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Two Heterogeneous Service Queue With Second Optional Repair Subject to Reneging During Compulsory Vacation and Breakdown Time

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Abstract: This paper deals with a single server batch arrival queue, two stages of heterogeneous service with different (arbitrary) service time distribution subject to random breakdowns followed by a repair and compulsory server vacations with general (arbitrary) vacation periods. After first stage service the server must provide second stage service. However after the completion of each second stage service the server will take compulsory vacation. The system may breakdown at random and it must be send to repair process immediately. If the server could not be repaired with first essential repair, subsequent repairs are needed for the restoration of the server. Both first essential repair and second optional repair times follow exponential distribution. We consider reneging to occur when the server is unavailable during the system breakdown or vacation periods. The steady state solutions have been found by using supplementary variable technique.

Keywords: Batch Arrivals, Breakdowns, Steady state, First essential repair, Second optional repair, Reneging.

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1. Introduction

Batch queueing models have been analyzed in the past by several authors. Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Application of vacation models can be found in production line systems, designing local area networks and data communication systems. The application of the models in a real life situation in a foundry involving sand casting. In sand blasting department of a foundry, the orders are come in batch. Once the process is started, the operation has to be continued successively for two services. First service is making the sand molded casting, second service is dyeing. After completing a blasting operation, then the operator stops the blasting process and performs the knock out process, which involves removing the casting from the sand mould. In terms of queueing terminology, the knock out process may be named as a vacation (compulsory). The moulding machine may breakdown at random and it must be send to repair process immediately. If the operator could not be repaired with first essential repair, subsequent repairs are needed for the restoration of the server. We consider reneging to occur when the operator is unavailable during the system breakdown or vacation periods

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2. Mathematical Description of the Model

We assume the following to describe the queueing model of our study. Let $\lambda k_i dt$, $i = 1, 2, 3, \ldots$ be the first order probability of arrival of 'I' customers in batches in the system during a short period of time $(t, t + dt]$, where $0 \leq k_i \leq 1$; $\sum_{i=}^{\infty} k_i = 1$, $\lambda > 0$ is the mean arrival rate of batches. The single server provides the two stages of heterogeneous service to all arriving customers. Let $B_1(v)$ and $b_1(v)$ be the distribution function and the density function of the two stages of heterogeneous service times respectively. After first stage service the server must provide second stage service. The service is assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$. Let $\mu_j(x)dx$ be the conditional probability of completion of the j$^{\text{th}}$ stage of service during the interval $(x, x + d]$ given that elapsed time is x, so that

$$\mu_j(x) = \frac{bj(x)}{1 - B_j(x)}, \ j = 1, 2$$

and therefore,

$$bj\,(x) = \mu_j(x) e^{-\int\limits_0^{\infty} \mu_j(x)dx}, \quad j = 1, 2$$

after the completion of each second stage service the server will take compulsory vacation. The vacation periods follow general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\eta(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that elapsed vacation time is x, so that

$$\eta(x) = \frac{v(x)}{1 - V(x)}$$

and therefore, $v(s) = \eta(s) e^{-\int\limits_0^{\infty} \eta(x)dx}$. The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$, as soon as the server is broken down, it is immediately sent for repair where in the repairman or repairing apparatus provides the essential repair (FER). After the completion of FER, the server may opt for second optional repair (SOR) with probability r or may join the system with complementary probability 1-r to render the service to the customers. The repair process provides two types of repair in which the first type of repair is essential and the second type of repairs is optional. Both exponentially distributed with mean $\frac{1}{\beta_1}$ and $\frac{1}{\beta_2}$. After completion of the required repair, the server provides service with the same efficiency as before failure according to FCFS discipline. Assuming that the batch arrival units get impatient after joining the queue and leave the system without getting service. This behavior of customer recognizes as reneging. The probability that an arriving customer reneging during the breakdown or during the time when the server takes vacation. Reneging is assumed to follows exponential distribution with probability ν.

