}(G)) = \sum_{uv\in E(G)} [(2d_G(u))(2d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(m+1)(m+1)]^\alpha + \sum_{ue\in E(S(G))} [2d_G(u)(m+1)]^\alpha$$
$$= 4^\alpha R_\alpha(G) + (m+1)^{2\alpha}\binom{m}{2} + [2(m+1)]^\alpha \sum_{u\in V(G)} d_G(u)[d_G(u)]^\alpha.$$

(6). $$R_\alpha(T^{-1+}(G)) = \sum_{uv\in E(\overline{G})} [(n-1)(n-1)]^\alpha + \sum_{e_ie_j\in E(K_m)} [(m+1)(m+1)]^\alpha + \sum_{ue\in E(S(G))} [(n-1)(m+1)]^\alpha$$
$$= (n-1)^{2\alpha}\left[\binom{n}{2} - m\right] + (m+1)^{2\alpha}\binom{m}{2} + 2m[(n-1)(m+1)]^\alpha.$$

$\square$

Theorem 2.7. *Let G be a graph of order n and size m. Then*

(1). $R_\alpha(T^{10-}(G)) = \sum_{u,v\in V(G)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + (n-2)^\alpha \sum_{u\in V(G)} [m-d_G(u)]\,[n+m-1-d_G(u)]^\alpha$

(2). $R_\alpha(T^{+0-}(G)) = m^{2\alpha+1} + [m(n-2)]^{\alpha+1}$

(3). $R_\alpha(T^{-0-}(G)) = \sum_{u\nsim v} [(n+m-1-2d_G(u))(n+m-1-2d_G(v))]^\alpha + (n-2)^\alpha \sum_{u\in V(G)} [m-d_G(u)]\,[n+m-1-2d_G(u)]^\alpha$

(4). $R_\alpha(T^{01-}(G)) = (n+m-3)^{2\alpha}\binom{m}{2} + (n+m-3)^\alpha \sum_{u\in V(G)} [m-d_G(u)]^{\alpha+1}$

(5). $R_\alpha(T^{11-}(G)) = \sum_{u,v\in V(G)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + (n+m-3)^{2\alpha}\binom{m}{2}$
$+ (n+m-3)^\alpha \sum_{u\in V(G)} [m-d_G(u)]\,[n+m-1-d_G(u)]^\alpha$

(6). $R_\alpha(T^{+1-}(G)) = m^{2\alpha+1} + (n+m-3)^{2\alpha}\binom{m}{2} + m^{\alpha+1}(n-2)(n+m-3)^\alpha$

(7). $$R_\alpha(T^{-1-}(G)) = \sum_{u\nsim v;\ u,v\in V(G)} [(n+m-1-2d_G(u))(n+m-1-2d_G(v))]^\alpha + (n+m-3)^{2\alpha}\binom{m}{2} + (n+m-3)^\alpha \sum_{u\in V(G)} [m-d_G(u)]\,[n+m-1-2d_G(u)]^\alpha$$

Proof.

(1). $$R_\alpha(T^{10-}(G)) = \sum_{uv\in E(K_n)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + \sum_{u\nsim e}[n+m-1-d_G(u)(n-2)]^\alpha$$
$$= \sum_{u,v\in V(G)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + (n-2)^\alpha \sum_{u\in V(G)} [m-d_G(u)][n+m-1-d_G(u)]^\alpha.$$

(2). $$R_\alpha(T^{+0-}(G)) = \sum_{uv\in E(G)} [m\cdot m]^\alpha + \sum_{u\nsim e}[m(n-2)]^\alpha = m^{2\alpha+1} + m(n-2)[m(n-2)]^\alpha.$$

(3). $$R_\alpha(T^{-0-}(G)) = \sum_{uv\in E(\overline{G})} [(n+m-1-2d_G(u))(n+m-1-2d_G(v))]^\alpha + \sum_{u\nsim e}[(n+m-1-2d_G(u))(n-2)]^\alpha$$
$$= \sum_{u\nsim v}[(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + (n-2)^\alpha \sum_{u\in V(G)} [m-d_G(u)][n+m-1-2d_G(u)]^\alpha.$$

$$(4).\ R_\alpha(T^{01-}(G)) = \sum_{e_ie_j\in E(K_m)} [(n+m-3)(n+m-3)]^\alpha + \sum_{u\nsim e}\{[m-d_G(u)](n+m-3)\}^\alpha$$

