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Dynamical Behaviour of Prey Predators Model with Time Delay

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Abstract: This paper describes a Prey and Two predators Ecological model. Continuous type gestational delay is incorporated in the interaction of Prey and first predator is taken for investigation. The system dynamics is studied at its equilibrium points. We construct a suitable Lyapunov function for global stability. The effect of Time delay on the dynamical behiviour of the system is studied. Using Numerical simulation, it is shown that the delay arguments with different kernels exhibit rich dynamics.

MSC: 34DXX.

Keywords: Equilibrium points, Local stability, Global stability, Time delay.

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1. Introduction

Mathematical modeling in biology and Ecology gains a lot of importance in recent decades. The results are in this field are very interesting. The models in population ecology are studied Lokta [1] and Volterra [2]. Most of the models in biology, medicine, epidemiology, ecology are widely discussed by Kapur [3, 4]. Authors [5–7] studied the stability and complexity of in population dynamics with different interaction in ecological models. These models are widely represented by differential equations. Braun [8] and Simon's [9] explain the applications of differential equations. In last few decades, Delay differential equations become popular in biology and ecology models. Time delays are naturally occurs in every biological and ecological phenomenon. These delays are significant in stability analysis. A delay can switch over from stable equilibrium to unstable or vice versa. A detailed time delay interactions are briefly explained by Cushing, J.M [10], Sreehari Rao [11], Gopalaswamy. K [12]. Paparao [13–17] studied the stability analysis of three species ecological models with time delay in prey, predator and competitor. Time delays in growth response are playing a major role in describing the stability of the systems. In the present paper is we study the stability analysis of a three species model with prey and two predators. We include the time delay on the interaction of prey and first predator. Local and global stability analysis is carried out at the equilibrium points. The effect of time delay arguments with different kernel strength is studied by numerical simulation in support of stability analysis using MAT LAB simulation. It is shown that the delay arguments exhibit rich dynamics.

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2. Mathematical Model

A three species ecological model with a prey and two predators are considered for investigation. Two predators namely first predator (N_2), second predator (N_3) are competing for the same prey (N_1). A time delay is introduced in the interaction of prey and first predator (Gestation period of the predator) Death rates of three populations are also considered for investigation .Keeping the above aspects in view, the model is characterized by the following system of integro-differential equations.

$$\begin{aligned}
\frac{dN_1}{dt} &= a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1 \int_{-\infty}^{t} k_2(t-u)N_2(u)du - \alpha_{13}N_1N_3 - d_1N_1 \\
\frac{dN_2}{dt} &= a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_2 \int_{-\infty}^{t} k_1(t-u)N_1(u)du - \alpha_{23}N_2N_3 - d_2N_2 \\
\frac{dN_3}{dt} &= a_3N_3 - \alpha_{33}N_3^2 + \alpha_{31}N_1N_3 - \alpha_{32}N_2N_3 - d_3N_3
\end{aligned} \tag{1}$$

Where the parameters in the above model is described as follows

2.1. Nomenclature

S.No	Parameter	Description
1	N_1, N_2 & N_3	Population strengths of prey, first predator and a second predator respectively
2	a_1, a_2, a_3	Growths rates of prey, first predator and second predator respectively
3	α_{ii} $(i = 1, 2, 3)$	Internal competition rates of prey ,first predator and second predator respectively (negative values)
4	α_{12}	Interaction coefficient of prey and first predator (negative value)
5	α_{21}	Interaction coefficient of first predator and prey (positive value)
6	α_{23}	Interaction coefficient of first predator and second predator(negative value)
7	α_{32}	Interaction coefficient of second predator and first predator (negative value)
7	α_{13}	Interaction coefficient of prey and second predator (negative value)
9	α_{31}	Interaction coefficient of second predator and prey (positive value)
10	d_1, d_2, d_3	Death rates of prey, first predator and a second predator
11	$k_1(t-u)$ & $k_2(t-u)$	Delay kernels of prey and first predator influence at time t.

Choose the kernels k_1 and k_2 such that

$$\int_0^{\infty} k_1(z)dz = 1, \quad \int_0^{\infty} k_2(z)dz = 1, \quad \int_0^{\infty} zk_1(z)dz < \infty, \quad \int_0^{\infty} zk_2(z)dz < \infty \tag{2}$$

By the normalization the system of equation (1) becomes

$$\begin{aligned}
\frac{dN_1}{dt} &= a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1 \int_{0}^{\infty} k_2(z)N_2(t-z)dz - \alpha_{13}N_1N_3 - d_1N_1 \\
\frac{dN_2}{dt} &= a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_2 \int_{0}^{\infty} k_1(z)N_1(t-z)dz - \alpha_{23}N_2N_3 - d_2N_2 \\
\frac{dN_3}{dt} &= a_3N_3 - \alpha_{33}N_3^2 + \alpha_{31}N_1N_3 - \alpha_{32}N_2N_3 - d_3N_3
\end{aligned} \tag{3}$$

3. Equilibrium States

By equating $\frac{dN_i}{dt} = 0$, $i = 1, 2, 3$ we get the following eight equilibrium states.

I. The Extinct state

$$E_1: \ \overline{N_1} = 0, \ \overline{N_2} = 0, \ \overline{N_3} = 0. \tag{4}$$

II. Semi Extinct

A: The state in which two of three species extinct and one survive

$$E_2:\ \overline{N_1}=0,\ \overline{N_2}=\frac{a_2-d_2}{\alpha_{22}},\ \overline{N_3}=0. \tag{5}$$

$$E_3:\ \overline{N_1}=0,\ \overline{N_2}=0,\ \overline{N_3}=\frac{a_3-d_3}{\alpha_{33}}. \tag{6}$$

$$E_4:\ \overline{N_1}=\frac{a_1-d_1}{\alpha_{11}},\ \overline{N_2}=0,\ \overline{N_3}=0 \tag{7}$$

B: Only one species Extinct and two are survive

$$E_5:\ \overline{N_1}=\frac{(a_1-d_1)\alpha_{22}-\alpha_{12}(a_2-d_2)}{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}},\ \overline{N_2}=\frac{(a_2-d_2)\alpha_{11}+\alpha_{21}(a_1-d_1)}{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}},\ \overline{N_3}=0$$