3. Definitions and Equations Governing the System

We define $P_n^{(i)}\,(x, t) =$ Probability that at time t, the server is active providing i$^{\text{th}}$ service and there are n $(n \geq 1)$ customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(i)}(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in i$^{\text{th}}$ service irrespective of the value of x. $V_n\,(x, t) =$ Probability that at time t, the server is on vacation with elapsed vacation time x, and there are n $(n \geq 1)$ customers waiting in the queue for service. Consequently $V_n\,(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

$R_n^{(1)}\,(x, t) =$ Probability that at time t, the server is inactive due to breakdown and the system is under first essential repair while there are in n $(n \geq 0)$ customers in the queue.

$R_n^{(2)}(x,t)$ = Probability that at time t, the server is inactive due to breakdown and the system is under second optional repair while there are in n $(n \geq 0)$ customers in the queue.

$Q(t)$ = Probability that at time t, there are no customers in the system and the server is idle but available in the system.

The model is then, governed by the following set of differential-difference equations

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_0^{(1)}(x,t) = 0 \tag{1}$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x,t) = \lambda\sum_{i=1}^{\infty}K_iP_{n-i}^{(1)}(x,t), \quad n \geq 1 \tag{2}$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha)P_0^{(2)}(x,t) = 0 \tag{3}$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha)P_n^{(2)}(x,t) = \lambda\sum_{i=1}^{\infty}K_iP_{n-i}^{(2)}(x,t), \quad n \geq 1 \tag{4}$$

$$\frac{\partial}{\partial t}V_0(t) + \frac{\partial}{\partial x}V_0(x,t) + (\lambda + \eta(x))V_0(x,t) = \nu V_0(x,t) \tag{5}$$

$$\frac{\partial}{\partial t}V_n(x,t) + \frac{\partial}{\partial x}V_n(x,t) + (\lambda + \eta(x))V_n(x,t) = \lambda\sum_{i=1}^{\infty}K_iV_{n-i}(x,t) + \nu V_{n+1}(x,t), \quad n \geq 1 \tag{6}$$

$$\frac{\partial}{\partial t}R_0^{(1)}(x,t) + \frac{\partial}{\partial x}R_0^{(1)}(x,t) + (\lambda + \beta_1)R_0^{(1)}(x,t) = \nu R_0^{(1)}(x,t) \tag{7}$$

$$\begin{aligned}\frac{\partial}{\partial t}R_n^{(1)}(x,t) + \frac{\partial}{\partial x}R_n^{(1)}(x,t) + (\lambda + \beta_1)R_n^{(1)}(x,t) &= \lambda\sum_{i=1}^{n}K_iR_{n-i}^{(1)}(x,t) + \nu R_n^{(1)}(x,t)\\ &\quad + \alpha\int_0^{\infty}P_{n-1}^{(1)}(x,t) + \alpha\int_0^{\infty}P_{n-1}^{(2)}(x,t), \quad n \geq 1\end{aligned} \tag{8}$$

$$\frac{\partial}{\partial t}R_0^{(2)}(x,t) + \frac{\partial}{\partial x}R_0^{(2)}(x,t) + (\lambda + \beta_2)R_0^{(2)}(x,t) = \nu(r\beta_1)R_0^{(1)}(x,t) \tag{9}$$

$$\frac{\partial}{\partial t}R_n^{(2)}(x,t) + \frac{\partial}{\partial x}R_n^{(2)}(x,t) + (\lambda + \beta_2)R_n^{(2)}(x,t) = \lambda\sum_{i=1}^{n}K_iR_{n-i}^{(2)}(x,t) + \nu(r\beta_1)R_n^{(1)}(x,t), \quad n \geq 1 \tag{10}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + (1-r)\beta_1R_0^{(1)}(t) + \beta_2R_0^{(2)}(t) + \int_0^{8}V_0(x,t)\eta(x)\,dx \tag{11}$$

Equations are to be solved subject to the following boundary conditions:

$$P_0^{(1)}(0,t) = \lambda c_1Q(t) + (1-r)\beta_1R_1^{(1)}(t) + \beta_2R_1^{(2)}(t) + \int_0^{\infty}V_n(x,t)\eta(x)\,dx \tag{12}$$

$$P_n^{(1)}(0,t) = \lambda c_{n+1}Q(t) + (1-r)\beta_1R_{n+1}^{(1)}(t) + \beta_2R_{n+1}^{(2)}(t) + \int_0^{\infty}V_{n+1}(x,t)\eta(x)\,dx \tag{13}$$

$$P_n^{(2)}(0,t) = \int_0^{\infty}P_n^{(1)}(x,t)\mu_1(x)\,dx, \quad n \geq 1 \tag{14}$$

$$V_n(0,t) = \int_0^{\infty}P_n^{(2)}(x,t)\mu_2(x)\,dx, \quad n \geq 1 \tag{15}$$