$$= (n+m-3)^{2\alpha}\binom{m}{2} + (n+m-3)^\alpha \sum_{u\in V(G)} [m-d_G(u)]\ [m-d_G(u)]^\alpha.$$

$$(5).\ R_\alpha(T^{11-}(G)) = \sum_{uv\in E(K_n)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(n+m-3)(n+m-3)]^\alpha$$

$$+\sum_{u\nsim e}[n+m-1-d_G(u)(n+m-3)]^\alpha$$

$$= \sum_{u,v\in V(G)} [(n+m-1-d_G(u))(n+m-1-d_G(v))]^\alpha$$

$$+ (n+m-3)^{2\alpha}\binom{m}{2} + (n+m-3)^\alpha \sum_{u\in V(G)} [m-d_G(u)][n+m-1-d_G(u)]^\alpha.$$

$$(6).\ R_\alpha(T^{+1-}(G)) = \sum_{uv\in E(G)} [m\cdot m]^\alpha + \sum_{e_ie_j\in E(K_m)} [(n+m-3)(n+m-3)]^\alpha + \sum_{u\nsim e}[m(n+m-3)]^\alpha$$

$$= m^{2\alpha+1} + (n+m-3)^{2\alpha}\binom{m}{2} + m^{\alpha+1}(n-2)(n+m-3)^\alpha$$

$$(7).\ R_\alpha(T^{-1-}(G)) = \sum_{uv\in E(\overline{G})} [(n+m-1-2d_G(u))(n+m-1-2d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(n+m-3)(n+m-3)]^\alpha$$

$$+\sum_{u\nsim e}[(n+m-1-2d_G(u))(n+m-3)]^\alpha$$

$$= \sum_{u\nsim v;\ u,v\in V(G)} [(n+m-1-2d_G(u))(n+m-1-2d_G(v))]^\alpha + (n+m-3)^{2\alpha}\binom{m}{2}$$

$$+ (n+m-3)^\alpha \sum_{u\in V(G)} [m-d_G(u)][n+m-1-2d_G(u)]^\alpha.$$

□

Theorem 2.8. *Let G be an $(n,\, m)$ graph and $\alpha < 0$. Then*

(1). $$R_\alpha(T^{0+1}(G)) \le (n-2+2\delta)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m^\alpha n \sum_{uv\in E(G)} [n-2+d_G(u)+d_G(v)]^\alpha$$

$$R_\alpha(T^{0+1}(G)) \ge (n-2+2\Delta)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m^\alpha n \sum_{uv\in E(G)} [n-2+d_G(u)+d_G(v)]^\alpha$$

(2). $$R_\alpha(T^{1+1}(G)) \le \binom{n}{2}(n+m-1)^{2\alpha} + [n-2+2\delta]^\alpha\left[\frac{1}{2}M_1 - m\right] + (n+m-1)^\alpha n \sum_{uv\in E(G)} [n-2+d_G(u)+d_G(v)]^\alpha$$

$$R_\alpha(T^{1+1}(G)) \ge \binom{n}{2}(n+m-1)^{2\alpha} + [n-2+2\Delta]^\alpha\left[\frac{1}{2}M_1 - m\right] + (n+m-1)^\alpha n \sum_{uv\in E(G)} [n-2+d_G(u)+d_G(v)]^\alpha$$

(3). $$R_\alpha(T^{++1}(G)) \le \sum_{uv\in E(G)} [(m+d_G(u))(m+d_G(v))]^\alpha + [n-2+2\delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + nm(m+\delta)^\alpha\,[n-2+2\delta]^\alpha$$

$$R_\alpha(T^{++1}(G)) \ge \sum_{uv\in E(G)} [(m+d_G(u))(m+d_G(v))]^\alpha + [n-2+2\Delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + nm(m+\Delta)^\alpha\,[n-2+2\Delta]^\alpha$$

(4). $$R_\alpha(T^{-+1}(G)) \le [n+m-1-\delta]^{2\alpha}\left[\binom{n}{2} - m\right] + [n-2+2\delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + mn[n+m-1-\delta]^\alpha[n-2+2\Delta]^\alpha$$

$$R_\alpha(T^{-+1}(G)) \ge [n+m-1-\Delta]^{2\alpha}\left[\binom{n}{2} - m\right] + [n-2+2\Delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + mn[n+m-1-\delta]^\alpha[n-2+2\Delta]^\alpha.$$

Proof.