This case arise only when

$$(a_1-d_1)\alpha_{22}>\alpha_{12}(a_2-d_2) \tag{8}$$

$$E_6:\ \overline{N}_1=\frac{(a_1-d_1)\,\alpha_{33}-(a_3-d_3)\alpha_{13}}{\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}},\ \overline{N}_2=0,\ \overline{N}_3=\frac{(a_1-d_1)\alpha_{31}+(a_3-d_3)\alpha_{11}}{\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31}} \tag{9}$$

Exist only when

$$(a_1-d_1)\,\alpha_{33}>(a_3-d_3)\,\alpha_{13} \tag{10}$$

$$E_7:\ \overline{N}_1=0,\ \overline{N}_2=\frac{(a_2-d_2)\,\alpha_{33}-(a_3-d_3)\,\alpha_{23}}{\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32}},\ \overline{N}_3=\frac{(a_3-d_3)\,\alpha_{22}-(a_2-d_2)\,\alpha_{32}}{\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32}} \tag{11}$$

Exists only when

$$(a_2-d_2)\,\alpha_{33}>(a_3-d_3)\,\alpha_{23},\ (a_3-d_3)\,\alpha_{22}>(a_2-d_2)\,\alpha_{32}\ \&\ \alpha_{22}\alpha_{33}>\alpha_{23}\alpha_{32} \tag{12}$$

C: Co-existing state

$$\begin{aligned}
E_8:\ \overline{N}_1&=\frac{(a_1-d_1)\,(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+(a_2-d_2)\,(\alpha_{13}\alpha_{32}-\alpha_{12}\alpha_{33})+(a_3-d_3)\,(\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{22})}{\alpha_{11}\,(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}\,(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+\alpha_{13}\,(\alpha_{31}\alpha_{22}-\alpha_{21}\alpha_{32})}\\
\overline{N}_2&=\frac{(a_1-d_1)\,(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+(a_2-d_2)\,(\alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31})-(a_3-d_3)\,(\alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21})}{\alpha_{11}\,(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}\,(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+\alpha_{13}\,(\alpha_{31}\alpha_{22}-\alpha_{12}\alpha_{32})},\\
\overline{N}_3&=\frac{(a_1-d_1)\,(\alpha_{22}\alpha_{31}-\alpha_{21}\alpha_{32})-(a_2-d_2)\,(\alpha_{11}\alpha_{32}+\alpha_{12}\alpha_{31})+(a_3-d_3)\,(\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21})}{\alpha_{11}\,(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}\,(\alpha_{21}\alpha_{33}-\alpha_{31}\alpha_{23})+\alpha_{13}\,(\alpha_{31}\alpha_{22}-\alpha_{12}\alpha_{32})}.
\end{aligned} \tag{13}$$

This equilibrium state exist only when,

$$\overline{N}_1>0,\ \ \overline{N}_2\Rightarrow 0,\ \overline{N}_3>0 \tag{14}$$

4. Stability Analysis

We discuss the system stability for E_5, E_6, E_7 & E_8.

4.1. Stability analysis of equilibrium point E_5

The variational matrix for linearized system of (1) is given by

$$J=\begin{bmatrix} a_1-2\alpha_{11}\overline{N_1}-\alpha_{12}\overline{N_2}-d_1 & -\alpha_{12}\overline{N_1}k_2^*(\lambda) & -\alpha_{13}\overline{N_1}\\ \alpha_{21}\overline{N_2}k_1^*(\lambda) & a_2-2\alpha_{22}\overline{N_2}+\alpha_{21}\overline{N_1}-d_2 & -\alpha_{23}\overline{N_2}\\ 0 & 0 & a_3+\alpha_{31}\overline{N_1}-\alpha_{32}\overline{N_2}-d_3 \end{bmatrix} \tag{15}$$

The Characteristic equation of the system is

$$\left(\lambda-\left(a_3+\alpha_{31}\overline{N_1}-\alpha_{23}\overline{N_2}-d_3\right)\right)\left[\begin{array}{l}\left(\lambda-\left(a_2-2\alpha_{22}\overline{N_2}+\alpha_{21}\overline{N_1}-d_2\right)\right)\left(\lambda-\left(a_1-2\alpha_{11}\overline{N_1}-\alpha_{12}\overline{N_2}-d_1\right)\right)\\+\alpha_{12}\alpha_{21}\overline{N_1N_2}k_1^*(\lambda)k_2^*(\lambda)\end{array}\right]=0 \quad (16)$$

The system is stable if the following conditions are satisfied

$(i).$ $\left(a_3+\alpha_{31}\overline{N_1}\right)<\left(d_3+\alpha_{23}\overline{N_2}\right)$

$(ii).$ $\left(a_1+a_2+\alpha_{21}\overline{N_1}<2\alpha_{11}\overline{N_1}+2\alpha_{22}\overline{N_2}+\alpha_{12}\overline{N_2}+d_1+d_2\right)$

$$(iii).\ \left(\begin{array}{l}a_1a_2+a_1\alpha_{21}\overline{N_1}+\alpha_{12}\alpha_{21}\overline{N_1N_2}k_1^*(\lambda)k_2^*(\lambda)+3\alpha_{11}\alpha_{22}\overline{N_1N_2}+2\alpha_{12}\alpha_{22}\overline{N_2}^2+a_1d_2\\>2a_2\alpha_{11}\overline{N_1}+2a_1\alpha_{22}\overline{N_2}+a_2\alpha_{12}\overline{N_2}+2\alpha_{11}\alpha_{21}\overline{N_1}^2+2d_2\alpha_{11}\overline{N_1}+d_1d_2+d_2\alpha_{12}\overline{N_2}\end{array}\right) \quad (17)$$