4. Probability Generating Functions of the Queue Size

We define the probability generating functions,

$$\begin{gathered}P^i(x,t) = \sum_{n=0}^{\infty}P^i(x,z,t)z^n; \quad P^i(z,t) = \sum_{n=0}^{\infty}P^i(t)z^n; \quad i = 1,2\\ V(x,z,t) = \sum_{n=0}^{\infty}V_n^{(i)}(x,t)z^n, \quad V(z,t) = \sum_{n=0}^{\infty}V_n^{(i)}(t)z^n, \quad R^i(z,t) = \sum_{n=0}^{\infty}R^i(t)z^n; \quad i = 1,2\end{gathered} \tag{16}$$

Taking Laplace transforms of equations (1) to (15)

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x,s) + (s + \lambda + \mu_1(x) + \alpha)\overline{P}_0^{(1)}(x,s) = 0 \tag{17}$$

$$\frac{\partial}{\partial x}\overline{P}_{n}^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_{n}^{(1)}(x,s)=\lambda\sum_{i=1}^{n-1}K_i\overline{P}_{n-i}^{(1)}(x,s),\quad n\geq 1 \tag{18}$$

$$\frac{\partial}{\partial x}\overline{P}_{0}^{(2)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_{0}^{(2)}(x,s)=0 \tag{19}$$

$$\frac{\partial}{\partial x}\overline{P}_{n}^{(2)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_{n}^{(2)}(x,s)=(x,s)\,\lambda\sum_{i=1}^{n-1}K_i\overline{P}_{n-i}^{(2)}(x,s),\quad n\geq 1 \tag{20}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s)+(s+\lambda+\eta(x))\,\overline{V}_0(x,s)=\nu\overline{V}_0(x,s) \tag{21}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,s)+(s+\lambda+\eta(x))\overline{V}_n(x,s)=\nu\overline{V}_{n+1}(x,s)+\lambda\sum_{i=1}^{n-1}K_i\overline{V}_{n-i}(x,s),\quad n\geq 1 \tag{22}$$

$$\frac{\partial}{\partial x}\overline{R}_{0}^{(1)}(x,s)+(s+\lambda+\beta_1)\overline{R}_{0}^{(1)}(x,s)=\nu R_{0}^{(1)}(x,s) \tag{23}$$

$$\begin{aligned}\frac{\partial}{\partial x}\overline{R}_{n}^{(1)}(x,s)+(s+\lambda+\beta_1)\,\overline{R}_{n}^{(1)}(x,s)&=\nu R_{n}^{(1)}(x,s)+\lambda\sum_{i=1}^{n-1}K_i\overline{R}_{n-i}^{(1)}(x,s)\\&+\alpha\int_0^{\infty}\overline{P}_{n-1}^{(1)}(x,s)+\alpha\int_0^{\infty}\overline{P}_{n-1}^{(2)}\;(x,s),\quad n\geq 1\end{aligned} \tag{24}$$

$$\frac{\partial}{\partial x}\overline{R}_{0}^{(2)}(x,s)+(s+\lambda+\beta_2)\overline{R}_{0}^{(2)}(x,s)=\nu\,(r\beta_1)\,R_{0}^{(1)}(x,s) \tag{25}$$

$$\frac{\partial}{\partial x}\overline{R}_{n}^{(2)}(x,s)+(s+\lambda+\beta_2)\,\overline{R}_{n}^{(2)}(x,s)=\lambda\sum_{i=1}^{n-1}K_i\overline{R}_{n-i}^{(2)}(x,s)+\nu\,(r\beta_1)\,\overline{R}_{n}^{(1)}(x,s) \tag{26}$$

$$(s+\lambda)\overline{Q}(x,s)=(1-r)\beta_1\overline{R}_{0}^{(1)}(x,s)+\beta_2\overline{R}_{0}^{(2)}(x,s)+\int_0^{\infty}\overline{V}_0(x,s)\,\eta(x)\,dx \tag{27}$$