(1). $$R_\alpha(T^{0+1}(G)) = \sum_{e_ie_j\in E(T^{0+1}(G))\cap E(L(G))} [d_{T^{0+1}(G)}(e_i)d_{T^{0+1}(G)}(e_j)]^\alpha + \sum_{ue\in E(K_{n,m})} [d_{T^{0+1}(G)}(u)d_{T^{0+1}(G)}(e)]^\alpha$$
$$= \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw} [(n-2+d_G(u)+d_G(v))(n-2+d_G(v)+d_G(w))]^\alpha$$
$$+ \sum_{ue\in E(K_{n,m})} \{m[n-2+d_G(u)+d_G(v)]\}^\alpha$$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$, we have

$$R_\alpha(T^{0+1}(G)) \leq (n-2+2\delta)^{2\alpha}\left[\frac{1}{2}M_1-m\right] + m^\alpha n \sum_{uv\in E(G)} [n-2+d_G(u)+d_G(v)]^\alpha$$

Similarly, we can compute the other side inequality.

(2). $$R_\alpha(T^{1+1}(G)) = \sum_{uv\in E(K_n)} [(n+m-1)(n+m-1)]^\alpha + \sum_{ue\in E(K_{n,m})} [(n+m-1)(n-2+d_G(u)+d_G(v))]^\alpha$$
$$+ \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw} \{[n-2+d_G(u)+d_G(v)][n-2+d_G(v)+d_G(w)]\}^\alpha$$
$$= \left[\sum_{uv\in E(K_n)} (n+m-1)^{2\alpha}\right] + (n+m-1)^\alpha \sum_{ue\in E(K_{n,m})} [n-2+d_G(u)+d_G(v)]^\alpha$$
$$+ \sum_{e_i\sim e_j,e_i=uv,e_j=vw} \{[n-2+d_G(u)+d_G(v)][n-2+d_G(v)+d_G(w)]\}^\alpha$$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$, we have

$$R_\alpha(T^{1+1}(G)) \leq \binom{n}{2}(n+m-1)^{2\alpha} + (n-2+2\delta)^{2\alpha}\left[\frac{1}{2}M_1-m\right] + (n+m-1)^\alpha n \sum_{uv\in E(G)} [n-2+d_G(u)+d_G(v)]^\alpha$$

Similarly, we can compute the other side inequality.

(3). $$R_\alpha(T^{++1}(G)) = \sum_{uv\in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + \sum_{ue\in E(K_{n,m})} \{[m+d_G(u)][n-2+d_G(v)+d_G(w)]\}^\alpha$$
$$+ \sum_{e_i\sim e_j,e_i=uv,e_j=vw} \{[n-2+d_G(u)+d_G(v)][n-2+d_G(v)+d_G(w)]\}^\alpha$$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$, we have

$$R_\alpha(T^{++1}(G)) \leq \sum_{uv\in E(G)} [(m+d_G(u))(m+d_G(v))]^\alpha + (n-2+2\delta)^{2\alpha}\left[\frac{1}{2}M_1-m\right] + nm(m+\delta)^\alpha(n-2+2\delta)^\alpha$$

Similarly, we can compute the other side inequality.

(4). $$R_\alpha(T^{-+1}(G)) = \sum_{uv\in E(\overline{G})} \{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha$$
$$+ \sum_{ue\in E(K_{n,m}),e=vw} \{[n+m-1-d_G(u)][n-2+d_G(v)+d_G(w)]\}^\alpha$$
$$+ \sum_{e_i\sim e_j,e_i=uv,e_j=vw} \{[n-2+d_G(u)+d_G(v)][n-2+d_G(v)+d_G(w)]\}^\alpha$$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$, we have

$$R_\alpha(T^{-+1}(G)) \leq (n+m-1-\delta)^{2\alpha}\left[\binom{n}{2}-m\right] + (n-2+2\delta)^{2\alpha}\left[\frac{1}{2}M_1-m\right] + mn(n+m-1-\delta)^\alpha(n-2+2\Delta)^\alpha$$

Similarly, we can compute the other side inequality. □

Theorem 2.9. *Let G be an $(n,\ m)$ graph and $\alpha<0$. Then*

(1). $R_\alpha(T^{0-1}(G)) \leq (n+m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right] + m^\alpha n \sum_{uv\in E(G)} [n+m+1-d_G(u)-d_G(v)]^\alpha$