4.2. Stability analysis of equilibrium point E_6

The Jacobian matrix for this case is

$$J=\begin{bmatrix}a_1-2\alpha_{11}\overline{N_1}-\alpha_{13}\overline{N_3}-d_1 & -\alpha_{12}\overline{N_1}k_2^*(\lambda) & -\alpha_{13}\overline{N_1}\\ 0 & a_2+\alpha_{21}\overline{N_1}-\alpha_{23}\overline{N_3}-d_2 & 0\\ \alpha_{31}\overline{N_1} & -\alpha_{32}\overline{N_2} & a_3-2\alpha_{33}\overline{N_3}+\alpha_{31}\overline{N_1}-d_3\end{bmatrix} \quad (18)$$

The characteristic equation is

$$\left(\lambda-\left(a_2+\alpha_{21}\overline{N_1}-\alpha_{23}\overline{N_3}-d_2\right)\right)\left[\begin{array}{l}\left(\lambda-\left(a_3-2\alpha_{33}\overline{N_3}+\alpha_{31}\overline{N_1}-d_3\right)\right)\left(\lambda-\left(a_1-2\alpha_{11}\overline{N_1}-\alpha_{13}\overline{N_3}-d_1\right)\right)\\+\alpha_{13}\alpha_{31}\overline{N_1N_3}\end{array}\right]=0 \quad (19)$$

The system is stable if the following conditions are satisfied

$(i).$ $(a_2+\alpha_{21}\overline{N_1})<\left(d_2+\alpha_{23}\overline{N_3}\right)$

$(ii).$ $\left(a_1+a_3+\alpha_{31}\overline{N_1}<2\alpha_{11}\overline{N_1}+2\alpha_{33}\overline{N_3}+\alpha_{13}\overline{N_3}+d_1+d_2\right)$

$$(iii).\ \left(\begin{array}{l}a_1a_3+a_1\alpha_{31}\overline{N_1}+4\alpha_{11}\alpha_{33}\overline{N_1N_3}+2\alpha_{13}\alpha_{33}\overline{N_3}^2+a_1d_3\\>2a_3\alpha_{11}\overline{N_1}+2a_1\alpha_{33}\overline{N_3}+a_3\alpha_{13}\overline{N_3}+2\alpha_{11}\alpha_{31}\overline{N_1}^2+2d_3\alpha_{11}\overline{N_1}+d_1d_3+d_3\alpha_{13}\overline{N_3}\end{array}\right) \quad (20)$$

4.3. Stability analysis of equilibrium point E_7

The variational matrix for this case is

$$J=\begin{bmatrix}a_1-\alpha_{12}\overline{N_2}-\alpha_{13}\overline{N_3}-d_1 & 0 & 0\\ \alpha_{21}\overline{N_2}k_1^*(\lambda) & a_2-2\alpha_{22}\overline{N_2}-\alpha_{23}\overline{N_3}-d_2 & -\alpha_{23}\overline{N_2}\\ \alpha_{31}\overline{N_1} & -\alpha_{32}\overline{N_2} & a_3-2\alpha_{33}\overline{N_3}-\alpha_{32}\overline{N_3}-d_3\end{bmatrix} \quad (21)$$

The characteristic equation is

$$\left(\lambda-\left(a_1-\alpha_{12}\overline{N_2}-\alpha_{13}\overline{N_3}-d_1\right)\right)\left[\begin{array}{l}\left(\lambda-\left(a_3-2\alpha_{33}\overline{N_3}-\alpha_{32}\overline{N_2}-d_3\right)\right)\\\left(\lambda-\left(a_2-2\alpha_{22}\overline{N_2}-\alpha_{23}\overline{N_3}-d_2\right)\right)-\alpha_{23}\alpha_{32}\overline{N_2N_3}\end{array}\right]=0 \quad (22)$$

The system is stable if the following conditions are satisfied

$(i).$ $\left(a_1-\alpha_{12}\overline{N_2}-\alpha_{13}\overline{N_3}-d_1\right)<0$

$(ii).$ $\left(a_2+a_3<2\alpha_{22}\overline{N_2}+2\alpha_{33}\overline{N_3}+\alpha_{23}\overline{N_3}+\alpha_{32}\overline{N_2}+d_2+d_3\right)$

$$(iii).\ \left(\begin{array}{l}a_1a_3+2\alpha_{23}\alpha_{32}\overline{N_3}^2+4\alpha_{22}\alpha_{33}\overline{N_2N_3}+2\alpha_{22}\alpha_{32}\overline{N_2}^2+a_2d_3\\>2a_3\alpha_{22}\overline{N_2}+2a_1\alpha_{33}\overline{N_3}+a_3\alpha_{23}\overline{N_3}+a_1\alpha_{32}\overline{N_2}+2d_3\alpha_{22}\overline{N_2}+d_2d_3+d_3\alpha_{23}\overline{N_3}\end{array}\right) \quad (23)$$

4.4. Stability of the equilibrium point E_8

Theorem 4.1. *The interior equilibrium point $E_8\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ is locally asymptotically stable if the condition (29) is satisfied.*

Proof. Let the variational matrix is given by

$$J=\begin{bmatrix} a_1-2\alpha_{11}\overline{N_1}-\alpha_{12}\overline{N_2}-\alpha_{13}\overline{N_3}-d_1 & -\alpha_{12}\overline{N_1}k_2^*(\lambda) & -\alpha_{13}\overline{N_1} \\ \alpha_{21}\overline{N_2}k_1^*(\lambda) & a_2-2\alpha_{22}\overline{N_2}+\alpha_{21}\overline{N_1}-\alpha_{23}\overline{N_3}-d_2 & -\alpha_{23}\overline{N_2} \\ \alpha_{31}\overline{N_3} & -\alpha_{32}\overline{N_3} & a_3-2\alpha_{33}\overline{N_3}+\alpha_{31}\overline{N_1}-\alpha_{32}\overline{N_3}-d_3 \end{bmatrix} \tag{24}$$