For the boundary conditions

$$\overline{P}_{0}^{(1)}(0,s)=\lambda c_1\overline{Q}(s)+(1-r)\beta_1\overline{R}_{1}^{(1)}(s)+\beta_2\overline{R}_{1}^{(2)}(s)+\int_0^{\infty}\overline{V}_1(x,s)\,\eta(x)\,dx \tag{28}$$

$$\overline{P}_{n}^{(1)}(0,s)=\lambda c_{n+1}\overline{Q}(s)+(1-r)\beta_1\overline{R}_{n+1}^{(1)}(s)+\beta_2\overline{R}_{n+1}^{(2)}(s)+\int_0^{\infty}\overline{V}_{n+1}(x,s)\,\eta(x)\,dx, \tag{29}$$

$$\overline{P}_{n}^{(2)}(0,s)=\int_0^{\infty}\overline{P}_{n}^{(1)}(x,s)\mu_1(x)\,dx,\quad n\geq 0 \tag{30}$$

$$\overline{V}_n(0,s)=\int_0^{\infty}\overline{P}_{1}^{(2)}(x,s)\,\mu_2(x)\,dx,\quad n\geq 0 \tag{31}$$

Multiplying (18) and (20) by z^n and summing over n from 1 to ∞, adding to equation (17) and (19) and using (16). Similar operations on equations (21) to (27), (28) to (31) yields

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,z,s)+[s+\lambda(1-K(z))+\mu_1(x)+a]\overline{P}^{(1)}(x,z,s)=0 \tag{32}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}(x,z,s)+[s+\lambda(1-K(z))+\mu_2(x)+a]\overline{P}^{(2)}(x,z,s)=0 \tag{33}$$

$$\frac{\partial}{\partial x}\overline{V}(x,z,s)+\left(s+\lambda(1-K(z)+\eta(x)+\nu-\frac{\nu}{z}\right)\overline{V}(x,z,s)=0 \tag{34}$$

$$[s+\lambda-\lambda K(z)+\beta_1+\nu-\frac{\nu}{z}]\overline{R}^{(1)}(z,s)=\alpha z\int_0^{\infty}\overline{P}^{(1)}(x,z,s)\,dx+\alpha z\int_0^{\infty}P^{(2)}(x,z,s)\,dx \tag{35}$$

$$\left[s+\lambda-\lambda K(z)+\beta_2+\nu-\frac{\nu}{z}\right]\overline{R}^{(2)}(z,s)=\nu-\frac{\nu}{z}\,(r\beta_1)\,\overline{R}^{(1)}(z,s) \tag{36}$$

$$\begin{aligned}z\overline{P}^{(1)}(0,z,s)&=(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\overline{Q}(s)\\&+(1-r)\beta_1\overline{R}^{(1)}(s)+\beta_2\overline{R}^{(2)}(s)+\int_0^{\infty}\overline{V}(x,s)\,\eta(x)\,dx,\quad n\geq 0\end{aligned} \tag{37}$$

$$\overline{P}^{(2)}(0,z,s)=\int_0^{\infty}\overline{P}^{(1)}(x,z,s)\,\mu_1(x)\,dx \tag{38}$$

$$\overline{V}(0,z,s)=\int_0^{\infty}\overline{P}^{(2)}(x,s)\,\mu_2(x)\,dx \tag{39}$$

Integrating equations (32) to (36) between 0 and x, we get

$$\overline{P}^{(1)}(x,z,s) = \overline{P}^{(1)}(0,z,s)\, e^{-(s+\lambda(1-K(z))+\alpha)x-\int\limits_{0}^{\infty}\mu_{1(t)dt}} \tag{40}$$

$$\overline{P}^{(2)}(x,z,s) = \overline{P}^{(2)}(0,z,s)\, e^{-(s+\lambda(1-K(z))+\alpha)x-\int\limits_{0}^{\infty}\mu_{2(t)dt}} \tag{41}$$

$$\overline{V}(x,z,s) = \overline{V}(0,z,s)\, e^{-\left(s+\lambda(1-K(z))+\nu-\frac{\nu}{z}\right)x-\int\limits_{0}^{\infty}\eta(x)dx} \tag{42}$$