$R_\alpha(T^{0-1}(G)) \geq (n+m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right] + m^\alpha n \sum_{uv\in E(G)} [n+2m+1-d_G(u)-d_G(v)]^\alpha$

(2). $R_\alpha(T^{1-1}(G)) \leq (n+m-1)^{2\alpha}\binom{n}{2} + (n+m-1-2\delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$

$+ (n+m-1)^\alpha n \sum_{uv\in E(G)} [n+m+1-d_G(u)-d_G(v)]^\alpha$

$R_\alpha(T^{1-1}(G)) \geq (n+m-1)^{2\alpha}\binom{n}{2} + (n+m-1-2\Delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$

$+ (n+m-1)^\alpha n \sum_{uv\in E(G)} [n+m+1-d_G(u)-d_G(v)]^\alpha$

(3). $R_\alpha(T^{+-1}(G)) \leq \sum_{uv\in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + (n+m-1-2\delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$

$+ mn(m+\delta)^\alpha(n+m-1-2\delta)^{2\alpha}$

$R_\alpha(T^{+-1}(G)) \geq \sum_{uv\in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + (n+m-1-2\Delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$

$+ mn(m+\Delta)^\alpha(n+m-1-2\Delta)^{2\alpha}$

(4). $R_\alpha(T^{--1}(G)) \leq (n+m-1-\delta)^{2\alpha}\left[\binom{n}{2}-m\right] + (n+m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$

$+ mn(n+m-1-\delta)^\alpha(n+m-1-2\delta)^\alpha$

$R_\alpha(T^{--1}(G)) \geq (n+m-1-\Delta)^{2\alpha}\left[\binom{n}{2}-m\right] + (n+m+1-2\Delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$

$+ mn(n+m-1-\Delta)^\alpha(n+m-1-2\Delta)^\alpha.$

Proof.

(1). $R_\alpha(T^{0-1}(G)) = \sum_{e_i\nsim e_j, e_i=uv, e_j=wx} \{[n+m+1-d_G(u)-d_G(v)][n+m+1-d_G(w)-d_G(x)]\}^\alpha$

$+ \sum_{ue\in E(K_{n,m}), e=uv} \{m[n+m+1-d_G(u)-d_G(v)]\}^\alpha$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$, we have

$$R_\alpha(T^{0-1}(G)) \leq (n+m+1-2\delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right] + m^\alpha n \sum_{uv\in E(G)} [n+m+1-d_G(u)-d_G(v)]^\alpha$$

Similarly, we can compute the other side inequality.

(2). $R_\alpha(T^{1-1}(G)) = \sum_{uv\in E(K_n)} [(n+m-1)(n+m-1)]^\alpha$

$+ \sum_{e_i\nsim e_j, e_i=uv, e_j=wx} [n+m+1-d_G(u)-d_G(v)][n+m+1-d_G(w)-d_G(x)]^\alpha$

$+ \sum_{ue\in E(K_{n,m}), e=uv} \{(n+m-1)[n+m+1-d_G(u)-d_G(v)]\}^\alpha$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$, we have

$$R_\alpha(T^{1-1}(G)) \leq (n+m-1)^{2\alpha}\binom{n}{2} + (n+m-1-2\delta)^{2\alpha}\left[\binom{m+1}{2}-\frac{1}{2}M_1\right]$$
$$+ (n+m-1)^\alpha n \sum_{uv\in E(G)} [n+m+1-d_G(u)-d_G(v)]^\alpha$$

Similarly, we can compute the other side inequality.

$$(3).\ R_\alpha(T^{+-1}(G)) = \sum_{uv\in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + \sum_{e_i \nsim e_j, e_i=uv, e_j=wx} \{[n+m+1-d_G(u)-d_G(v)][n+m+1-d_G(w)-d_G(x)]\}^\alpha + \sum_{ue\in E(K_{n,m}), e=uv} \{[m+d_G(u)][n+m+1-d_G(u)-d_G(v)]\}^\alpha$$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{+-1}(G)) \leq \sum_{uv\in E(G)} \{[m+d_G(u)][m+d_G(v)]\}^\alpha + [n+m-1-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + mn(m+\delta)^\alpha[n+m-1-2\delta]^{2\alpha}.$$

Similarly, we can compute the other side inequality.