With The characteristic equation

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0 \tag{25}$$

Where

$$\begin{aligned} b_1 &= -\left(\lambda_1+\lambda_2+\lambda_3\right), \\ b_2 &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3+\alpha_{13}\alpha_{31}\overline{N_1N_3}+\alpha_{12}\alpha_{21}\overline{N_1N_2}k_1^*(\lambda)k_2^*(\lambda)+\alpha_{23}\alpha_{32}\overline{N_2N_3} \\ b_3 &= (\alpha_{12}\alpha_{31}\alpha_{23}\overline{N_1N_2N_3}k_2^*(\lambda)+\alpha_{13}\alpha_{21}\alpha_{32}\overline{N_1N_2N_3}k_1^*(\lambda)+\lambda_1\lambda_2\lambda_3 \\ &\quad -\lambda_1\alpha_{32}\alpha_{23}\overline{N_2N_3}+\lambda_3\alpha_{12}\alpha_{21}\alpha_{32}\overline{N_1N_2}k_1^*(\lambda)k_2^*(\lambda)+\lambda_2\alpha_{13}\alpha_{31}\overline{N_1N_3}) \end{aligned} \tag{26}$$

And

$$\begin{aligned} \lambda_1 &= \left(a_1-2\alpha_{11}\overline{N_1}-\alpha_{12}\overline{N_2}-\alpha_{13}\overline{N_3}-d_1\right), \\ \lambda_2 &= \left(a_2-2\alpha_{22}\overline{N_2}+\alpha_{21}\overline{N_1}-\alpha_{23}\overline{N_3}-d_2\right), \\ \lambda_3 &= \left(a_3-2\alpha_{33}\overline{N_3}+\alpha_{31}\overline{N_1}-\alpha_{32}\overline{N_2}-d_3\right) \end{aligned}$$

Here

$$(b_1b_2-b_3)=\begin{pmatrix} -\lambda_1^2(\lambda_2+\lambda_3)-\lambda_2^2(\lambda_1+\lambda_3)-\lambda_3^2(\lambda_1+\lambda_2)-4\lambda_1\lambda_2\lambda_3 \\ -(\lambda_1+\lambda_2)(\alpha_{12}\alpha_{21}\overline{N_1N_2}k_1^*(\lambda)k_2^*(\lambda))-(\lambda_1+\lambda_3)\alpha_{13}\alpha_{31}\overline{N_1N_3}+(\lambda_2+\lambda_3)\alpha_{23}\alpha_{32}\overline{N_2N_3} \end{pmatrix} \tag{27}$$

By Routh-Hurwitz criteria, the system is stable if

$$b_1>0,\ (b_1b_2-b_3)>0 \text{ and } b_3\left(b_1b_2-b_3\right)>0. \tag{28}$$

If

$$\left(a_1<2\alpha_{11}\overline{N_1}+\alpha_{12}\overline{N_2}+\alpha_{13}\overline{N_3}\right),\ \left(a_2+\alpha_{21}\overline{N_1}<2\alpha_{22}\overline{N_2}+\alpha_{23}\overline{N_3}\right),\ \left(a_3+\alpha_{31}\overline{N_1}<2\alpha_{33}\overline{N_3}+\alpha_{32}\overline{N_2}\right) \tag{29}$$

Therefore the interior equilibrium point $E_8\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ is locally asymptotically stable if the condition (29) is satisfied. □

5. Global Stability

Theorem 5.1. *The interior equilibrium point $E_8\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ is globally asymptotically stable.*

Proof. Let the Lyapunov function be

$$V\left(N_1,N_2,N_3\right)=\sum_{i=1}^{3}N_i-\overline{N_i}-\overline{N_i}\log\left(\frac{N_i}{\overline{N_i}}\right)+\frac{1}{2}\alpha_{12}\int_0^\infty k_1\left(z\right)\int_{t-z}^{t}\left[N_2-\overline{N_2}\right]^2dudz+\frac{1}{2}\alpha_{21}\int_0^\infty k_2\left(z\right)\int_{t-z}^{t}\left[N_1-\overline{N_2}\right]^2dudz \tag{30}$$

The time derivative of 'V' along the solutions of equations (1) is

$$\begin{aligned}V^1\left(t\right)=&\sum_{i=1}^{3}\frac{\left[N_i-\overline{N_i}\right]}{N_i}N_i^1+\frac{1}{2}\alpha_{12}\int_0^\infty k_1\left(z\right)\left[N_2-\overline{N_2}\right]^2dz-\frac{1}{2}\alpha_{12}\int_0^\infty k_1\left(z\right)\left[N_2\left(t-z\right)-\overline{N_2}\right]^2dz\\&+\frac{1}{2}\alpha_{21}\int_0^\infty k_2\left(z\right)\left[N_1-\overline{N_1}\right]^2dz-\frac{1}{2}\alpha_{21}\int_0^\infty k_2\left(z\right)\left[N_1\left(t-z\right)-\overline{N_1}\right]^2dz\end{aligned} \tag{31}$$

From the relation of (2) we have

$$\begin{aligned}V^1\left(t\right)=&\left[N_1-\overline{N_1}\right]\left(a_1-\alpha_{11}N_1-\alpha_{12}\int_0^\infty k_1\left(z\right)N_2(t-z)dz-\alpha_{13}N_3-d_1\right)\\&+\left[N_2-\overline{N_2}\right]\left(a_2-\alpha_{22}N_2+\alpha_{21}N_1\int_0^\infty k_2\left(z\right)N_1(t-z)dz-\alpha_{23}N_3-d_2\right)\\&+\left[N_3-\overline{N_3}\right]\left(a_3-\alpha_{32}N_3-\alpha_{31}N_1-\alpha_{32}N_2-d_3\right)+\frac{1}{2}\alpha_{12}\left[N_2-\overline{N_2}\right]^2+\frac{1}{2}\alpha_{21}\left[N_1-\overline{N_1}\right]^2\\&-\frac{1}{2}\alpha_{12}\int_0^\infty k_1\left(z\right)\left[N_2\left(t-z\right)-\overline{N_2}\right]^2dz-\frac{1}{2}\alpha_{21}\int_0^\infty k_2\left(z\right)\left[N_1\left(t-z\right)-\overline{N_1}\right]^2dz\end{aligned}$$