$$\overline{R}^{(1)}(x,z,s) = \overline{R}^{(1)}(0,z,s)\, e^{-\left(s+\lambda(1-K(z))+\nu-\frac{\nu}{z}\right)x-\int\limits_{0}^{\infty}\beta_{(t)dt}} \tag{43}$$

$$\overline{R}^{(2)}(x,z,s) = \overline{R}^{(2)}(0,z,s)\, e^{-\left(s+\lambda(1-K(z))+\nu-\frac{\nu}{z}\right)x-\int\limits_{0}^{\infty}\beta_{(t)dt}} \tag{44}$$

Again integrating equation (40) w.r.to x, we have

$$\overline{P}^{(1)}(z,s) = \overline{P}^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1\left(s+\lambda(1-K(z))+\alpha\right)}{(s+\lambda(1-K(z))+\alpha)}\right] \tag{45}$$

where

$$\overline{B}_1\left(s+\lambda(1-C(z))+a\right) = \int_0^\infty e^{-(s+\lambda(1-K(z))+\alpha)x} d\overline{B}_1(x)$$

is the Laplace transform of service time. Multiply equation (40) by $\mu_1(x)$ and integrate with respect to x

$$\int_0^\infty \overline{P}^{(1)}(x,z,s)\,\mu_1(x)\,dx = \overline{P}^{(1)}(0,z,s)\,\overline{B}_1\left(s+\lambda(1-K(z))+a\right) \tag{46}$$

Again integrating equation (41) w.r.to x, we have

$$\overline{P}^{(2)}(z,s) = \overline{P}^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2\left(s+\lambda(1-C(z))+\alpha\right)}{(s+\lambda(1-C(z))+\alpha)}\right] \tag{47}$$

where

$$\overline{B}_2\left(s+\lambda(1-K(z))+\alpha\right) = \int_0^\infty e^{-(s+\lambda(1-K(z))+\alpha)x} d\overline{B}_2(x)$$

Using equation (41) by $\mu_2(x)$ and integrating with respect to x

$$\int_0^8 \overline{P}^{(2)}(x,z,s)\,\mu_2(x)\,dx = \overline{P}^{(2)}(0,z,s)\overline{B}_2\left(s+\lambda(1-K(z))+\alpha\right) \tag{48}$$

Again integrating equation (42)

$$\overline{V}(z,s) = \overline{V}(0,z,s)\left[\frac{1-\overline{V}\left(s+\lambda\left(1-K(z)\right)+\nu-\frac{\nu}{z}\right)}{\left(s+\lambda\left(1-K(z)\right)+\nu-\frac{\nu}{z}\right)}\right] \tag{49}$$

where

$$\overline{V}(s+\lambda(1-K(z))+\nu-\frac{\nu}{z}) = \int_0^\infty e^{-\left(s+\lambda(1-K(z)+\nu-\frac{\nu}{z})\right)x} d\overline{V}(x)$$

Multiply (42) by $?\,(x)$ integrating with respect to x

$$\int_0^\infty \overline{V}(x,z,s)(x)\,\eta(x)\,dx = \overline{V}(0,z,s)\,\overline{V}\left(s+\lambda(1-K(z)+\alpha\right) \tag{50}$$

Equation (47) becomes

$$\overline{P}^{(2)}(z,s) = \overline{P}^{(1)}(0,z,s)\left[\frac{\overline{B}_1\left(s+\lambda\left(1-K(z)\right)+\alpha\right)\left(1-\overline{B}_2\left(s+\lambda\left(1-K(z)\right)+\alpha\right)\right.}{s+\lambda\left(1-K(z)\right)+\alpha}\right] \tag{51}$$

Using (48) to (50) in (39)

$$\overline{V}(0,z,s)=\overline{P}^{(1)}(0,z,s)\,\overline{B}_1(s+\lambda(1-K(z))+\alpha)+\overline{P}^{(2)}(0,z,s)\,\overline{B}_1(s+\lambda(1-K(z))+\alpha)\,\overline{B}_2(s+\lambda(1-K(z))+\alpha) \tag{52}$$