$$(4).\ R_\alpha(T^{--1}(G)) = \sum_{uv\in E(\overline{G})} \{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha + \sum_{e_i \nsim e_j, e_i=uv, e_j=wx} \{[n+m+1-d_G(u)-d_G(v)][n+m+1-d_G(w)-d_G(x)]\}^\alpha + \sum_{ue\in E(K_{n,m}), e=uv} \{[n+m-1-d_G(u)][n+m+1-d_G(u)-d_G(v)]\}^\alpha$$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{--1}(G)) \leq (n+m-1-\delta)^{2\alpha}\left[\binom{n}{2} - m\right] + [n+m+1-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + mn(n+m-1-\delta)^\alpha[n+m-1-2\delta]^\alpha$$

Similarly, we can prove the other side inequality. $\square$

Theorem 2.10. *Let G be an (n, m) graph and $\alpha < 0$. Then*

$$(1).\ R_\alpha(T^{0-+}(G)) \leq [m+3-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + \sum_{u\in V(G)} [d_G(u)^\alpha + d_G(v)^\alpha][m+3-d_G(u)-d_G(v)]^\alpha$$

$$R_\alpha(T^{0-+}(G)) \geq [m+3-2\Delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + \sum_{u\in V(G)} [d_G(u)^\alpha + d_G(v)^\alpha][m+3-d_G(u)-d_G(v)]^\alpha$$

$$(2).\ R_\alpha(T^{1-+}(G)) \leq [n-1+\delta]^{2\alpha}\binom{n}{2} + (m+3-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + 2m(n-1+\delta)^\alpha[m+3-2\delta]^\alpha$$

$$R_\alpha(T^{1-+}(G)) \geq [n-1+\Delta]^{2\alpha}\binom{n}{2} + (m+3-2\Delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + 2m(n-1+\Delta)^\alpha[m+3-2\Delta]^\alpha$$

$$(3).\ R_\alpha(T^{1++}(G)) \leq [n-1+\delta]^{2\alpha}\binom{n}{2} + (2\delta)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + 2^{2\alpha+1}m\,\delta^\alpha(n-1+\delta)^\alpha$$

$$R_\alpha(T^{1++}(G)) \geq [n-1+\Delta]^{2\alpha}\binom{n}{2} + (2\Delta)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + 2^{2\alpha+1}m\,\Delta^\alpha(n-1+\Delta)^\alpha$$

$$(4).\ R_\alpha(T^{0+-}(G)) \leq [n-4+2\delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m(n-2)(m-\delta)^\alpha[n-4+2\delta]^\alpha$$

$$R_\alpha(T^{0+-}(G)) \geq [n-4+2\Delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m(n-2)(m-\Delta)^\alpha[n-4+2\Delta]^\alpha$$

(5). $R_\alpha(T^{1+-}(G)) \leq (n+m-1-\delta)^{2\alpha}\binom{n}{2} + 2^\alpha[n-4+2\delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m(n-2)(n+m-1-\delta)^\alpha[n-4+2\delta]^\alpha$

$$R_\alpha(T^{1+-}(G)) \geq (n+m-1-\Delta)^{2\alpha}\binom{n}{2} + 2^\alpha[n-4+2\Delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m(n-2)(n+m-1-\Delta)^\alpha[n-4+2\Delta]^\alpha$$

(6). $R_\alpha(T^{0--}(G)) \leq [n+m-1-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + m(n-2)(m-\delta)^\alpha(n+m-1-2\delta)^\alpha$

$$R_\alpha(T^{0--}(G)) \geq [n+m-1-2\Delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + m(n-2)(m-\Delta)^\alpha(n+m-1-2\Delta)^\alpha$$

(7). $R_\alpha(T^{1--}(G)) \leq (n+m-1-\delta)^{2\alpha}\binom{n}{2} + [n+m-1-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right].$

$$+ m(n-2)(n+m-1-\delta)^\alpha(n+m-1-2\delta)^\alpha$$

$$R_\alpha(T^{1--}(G)) \geq (n+m-1-\Delta)^{2\alpha}\binom{n}{2} + [n+m-1-2\Delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right]$$

$$+ m(n-2)(n+m-1-\Delta)^\alpha(n+m-1-2\Delta)^\alpha$$

Proof.