By proper choice of a_1, a_2 and a_3

$$\begin{aligned}a_1&=\alpha_{11}\overline{N_1}+\alpha_{13}N_3+\alpha_{12}\int_0^\infty k_1\left(z\right)N_2(t-z)dz+d_1\\a_2&=\alpha_{22}\overline{N_2}+\alpha_{23}\overline{N_3}-\alpha_{21}\int_0^\infty k_2\left(z\right)\overline{N}_1(t-z)dz+d_2\\a_3&=\alpha_{33}\overline{N_3}+\alpha_{31}\overline{N}_1+\alpha_{32}\overline{N}_2+d_3\\&=-\alpha_{11}\left(N_1-\overline{N_1}\right)^2-\alpha_{22}\left(N_2-\overline{N_2}\right)^2-\alpha_{33}\left(N_3-\overline{N_3}\right)^2+\left(\alpha_{13}+\alpha_{31}\right)\left(N_1-\overline{N_1}\right)\left(N_3-\overline{N_3}\right)\\&+\left(\alpha_{23}+\alpha_{32}\right)\left(N_2-\overline{N_2}\right)\left(N_3-\overline{N_3}\right)+\frac{1}{2}\alpha_{12}\left[N_2-\overline{N_2}\right]^2+\frac{1}{2}\alpha_{21}\left[N_1-\overline{N_1}\right]^2\\&-\frac{1}{2}\alpha_{12}\int_0^\infty k_1\left(z\right)\left[N_2\left(t-z\right)-\overline{N_2}\right]^2dz-\frac{1}{2}\alpha_{21}\int_0^\infty k_2\left(z\right)\left[N_1\left(t-z\right)-\overline{N_1}\right]^2dz\end{aligned} \tag{32}$$

Using the inequality

$$\begin{aligned}ab\le&\frac{a^2+b^2}{2},\int_0^\infty k_1\left(z\right)\left[N_2\left(t-z\right)-\overline{N_2}\right]^2\le\int_0^\infty k_1\left(z\right)dz=1,\ \int_0^\infty k_2\left(z\right)\left[N_1\left(t-z\right)-\overline{N_1}\right]^2\le\int_0^\infty k_2\left(z\right)dz=1,\\&=-\alpha_{11}\left(N_1-\overline{N_1}\right)^2-\alpha_{22}\left(N_2-\overline{N_2}\right)^2-\alpha_{33}\left(N_3-\overline{N_3}\right)^2+\frac{\left(\alpha_{13}+\alpha_{31}\right)}{2}\left[\left(N_1-\overline{N_1}\right)^2+\left(N_3-\overline{N_3}\right)^2\right]\\&+\frac{1}{2}\alpha_{12}\left[N_2-\overline{N_2}\right]^2+\frac{1}{2}\alpha_{21}\left[N_1-\overline{N_1}\right]^2+\frac{\left(\alpha_{23}+\alpha_{32}\right)}{2}\left[\left(N_2-\overline{N_2}\right)^2+\left(N_3-\overline{N_3}\right)^2\right]-\frac{1}{2}\left(\alpha_{12}+\alpha_{21}\right)\\&\le-\left\|\left(\alpha_{11}-\frac{1}{2}\alpha_{13}-\frac{1}{2}\alpha_{31}-\frac{1}{2}\alpha_{21}\right)\right\|\left(N_1-\overline{N_1}\right)^2-\left\|\left(\alpha_{22}-\frac{1}{2}\alpha_{12}-\frac{1}{2}\alpha_{23}-\frac{1}{2}\alpha_{32}\right)\right\|\left(N_2-\overline{N_2}\right)^2\\&-\left\|\left(\alpha_{33}-\frac{1}{2}\alpha_{13}-\frac{1}{2}\alpha_{31}-\frac{1}{2}\alpha_{23}-\frac{1}{2}\alpha_{32}\right)\right\|\left(N_3-\overline{N_3}\right)^2-\frac{1}{2}\left\|\left(\alpha_{12}+\alpha_{21}\right)\right\|\\V^1\left(t\right)\le&-\mu\sum_{i=1}^{3}\left[N_i-\overline{N_i}\right]^2<0\end{aligned}$$

Where $\mu=\min\left(\alpha_{11}+\alpha_{22}+\alpha_{33}-\frac{1}{2}\alpha_{13}-\frac{1}{2}\alpha_{31}-\frac{1}{2}\alpha_{23}-\frac{1}{2}\alpha_{32}-\frac{1}{2}(\alpha_{12}+\alpha_{21})\right)$

$$\frac{dV}{dt}<0,$$

Therefore the system is globally stable at interior equilibrium $E_8\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$. □

6. Numerical Example

Let us define the two kernelsas follows $k_1(u) = e^{-\alpha u}$, $k_2(u) = e^{-\beta u}$, $\alpha > 0$, $\beta > 0$. The results are simulated for the system of equations (2) Using MAT LAB simulation.

Example 6.1. *Let* $a_1 = 6$, $a_{11} = 0.01$, $a_{12} = 0.45$, $a_{13} = 0.3$, $a_2 = 2.5$, $a_{21} = 0.43$, $a_{22} = 0.1$, $a_{23} = 0.32$, $a_3 = 3$, $a_{31} = 0.01$, $a_{32} = 0.12$, $a_{33} = 0.23$, $d_1 = 0.02$, $d_2 = 0.02$, $d_3 = 0.03$, $N_1 = 15$, $N_2 = 15$, $N_3 = 15$.

For above mentioned parameters with different delay kernel values of α and β, the graphs are plotted and observe the dynamics of the system. The rich dynamics is observed and is shown in the Table 1. The graphs for above kernels are shown below: odd numbered figures shown the time series evolution and even numbered figures shows respective phase portraits