Using (43)and (44) in equations (35) and (36) becomes

$$\overline{R}^{(1)}(z,s)=\frac{az\overline{P}^{(1)}(0,z,s)\left[(1-\overline{B}_1(s+\lambda(1-K(z))+\alpha)\,\overline{B}_2\left(s+\lambda(1-K(z))+\nu-\frac{\nu}{z}\right)\right]}{(s+\lambda(1-K(z))+\alpha)(s+\lambda(1-K(z))+\beta_1)} \tag{53}$$

$$\overline{R}^{(2)}(z,s)=\frac{r\beta_1\alpha z\overline{P}^{(1)}(0,z,s)\left[1-\overline{B}_1(s+\lambda(1-K(z))+\alpha)\,\overline{B}_1\left(s+\lambda(1-K(z))+\nu-\frac{\nu}{z}\right)\right]}{(s+\lambda(1-K(z))+\alpha)(s+\lambda(1-K(z))+\beta_1)(s+\lambda(1-K(z))+\beta_2)} \tag{54}$$

Solving equation for $\overline{P}^{(1)}(0,z,s)$

$$\overline{P}^{(1)}(0,z,s)=\frac{f_1(z)[(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\,\overline{Q}(s)]}{Dr} \tag{55}$$

$$\overline{P}^{(1)}(0,z,s)=\frac{f_1(z)\left[(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\,\overline{Q}(s)\right]\overline{B}_1(f_1(z))}{Dr} \tag{56}$$

Where

$$Dr=f_1(z)\{z-((1-p)-p\overline{V}(s+\lambda(1-K(z))+\alpha))-\alpha z[(1-r_1)\,\beta_1\overline{f}_3(z)+r\beta_1\beta_2][1-\overline{f}_1(z)] \tag{57}$$

Where

$$\begin{aligned}\overline{f}_1(z)&=s+\lambda(1-C(z))+\alpha\\ \overline{f}_2(z)&=s+\lambda(1-C(z))+\beta_1\\ \overline{f}_3(z)&=s+\lambda(1-C(z))+\beta_2\end{aligned} \tag{58}$$

Substituting the value of $P^{(1)}(0,z)$ from equation (46) into equations (50) to (56) we get

$$\overline{P}^{(1)}(z,s)=\frac{[1-\overline{B}_1\overline{f}_1(z)]}{Dr}[(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\,\overline{Q}(s) \tag{59}$$

$$\overline{P}^{(2)}(z,s)=\frac{r\overline{B}_1\overline{f}_1(z)\,[1-\overline{B}_2\overline{f}_1(z)]]}{Dr}[(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\,\overline{Q}(s) \tag{60}$$

$$\overline{V}(z,s)=\frac{pf_1(z)[1-\overline{B}_1\left(\overline{f}_1(z)\right)\overline{B}_2\left(\overline{f}_1(z)\right)}{Dr}\left[\frac{1-\overline{V}(s+\lambda(1-K(z))+\beta_1)}{(s+\lambda(1-K(z))+\beta_1)}\right] \tag{61}$$

$$\overline{R}_q^{(1)}(z,s)=\frac{a\beta_2zf_1(z)[1-\overline{B}_1\left(\overline{f}_1(z)\,\overline{B}_2\left(\overline{f}_1(z)\right)\right)][(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\,\overline{Q}(s)]}{DR} \tag{62}$$

$$\overline{R}_q^{(2)}(z,s)=\frac{ra\beta_1z[1-\overline{B}_1\left(\overline{f}_1(z)\,\overline{B}_2\left(\overline{f}_1(z)\right)\right)][(1-s)\,\overline{Q}(s)+\lambda(K(z)-1)\,\overline{Q}(s)]}{Dr} \tag{63}$$

Where Dr is given in (57)