(1).
$$R_\alpha(T^{0-+}(G)) = \sum_{e_ie_j\in E(\overline{L(G)}), e_i=uv, e_j=wx} \{[m+3-d_G(u)-d_G(v)][m+3-d_G(w)-d_G(x)]\}^\alpha$$
$$+ \sum_{ue\in E(S(G)), e=uv} [d_G(u)(m+3-d_G(u)-d_G(v))]^\alpha$$
$$= \sum_{e_i\nsim e_j, e_i=uv, e_j=wx} \{[m+3-d_G(u)-d_G(v)][m+3-d_G(w)-d_G(x)]\}^\alpha$$
$$+ \left[\sum_{u\in V(G)} [d_G(u)^\alpha + d_G(v)^\alpha][m+3-d_G(u)-d_G(v)]^\alpha\right]$$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{0-+}(G)) \leq (m+3-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + \sum_{u\in V(G)} [d_G(u)^\alpha + d_G(v)^\alpha][m+3-d_G(u)-d_G(v)]^\alpha.$$

Similarly, we can compute the other side inequality.

(2).
$$R_\alpha(T^{1-+}(G)) = \sum_{uv\in E(K_n)} \{[n-1+d_G(u)][n-1+d_G(v)]\}^\alpha$$
$$+ \sum_{e_i\nsim e_j, e_i=uv, e_j=wx} \{[m+3-d_G(u)-d_G(v)][m+3-d_G(w)-d_G(x)]\}^\alpha$$
$$+ \sum_{ue\in E(S(G)), e=uv} \{[n-1+d_G(u)][m+3-d_G(u)-d_G(v)]\}^\alpha$$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{1-+}(G)) \leq (n-1+\delta)^{2\alpha}\binom{n}{2} + (m+3-2\delta)^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + 2m(n-1+\delta)^\alpha(m+3-2\delta)^\alpha.$$

(3).
$$R_\alpha(T^{1++}(G)) = \sum_{uv\in E(K_n)} \{[n-1+d_G(u)][n-1+d_G(v)]\}^\alpha + \sum_{e_i\sim e_j, e_i=uv, e_j=vw} \{[d_G(u)+d_G(v)][d_G(v)+d_G(w)]\}^\alpha$$
$$+ \sum_{ue\in E(S(G)), e=uv} \{[n-1+d_G(u)][d_G(u)+d_G(v)]\}^\alpha$$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{1++}(G)) \leq (n-1+\delta)^{2\alpha}\binom{n}{2} + (2\delta)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + 2^{2\alpha+1}m\,\delta^\alpha(n-1+\delta)^\alpha.$$

(4). $R_\alpha(T^{0+-}(G)) = \sum_{e_i \sim e_j, e_i=uv, e_j=vw} \{[n-4+d_G(u)+d_G(v)][n-4+d_G(v)+d_G(w)]\}^\alpha$

$+ \sum_{u \nsim e, e=vw} \{[m-d_G(u)][n-4+d_G(v)+d_G(w)]\}^\alpha$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{0+-}(G)) \leq (n-4+2\delta)^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m(n-2)[m-\delta]^\alpha(n-4+2\delta)^\alpha.$$

(5). $R_\alpha(T^{1+-}(G)) = \sum_{uv \in E(K_n)} \{[n+m-1-d_G(u)][n+m-1-d_G(v)]\}^\alpha$

$+ \sum_{e_i \sim e_j, e_i=uv, e_j=vw} \{[n-4+d_G(u)+d_G(v)][n-4+d_G(v)+d_G(w)]\}^\alpha$

$+ \sum_{u \nsim e, e=vw} \{[n+m-1-d_G(u)][n-4+d_G(v)+d_G(w)]\}^\alpha$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{1+-}(G)) \leq [n+m-1-\delta]^{2\alpha}\binom{n}{2} + [n-4+2\delta]^{2\alpha}\left[\frac{1}{2}M_1 - m\right] + m(n-2)(n+m-1-\delta)^\alpha[n-4+2\delta]^\alpha.$$

(6). $R_\alpha(T^{0--}(G)) = \sum_{e_i \nsim e_j, e_i=uv, e_j=wx} \{[n+m-1-d_G(u)-d_G(v)][n+m-1-d_G(w)-d_G(x)]\}^\alpha$

$+ \sum_{u \nsim e, e=vw} \{[m-d_G(u)][n+m-1-d_G(v)-d_G(w)]\}^\alpha$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{0--}(G)) \leq [n+m-1-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + m(n-2)(m-\delta)^\alpha(n+m-1-2\delta)^\alpha.$$

(7). $R_\alpha(T^{1--}(G)) = \sum_{uv \in E(K_n)} [(n+m-1-d_G(u)(n+m-1-d_G(v)]^\alpha$