S.No	Parameters values α and β and Converging equilibrium point E	Nature of system
1	$\alpha = 0.5$, $\beta = 0.05$ E (0.08, 4.62, 10.45)	The system is asymptotically stable converging to a fixed equilibrium point which exhibits periodic oscillations and limit cycle behaviour .
2	$\alpha = 0.05$, $\beta = 0.5$ E (2.01, 0.12, 12.9)	The system is asymptotically stable converging to a fixed equilibrium point which exhibits periodic oscillations and limit cycle behaviour.
3	$\alpha = 0.05$, $\beta = 0.05$ E (0, 0, 13)	The prey and first predator populations are extinct. The second predator population stabilizes at a fixed point. The system exhibits limit cycles and periodic oscillations forms an asymptotically stable system.
4	$\alpha = 0.005$, $\beta = 1.5$ E (6, 0, 13)	The first predator population is extinct. The prey and second predator populations are exist. The system is asymptotically stable with periodic solutions.
5	$\alpha = 0.5$, $\beta = 1.5$ E (5.4, 2.67, 11.75)	The system is asymptotically stable and exhibit oscillatory behavior up to the time lag $t = 40$, later it stabilizes and converging to fixed equilibrium point.
6	$\alpha = 0.05$, $\beta = 1.5$ E (6, 0, 13)	The first predator population is extinct, hence the prey and second predator populations are surviving due to the lag in prey, predator interaction, the prey population exhibit oscillatory behaviour and exhibit limit cycles forms a stable system.
7	$\alpha = 1.5$, $\beta = 0.5$ E (0.8, 15, 6)	The system is asymptotically stable and exhibit oscillatory behavior up to the time lag $t = 20$, later it stabilizes and converging to fixed equilibrium point.
8	$\alpha = 1.5$, $\beta = 0.05$ E (0, 14, 5)	The Prey species is almost extinct due to lime lags in first predator, still it is serving. This lag also helps to sustain second predator. Hence the system exhibits stable behaviour.
9	$\alpha = 1.5$, $\beta = 0.005$ E (0, 0, 0)	The three populations are extinct and the system is stable and converging to origin.

Table 1.

For the above mentioned kernels the system is asymptotically stable and produces periodic solutions for the kernels shown in above table. The kernels for $\alpha = 1.5$, $\beta = 0.005$, the three populations are extinct. So delay has significant impact on the dynamics of the system.