5. The Steady State Analysis

By using well known Tauberian property,

$$\lim_{s\to 0}s\overline{f}(s)=\lim_{t\to\infty}f(t) \tag{64}$$

Multiply both sides of equation (59) to (63) using (64)

$$P^{(1)}(z)=\frac{\left[1-B_1\left(\overline{f}_1(z)\right)\right][\lambda(K(z)-1)\,Q]}{DR} \tag{65}$$

$$P^{(2)}(z) = \frac{B_1(\overline{f}_1(z))[1 - B_2(\overline{f}_1(z))][\lambda(K(z) - 1)Q}{DR} \tag{66}$$

$$\overline{V}(z) = \frac{pf_1(z)[1 - \overline{B}_1(\overline{f}_1(z))\overline{B}_2(\overline{f}_1(z))]}{Dr}\left[\frac{1 - \overline{f}_2(z)}{\overline{f}_2(z)}\right]Q \tag{67}$$

$$\overline{R}_q^{(1)}(z) = \frac{az\beta_2 f_3(z)[1 - \overline{B}_1(\overline{f}_1(z))\overline{B}_2(\overline{f}_1(z))][\lambda(K(z) - 1)Q]}{DR} \tag{68}$$

$$\overline{R}_q^{(2)}(z) = \frac{ra\beta_1 z[1 - \overline{B}_1(\overline{f}_1(z))\overline{B}_2(\overline{f}_1(z))][\lambda(K(z) - 1)Q]}{DR} \tag{69}$$

Let $P_q(z)$ denote the PGF of the queue size irrespective of the server state. Then adding equation (68) to (69) we obtain

$$P_q(z) = P^{(1)}(z) + P^{(2)}(z) + V_q(z) + R_q^{(1)}(z) + R_q^{(2)}(z) \tag{70}$$

In order to determine $P^{(1)}(z)$, $P^{(2)}(z)$, $V_q(z)$, $R_q^{(1)}(z)$, $R_q^{(2)}(z)$ completely, we have yet to determine the unknown Q which appears in the numerator of the right sides of equation (68) to (69). For that purpose, we shall use the normalizing condition.

$$P_q(1) + Q = 1 \tag{71}$$

$$P^{(1)}(1) = \frac{\lambda\beta_1\beta_2 QE(I)\left(1 - \overline{B}_1(\alpha)\right)}{dr} \tag{72}$$

$$P^{(2)}(1) = \frac{\lambda\beta_1\beta_2 QE(I)\overline{B}_1(\alpha)[1 - \overline{B}_2(a)}{dr} \tag{73}$$

$$V(1) = \frac{\lambda\alpha\beta_1\beta_2 QE(I)E(v)[1 - \overline{B}_1(a)\overline{B}_2(a)]}{dr}$$

$$\overline{R}_q^{(1)}(1) = \frac{\lambda\alpha\beta_2 QE(I)[1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]}{dr} \tag{74}$$

$$\overline{R}_q^{(2)}(1) = \frac{r\lambda\alpha\beta_2 QE(I)[1 - \overline{B}_1(a)\overline{B}_2(a)]]}{dr} \tag{75}$$

Where

$$dr = \alpha\beta_1\beta_2[\lambda E(I)] - \lambda E(I)[r\alpha\beta_1 + \alpha\beta_1 + \beta_1\beta_2 - \nu] - \alpha\beta_1\beta_2(E(I) - \nu)E(V)(1 - B_1(\alpha)B_2(\alpha)]$$

$P^{(1)}(1)$, $P^{(2)}(1)$, $V(1)$, $R_q^{(1)}(1)$, $R_q^{(2)}(1)$ and Q are the steady state probabilities that the server is providing two heterogeneous service, server under repair, server under compulsory vacation and idle respectively without regard to the number of customers in the queue. Now using equations (72) to (75) into the normalizing condition (71) and simplifying, we obtain

$$Q = 1 - \lambda E(I)\left[\frac{1}{\alpha[(1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]} + \frac{1}{\beta_1[(1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]} + \frac{r}{\beta_2[(1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]} + \frac{r}{\beta_2} - \frac{1}{\beta_1} - \frac{1}{\alpha} + \nu E(V)\right] \tag{76}$$

and the utilization factor ρ of the system is given by

$$\rho = \lambda E(I)\left[\frac{1}{\alpha[(1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]} + \frac{1}{\beta_1[(1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]} + \frac{r}{\beta_2[(1 - \overline{B}_1(\alpha)\overline{B}_2(\alpha)]} + \frac{r}{\beta_2} - \frac{1}{\beta_1} - \frac{1}{\alpha} + \nu E(V)\right] \tag{77}$$

Where $\rho < 1$ is the stability condition under which the steady states exits.

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