$+ \sum_{e_i \nsim e_j, e_i=uv, e_j=wx} [2(n+m-1)-d_G(u)-d_G(v)-d_G(w)-d_G(x)]^\alpha$

$+ \sum_{u \nsim e, e=vw} \{[n+m-1-d_G(u)][n+m-1-d_G(v)-d_G(w)]\}^\alpha$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$, we have

$$R_\alpha(T^{1--}(G)) \leq (n+m-1-\delta)^{2\alpha}\binom{n}{2} + [n+m-1-2\delta]^{2\alpha}\left[\binom{m+1}{2} - \frac{1}{2}M_1\right] + m(n-2)(n+m-1-\delta)^\alpha(n+m-1-2\delta)^\alpha.$$

One can analogously compute the other side inequalities of above graphs. □

3. Conclusion

In this paper, we have obtained the bounds for general product-connectivity index of some generalized xyz–point-line transformation graphs. In order to obtain sharp bounds, we compute the expressions in terms of order, size, maximum vertex degree, minimum vertex degree and summation over vertices (or edges) of G. Also note that, if $\alpha > 0$, then the opposite side inequality is valid.

Acknowledgement

This research is supported by UGC-SAP DRS-III, New Delhi, India for 2016-2021: F.510/3/DRS-III/2016(SAP-I) Dated: 29^{th} Feb. 2016.

References

[1] B. Basavanagoud, *Basic properties of generalized xyz–point-line transformation graphs*, J. Inf. Optim. Sci., 39(2)(2018), 561-580.

[2] B. Basavanagoud and P. V. Patil, *A criterion for (non-)planarity of the transformation graph* G^{xyz} *when* $xyz = -++$, J. Discrete Math. Sci. Crypt., 13(2010), 601-610.

[3] B. Basavanagoud, I. Gutman and C. S. Gali, *On second Zagreb index and coindex of some derived graphs*, Kragujevac J. Sci., 37(2015), 113-121.

[4] B. Basavanagoud and C. S. Gali, *Computing first and second Zagreb indices of generalized xyz−point-line transformation graphs*, J. Glob. Res. Math. Arch., 5(4)(2018), 100-122.

[5] B. Basavanagoud and C. S. Gali, *On general sum-connectivity index of generalized xyz−point-line transformation graphs*, International J. Math. Sci. Engg. Appls., 12(I)(2018), 123-142.

[6] B. Basavanagoud and C. S. Gali, *On the general product-connectivity index of transformation graphs*, Int. J. Sci. Res. Math. Stat. Sci., 5(5)(2018) 33-40.

[7] B. Bollobás and P.Erdös, *Graphs of extremal weights*, Ars Combinatoria, 50(1998), 225-233.

[8] A. Deng, A. Kelmans and J. Meng, *Laplacian spectra of regular graphs transformations*, Discrete Appl. Math., 161(2013), 118-133.

[9] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals, Total* $\pi-$*electron energy of alternant hydrocarbons*, Chem. Phys. Lett., 17(1972), 535-538.

[10] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, (1969).

[11] S. M. Hosamani and I. Gutman, *Zagreb indices of transformation graphs and total transformation graphs*, Appl. Math. Comput., 247(2014), 1156-1160.

[12] G. Indulal and A. Vijayakumar, *A note on energy of some graphs*, MATCH Commun. Math. Comput. Chem., 59(2008), 269-274.

[13] M. Karelson, *Graph Theory: Molecular Descriptors in QSAR/QSPR*, New York, USA: Wiley-Interscience, (2000).

[14] V. R. Kulli, *Graph Theory: College Graph Theory*, Vishwa International Publications, (2012).

[15] V. R. Kulli and B. Basavanagoud, *On the quasivertex-total graph of a graph*, J. Karnatak Univ. Sci., 42(1998), 1-7.

[16] X. Li and H. Zhao, *Trees with the first three smallest and largest generalized topological indices*, MATCH Commun . Math. Comput. Chem., 50(2004), 57-62.

[17] M. Randić, *On characterization of molecular branching*, J. of the Am. Chem. Soc., 97(1975), 6609-6615.

[18] E. Sampathkumar and S. B. Chikkodimath, *Semitotal graphs of a graph-I*, J. Karnatak Univ. Sci., 18(1973), 274-280.

[19] B. Wu and J. Meng, *Basic properties of total transformation graphs*, J. Math. Study, 34(2001), 109-116.