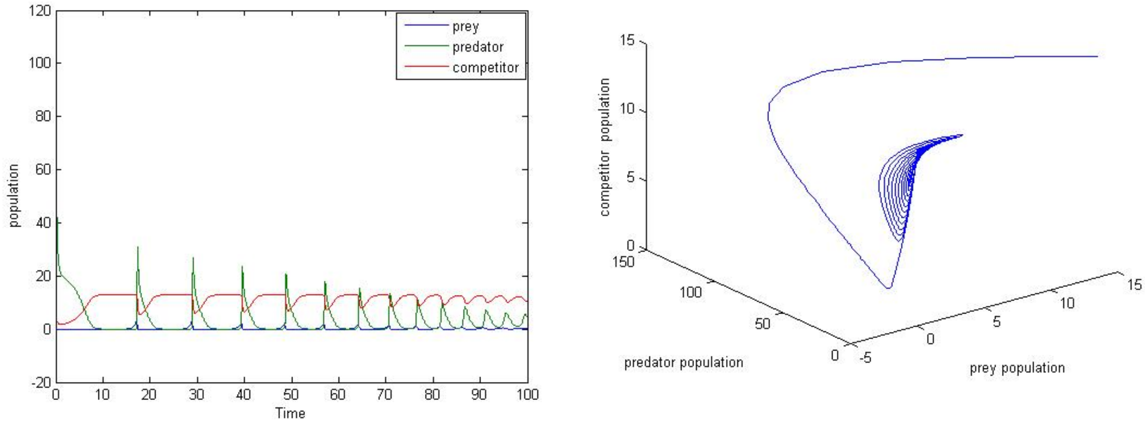

Figure 1. $\alpha = 0.5$, $\beta = 0.05$ **E (0.08, 4.62, 10.45)**

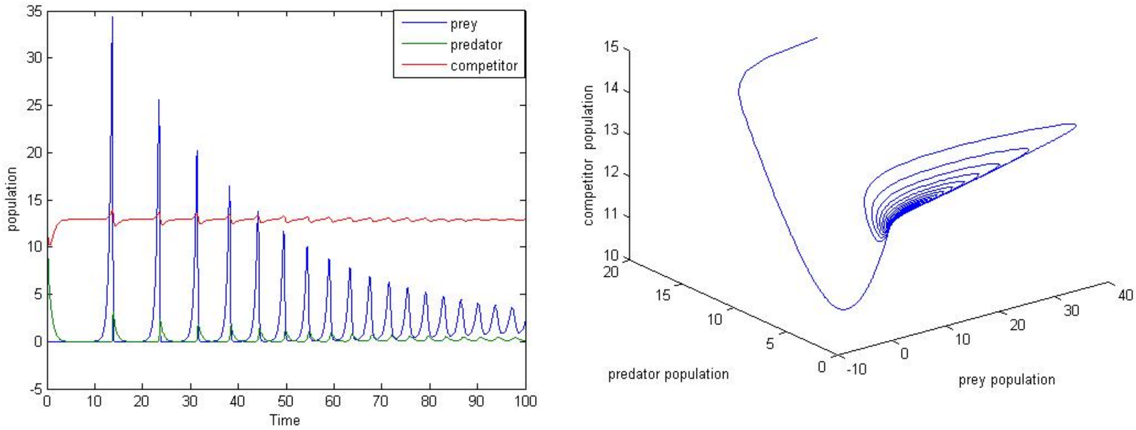

Figure 2. $\alpha = 0.05$, $\beta = 0.5$ **E (2.01, 0.12, 12.9)**

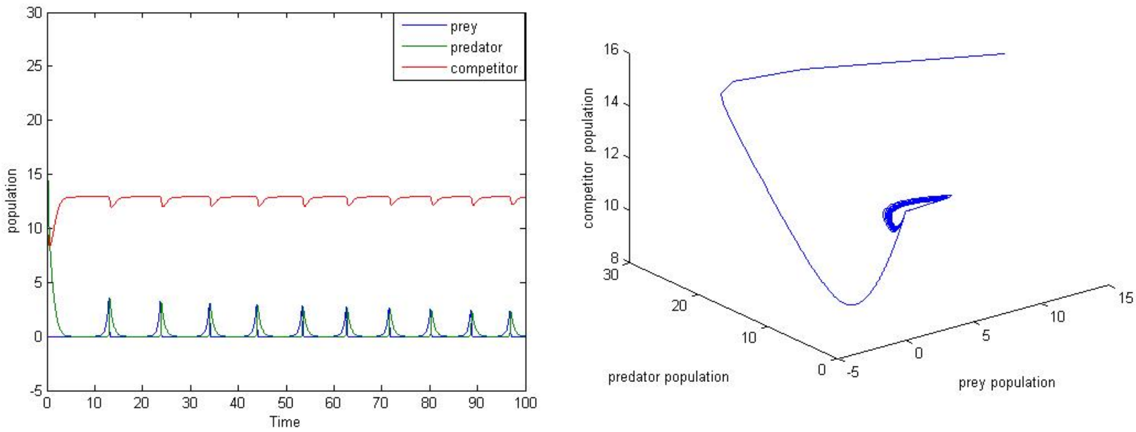

Figure 3. $\alpha = 0.05$, $\beta = 0.05$ **E (0, 0, 13)**

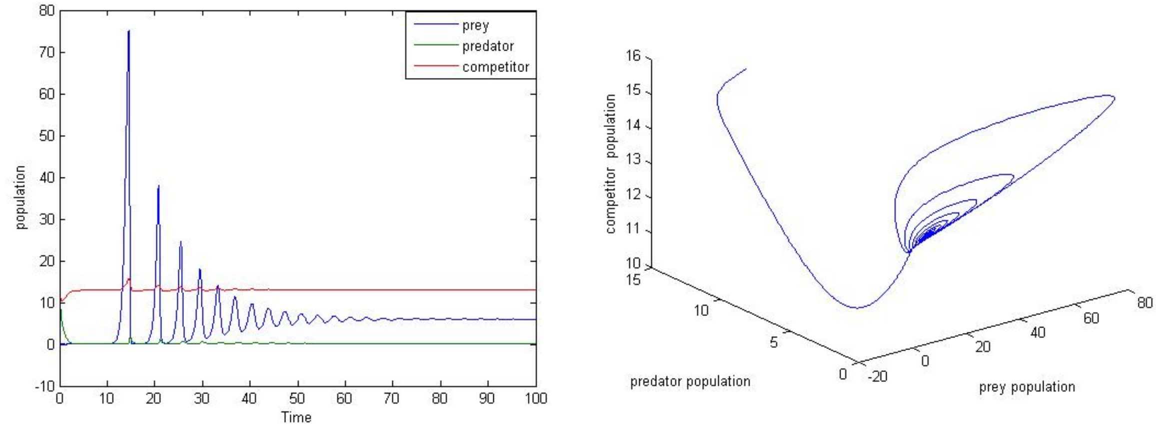

Figure 4. $\alpha = 0.005$, $\beta = 1.5$ **E (6, 0, 13)**

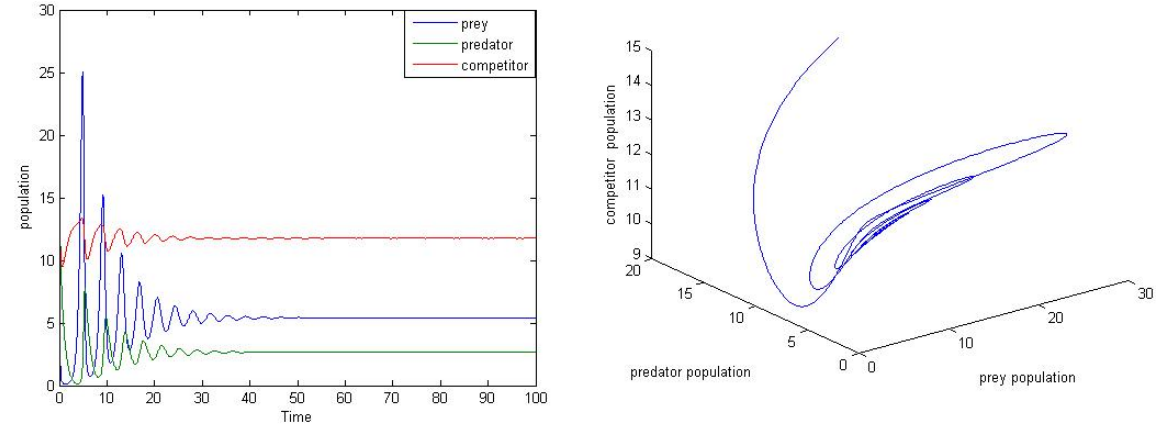

Figure 5. $\alpha = 0.5$, $\beta = 1.5$ **E (5.4, 2.67, 11.75)**

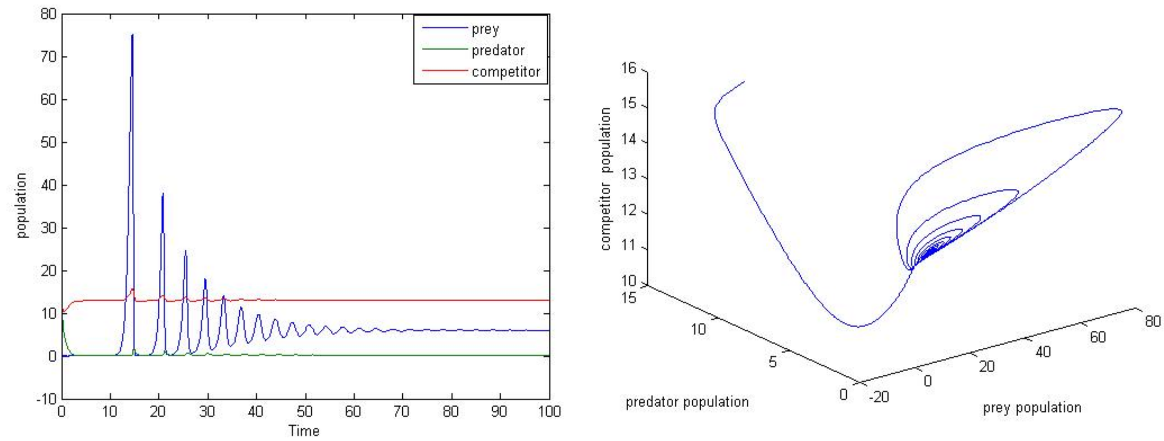

Figure 6. $\alpha = 0.05$, $\beta = 1.5$ **E (6, 0, 13)**

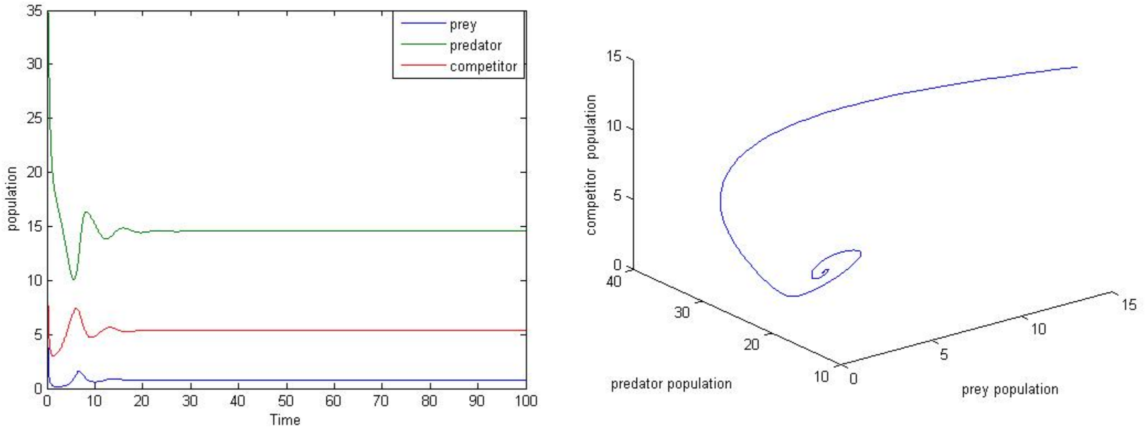

Figure 7. $\alpha = 1.5,\ \beta = 0.5$ **E (0.8, 15, 6)**

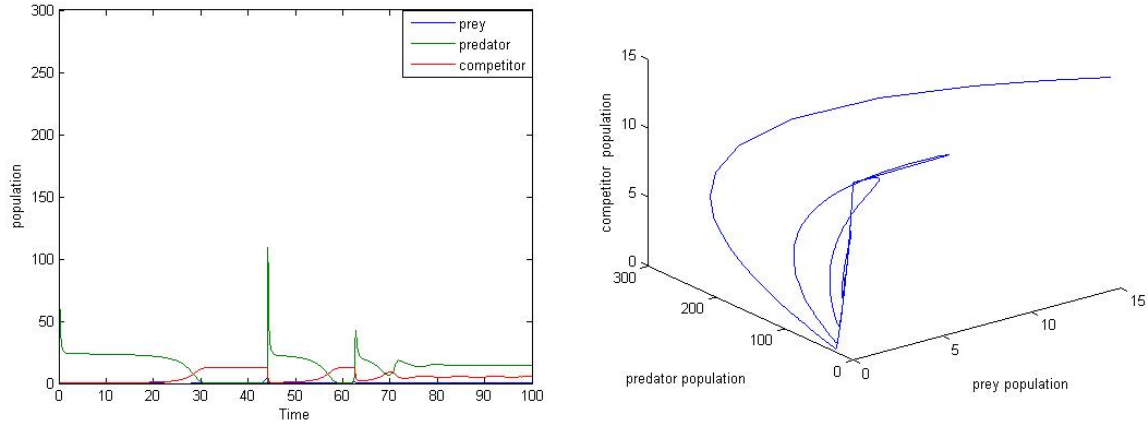

Figure 8. $\alpha = 1.5,\ \beta = 0.05$ **E (0, 14, 5)**

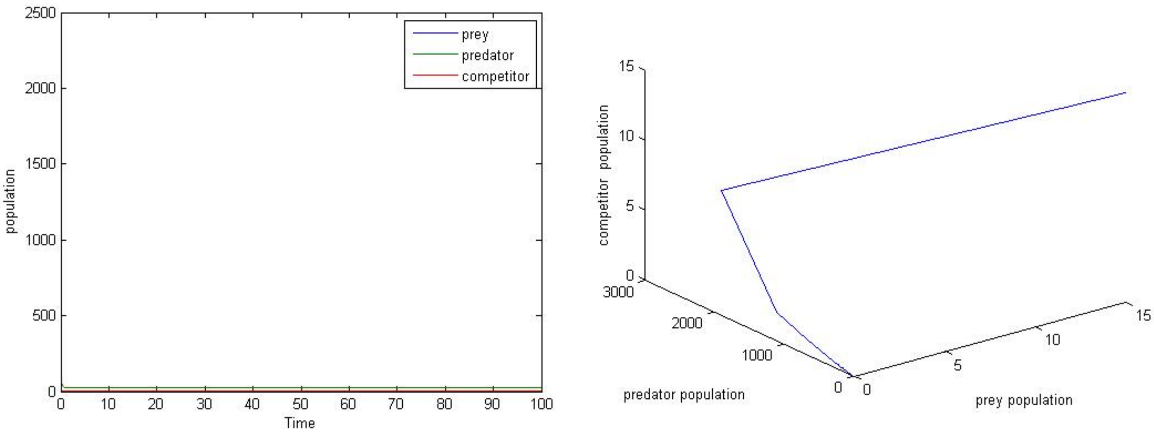

Figure 9. $\alpha = 1.5,\ \beta = 0.005$ **E (0, 0, 0)**

7. Conclusion

A three species ecological model with two Predators is considered for investigation. Here two predators are competing for the same Prey. The time delay is imposed on the prey and first predator species. The possible equilibrium points are identified .The system is conditionally stability for the equilibrium states E_5, E_6,E_7& E_8 .The global stability is studied by Lyapunov's function. The dynamics of the system is studied using numerical simulation in support of stability analysis. We consider a numerical examples in which the death rates of the populations are smaller than their birth rates. The impact of delay with different kernel strength is studied and observes the rich dynamics as shown in table1. The delay kernels are taken in [0.05, 1.5] for different combinations of two delay kernels, the systems is asymptotically stable, exhibit periodic solutions and limit cycles. Hence Delay arguments play a significant role in system dynamics.

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