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An Alternative Proof for an Inequality with Cyclic Fractions

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Abstract: It is known that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$ for any positive a,b,c. In this paper we provide an alternative proof for this inequality, and generalize it to an n-variable version.

MSC: 26D15.

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1. Background

While studying inequalities, we found that Cvetkovski introduced the following inequality in his book ([1], Exercise 304).

Proposition 1.1. *For any positive real numbers a,b,c, we have*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}.$$

The proof provided in the book is as follow.

Proof. Without loss of generality, we may assume that $c=min\{a,b,c\}$.

Since $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-3=\frac{1}{xy}(x-y)^2+\frac{1}{xz}(x-z)(y-z)$, the claimed inequality is equivalent to

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+c}-3,$$

or

$$\frac{1}{ab}(a-b)^2+\frac{1}{ac}(a-c)(b-c)\geq\frac{1}{(a+c)(b+c)}(a-b)^2+\frac{1}{(a+c)(a+b)}(b-c)(a-c).$$

After reorganizing the above expression, we only need to prove that

$$\left(\frac{1}{ab}-\frac{1}{(a+c)(b+c)}\right)(a-b)^2+\left(\frac{1}{ac}-\frac{1}{(a+c)(a+b)}\right)(a-c)(b-c)\geq 0.$$

Since $c=min\{a,b,c\}$, we have $\left(\frac{1}{ab}-\frac{1}{(a+c)(b+c)}\right)\geq 0$, $\left(\frac{1}{ac}-\frac{1}{(a+c)(a+b)}\right)\geq 0$, and $(a-c)(b-c)\geq 0$. Therefore, the last inequality is apparently true. □

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This proof is very technical, and may be hard to apply on versions of more than three variables. In the next section we will provide another proof. With the new method, we also successfully generalize the inequality to an n-variable case. In the new proof, we apply the AM-GM inequality. This inequality can be found in many inequality focused books. Here we refer to Cvetkovski's book [1] as our main source of notations.

Theorem 1.2 (AM-GM Inequality)**.** *Let $a_1, a_2, \cdots, a_n$ be positive real numbers. The numbers $AM = \frac{a_1+a_2+\cdots+a_n}{n}$ and $GM = \sqrt[n]{a_1 a_2 \cdots a_n}$ are called the arithmetic mean and geometric mean for the numbers $a_1, a_2, \cdots, a_n$, respectively, and we have $AM \geq GM$. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.*

2. An Alternative Proof

Instead of considering $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} - 3$ like the proof in [1], we consider $\frac{x}{y} - \frac{x+y}{y+z}$ in the new proof.

Proof. Since $\frac{x}{y} - \frac{x+y}{y+z} = \frac{xz-y^2}{y^2+yz}$, the original inequality, which can be arranged to

$$\left(\frac{a}{b} - \frac{a+b}{b+c}\right) + \left(\frac{b}{c} - \frac{b+c}{c+a}\right) + \left(\frac{c}{a} - \frac{c+a}{a+b}\right) \geq 0,$$

is now equivalent to

$$\frac{ac-b^2}{b^2+bc} + \frac{ba-c^2}{c^2+ca} + \frac{cb-a^2}{a^2+ab} \geq 0.$$

We notice that

$$\begin{aligned} \frac{ac-b^2}{b^2+bc} + \frac{ba-c^2}{c^2+ca} + \frac{cb-a^2}{a^2+ab} &= \frac{\frac{a}{b}-\frac{b}{c}}{\frac{b}{c}+1} + \frac{\frac{b}{c}-\frac{c}{a}}{\frac{c}{a}+1} + \frac{\frac{c}{a}-\frac{a}{b}}{\frac{a}{b}+1} \\ &= \left(\frac{\frac{a}{b}+1}{\frac{b}{c}+1} - 1\right) + \left(\frac{\frac{b}{c}+1}{\frac{c}{a}+1} - 1\right) + \left(\frac{\frac{c}{a}+1}{\frac{a}{b}+1} - 1\right), \end{aligned}$$

so we only need to prove that

$$\frac{\frac{a}{b}+1}{\frac{b}{c}+1} + \frac{\frac{b}{c}+1}{\frac{c}{a}+1} + \frac{\frac{c}{a}+1}{\frac{a}{b}+1} \geq 3.$$

Applying the AM-GM inequality, we have

$$\frac{\frac{a}{b}+1}{\frac{b}{c}+1} + \frac{\frac{b}{c}+1}{\frac{c}{a}+1} + \frac{\frac{c}{a}+1}{\frac{a}{b}+1} \geq 3\left[\left(\frac{\frac{a}{b}+1}{\frac{b}{c}+1}\right)\left(\frac{\frac{b}{c}+1}{\frac{c}{a}+1}\right)\left(\frac{\frac{c}{a}+1}{\frac{a}{b}+1}\right)\right]^{\frac{1}{3}} = 3$$

indeed. □

Notice that, in this proof we create a sum of cyclic complex fractions in the last inequality. That means if we generalize the original inequality to a random n-variable version, the technique in the proof can still apply. Therefore we have the following generalization.

Proposition 2.1. *For any positive integer n and positive real numbers $a_1, a_2, \cdots, a_n$, we have*

$$\sum_{i=1}^{n} \frac{a_i}{a_{i+1}} \geq \sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_{i+1} + a_{i+2}},$$

in which, $a_{n+1} = a_1$ and $a_{n+2} = a_2$.

The proof of Proposition 2.1 is almost identical to the last proof.

References

[1] Z.Cvetkovski, *Inequalities: Theorems, Techniques and Selected Problems*, Springer, (2012).